Bulk Topological Proximity Effect

Timothy H. Hsieh,[1,*] Hiroaki Ishizuka,[1] Leon Balents,[1] and Taylor L. Hughes[2]
[1]*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*
[2]*Department of Physics, Institute for Condensed Matter Theory, University of Illinois at Urbana–Champaign, Champaign, Illinois 61801, USA*
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Existing proximity effects stem from systems with a local order parameter, such as a local magnetic moment or a local superconducting pairing amplitude. Here, we demonstrate that despite lacking a local order parameter, topological phases also may give rise to a proximity effect of a distinctively inverted nature. We focus on a general construction in which a topological phase is extensively coupled to a second system, and we argue that, in many cases, the inverse topological order will be induced on the second system. To support our arguments, we rigorously establish this "bulk topological proximity effect" for all gapped free-fermion topological phases and representative integrable models of interacting topological phases. We present a terrace construction which illustrates the phenomenological consequences of this proximity effect. Finally, we discuss generalizations beyond our framework, including how intrinsic topological order may also exhibit this effect.

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Topological phases of matter cannot be characterized by a local order parameter, unlike conventional ordered phases with, e.g., magnetic order. Instead, one discriminates them from so-called trivial phases of matter by analyzing either their global properties, such as topological invariants [1–5] and entanglement fingerprints [6–20], or their boundaries, which often exhibit robust gapless states. Despite such subtleties in characterization, significant progress has been made in understanding the different varieties and phenomenology of topological phases. One broad class of states involves those with intrinsic topological order [21] which possess excitations with fractional statistics, topological ground state degeneracy, and long-range entanglement. Additionally, even systems without intrinsic topological order can support robust topological phases in the presence of symmetry. In particular, two gapped phases which cannot be smoothly connected without either closing the gap or breaking a given symmetry constitute distinct symmetry protected topological (SPT) phases [22–24]. Both classes of topological matter—intrinsic and SPT—have been realized, respectively, in fractional quantum Hall systems [25] and, more recently, in a variety of topological insulators [2].

Given the maturity of material synthesis and engineering processes, one can easily imagine fabricating topological materials in proximity to other systems. Indeed, depositing superconductors and magnets on 2D and 3D time-reversal invariant topological insulators has become a booming industry [26–29], as the superconducting proximity effect has featured prominently in proposals for realizing Majorana fermions in solid-state materials [26]. However, the opposite effect—how the topological state might, in turn, affect proximate systems—has not been explored to the same extent. In conventional proximity effects, such as those involving superconducting or magnetic phases, there is an order parameter which penetrates into a proximate material. Is this phenomenon different, or denied, for topological phases which do not carry a local order parameter?

In this Letter, we demonstrate that, despite lacking a local order parameter, topological phases can nonetheless exhibit a proximity effect in which a topologically nontrivial system causes a proximate system to become topologically nontrivial as well. In many cases, the induced topological phase of the proximate system will be an "inverse" of the original phase, to be made precise shortly. In other cases, the *entire* combined system can be driven into a new topological phase by simply increasing the intersystem coupling. To avoid confusion, we emphasize that this phenomenon is fundamentally different from the "topological proximity effect" discussed in Ref. [30], where the gapless boundary states of a three-dimensional topological insulator essentially move into a proximate thin metallic film. In contrast, the setup we envision involves coupling two *bulk* systems *of the same dimension* to each other, and inducing a full bulk topological phase; it is not just an effect on the boundary states. We will hereafter refer to this phenomenon as the bulk topological proximity effect (BTPE).

The structure of this Letter is as follows. We first detail the setup for the BTPE and present general arguments for why it is expected to work for a large class of topological phases and intersystem couplings. For concreteness, we demonstrate this phenomenon in a bilayer of free-fermion systems in which one system with Chern number -1 induces Chern number $+1$ in the second system. After rigorously generalizing to all free-fermion topological phases, we argue how this effect may be realized in coupled spin chains, and we explicitly demonstrate it in a one-dimensional SPT protected by $Z_2 \times Z_2$ symmetry.

Finally, we discuss generalizations of the BTPE, including applications to intrinsic topological order.

General framework.—Our setup is a composite system consisting of subsystems A and B, with identical Hilbert spaces. However, we will be interested in Hamiltonians for the combined system

$$H = H_A + H_B + H_{AB}, \tag{1}$$

in which H_A and H_B are very different: H_A is a gapped Hamiltonian with a topologically nontrivial ground state and an energy gap of order Δ_A, while H_B is *any* Hamiltonian with a much smaller energy scale $W_B \ll \Delta_A$. H_{AB} is a coupling between the subsystems with strength g_{AB} which, we assume, obeys $W_B \ll g_{AB} \ll \Delta_A$. The motivation for $g_{AB} \ll \Delta_A$ is that we would like to have a notion of A and B as two independent subsystems (meaning the joint ground state approximately factorizes: $|\psi^0_{AB}\rangle \approx |\psi^0_A\rangle|\psi^0_B\rangle$), and we require $W_B \ll g_{AB}$ so that subsystem B is "susceptible" to A's topological Hamiltonian. The precise nature of H_B is irrelevant in the strong intersystem coupling limit, and for simplicity we will focus on the extreme limit $H_B = 0$; in other words, without the coupling, system B is a set of independent degrees of freedom, as in a Kondo lattice model. Having a nonvanishing H_B can lead to richer phase diagrams in the intermediate coupling regime, and we give an example of this below.

Let us first focus on a particular (and large) class of topological Hamiltonians and a coupling H_{AB} that could be dominant in realistic systems. Specifically, we consider any Hamiltonian H_A which has an "invertible" topological ground state ψ_A; i.e., there must exist an inverse $\overline{\psi_A}$ such that the direct product of the two, $\psi_A \otimes \overline{\psi_A}$, is a topologically trivial state [31–33]. Examples of invertible states are all SPTs, including fermionic SPTs such as Chern insulators, and Kitaev or Majorana wires. As for the coupling H_{AB}, we focus on those whose ground state is a product state of maximally entangled A and B sites, as depicted in Fig. 1(c). Such couplings are prevalent and include, e.g., interlayer tunneling in bilayer fermion systems and antiferromagnetic exchange coupling for bilayer spin systems.

With these mild assumptions, at infinite coupling between A and B, the combined system is topologically trivial because it is a product state. As the coupling is decreased, naively, there could be a phase transition to a topologically nontrivial ground state. In this Letter, however, we demonstrate that the strong and weak coupling phases are smoothly connected for *all* free-fermion systems and a class of interacting integrable models below. Since this implies that the weak coupling phase of the composite system is also topologically trivial, system B must carry the inverse topological order as system A, even at weak coupling, thus realizing BTPE. From another perspective, system B "screens" the topological phase of system A so that the composite system is neutralized or trivial. The above holds for $H_B = 0$; if $H_B \neq 0$, system B can potentially overscreen A and induce new topological properties for the composite system.

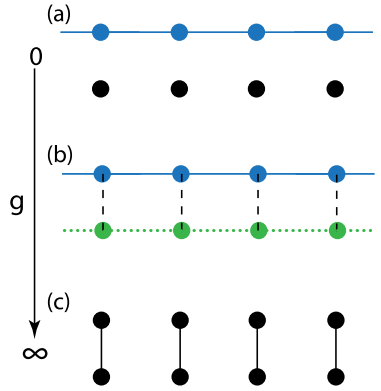

FIG. 1. Schematic of the bulk topological proximity effect. (a) The starting point is a topological state in system A (blue) and free, isolated degrees of freedom in system B (black). (b) At small coupling (the dashed lines), the inverse topological phase is induced in B, and the composite is trivial. (c) At infinite coupling, the system is a product state.

Free-fermion BTPE.—We now provide a concrete example of the topological proximity effect due to a Chern insulator. Specifically, we use the following tight-binding model for spin-1/2 fermions on the square lattice:

$$H_{CI}(\mu) = \sum_k h_{\alpha\beta}(k) c_{Ak\alpha} c_{Ak\beta}$$
$$h(k) = (\cos k_x + \cos k_y - \mu)\sigma^z + \sin k_x \sigma^x + \sin k_y \sigma^y. \tag{2}$$

Here, $c_{Ak\sigma}$ ($c^\dagger_{Ak\sigma}$) is the fermion annihilation (creation) operator with wave number k and spin σ on layer A and $\sigma^{x,y,z}$ are the Pauli spin matrices. For $0 < \mu < 2$, the ground state has the Chern number -1 and for $\mu > 2$, the ground state is trivial. The full Hamiltonian is given by

$$H_A = H_{CI}(\mu = 1), \qquad H_B = 0$$
$$H_{AB} = g \sum_i c^\dagger_{Ai\sigma} c_{Bi\sigma} + \text{H.c.}$$

For weak coupling, degenerate perturbation theory to second order provides an effective Hamiltonian for B given by

$$H^{\text{eff}}_B = \sum_k h^{\text{eff}}_{\alpha\beta}(k) c^\dagger_{Bk\alpha} c_{Bk\beta}, \tag{3}$$

$$h^{\text{eff}}(k) = -\frac{4g^2}{\Delta(k)} |\psi^{\text{ex}}_k\rangle\langle\psi^{\text{ex}}_k|, \tag{4}$$

where $|\psi^{\text{ex}}_k\rangle$ is the spinor of the excited state of $h(k)$, and $\Delta(k)$ is its energy difference from the ground state of $h(k)$. However, the conduction band of H_A must have the opposite Chern number of the valence band (if both bands are filled, then system A would be a trivial insulator). Thus, we find that the effective ground state of H_B has the Chern number $+1$.

In fact, in all free-fermion models of the above type, in which a nontrivial topological system A is tunnel coupled to a flat band in the band gap of system A, the strong and weak coupling limits are smoothly connected without any phase transition. This is because the full single-particle Hamiltonian has the form

$$H = \begin{pmatrix} H_A & g \\ g & 0 \end{pmatrix} = \frac{1}{2}(\mathbb{I} + \tau^z) \otimes H_A + g(\tau^x \otimes \mathbb{I}), \quad (5)$$

where the upper-left and lower-right blocks correspond to the local basis of systems A and B, respectively [for convenience, we have centered the band gap of A and the flat band(s) of B at zero energy]. However, even after H_A is diagonalized ($H_A^d = U H_A U^{-1}$), the form of the above matrix can be preserved by changing the basis of B with the same unitary U:

$$H = \mathsf{U}^{-1} \begin{pmatrix} H_A^d & g \\ g & 0 \end{pmatrix} \mathsf{U}, \quad (6)$$

with $\mathsf{U} = \mathbb{I} \otimes U$. The resulting eigenvalues are $E_n^{\pm} = (E_n^0/2) \pm \sqrt{(E_n^0/2)^2 + g^2}$, where E_n^0 represents the eigenvalues of H_A. Hence, all eigenvalues are repelled from $E = 0$ if $g \neq 0$, implying that there is no phase transition between strong and weak coupling. This unambiguously establishes the BTPE for all free-fermion topological phases.

The interpretation of topological "screening" can be explicitly shown from the ground state wave function

$$|\psi\rangle = \prod_{E_n<0} \left(\cos\frac{\theta_n}{2} |\varphi_n^{\text{fill}}\rangle_A + \sin\frac{\theta_n}{2} |\varphi_n^{\text{fill}}\rangle_B \right) \times \prod_{E_n>0} \left(\sin\frac{\theta_n}{2} |\varphi_n^{\text{ex}}\rangle_A - \cos\frac{\theta_n}{2} |\varphi_n^{\text{ex}}\rangle_B \right), \quad (7)$$

where $\tan\theta_n = g/E_n^0$ and $|\varphi_n^{\alpha}\rangle_a$ ($\alpha = \text{ex}, \text{fill}$, $a = A, B$) is the single-particle wave function for excited (ex) and filled (fill) states of H_A at $g = 0$; the index a denotes the layer that the electron is on. As $g \to 0$ ($\theta_n \to 0$), the ground state wave function is approximately the direct product of a $C = -1$ band of electrons in A and a $C = 1$ band in B, consistent with the perturbation argument above. For an arbitrary g, the Berry curvature for the bands in the first and second products are exactly the same as that of the filled and excited bands, respectively, of H_A at $g = 0$.

Since the composite system is trivial, the nontrivial nature of B is not immediately manifest. If A and B are strictly identical, then the composite system with boundary will have a gapped edge since the gapless mode of A hybridizes with the counterpropagating mode of B [see Figs. 2(a) and 2(b)]. However, in a terrace construction in which the boundaries of A and B are separated, as shown in Fig. 2(c), the gapless modes on the two different boundaries

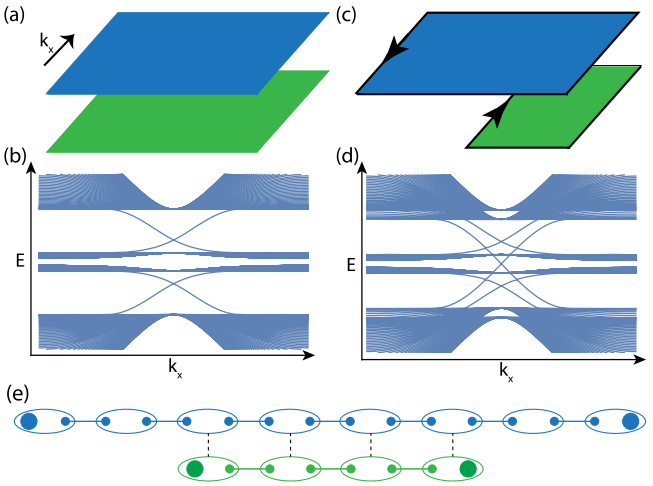

FIG. 2. Topological proximity effect of a Chern insulator. (a) A bilayer slab (in-)finite in the (x) y direction, with the spectrum shown in (b). (c) A "terrace" in which the left boundary of A is separated from that of B. (d) The resulting spectrum, with weakly coupled counterpropagating edge states in the bulk gap localized at the distinct boundaries. (e) Schematic of a Kitaev wire BTPE. Note the original Majorana zero modes (the large blue dots) and the new Majorana modes (the large green dots).

interact weakly and are exhibited clearly in Figs. 2(c) and 2(d). We note that, in this terrace construction, increasing the coupling shifts the weight of the edge mode initially on the B boundary onto the A side of the kink. This construction works for any system, and we illustrate it for a Kitaev chain in Fig. 2(e), in which Majorana bound states are induced at the ends of system B.

In realistic models, H_B is likely to be nonzero, but as long as the intersystem coupling g is much larger than the energy scale(s) in H_B, the BTPE of the above type will occur. However, the properties of the small-g regime depend on the nature of H_B, and they may exhibit a wide range of phases. To illustrate this in the Chern insulator example, we also considered $H_B = \gamma H_{CI}(\mu)$ of the form (2), but scaled down by a factor of γ ($0 < \gamma < 1$) relative to H_A [34]. Depending on the coupling g and the intrinsic phase of B, parametrized by μ, a variety of Chern numbers for the composite system can be achieved [see Fig. 3(a)]. First, consider the $g = 0$ limit: the total Chern number is simply the sum of the Chern numbers of each layer. Upon increasing the coupling, however, it is striking that the *composite* system can carry the *opposite* Chern number of the original topological phase. In this case, for a slab of the two systems, the gapless edge mode remarkably reverses direction at finite coupling [see Fig. 3(b)]. This effect can be thought of as an overscreening of the topological phase of H_A, where instead of canceling the Chern number it induces a composite Chern number of the opposite sign. Optimistically, one could design a device utilizing this effect: since the chirality is controlled by the interlayer coupling, one could imagine creating a low-dissipation,

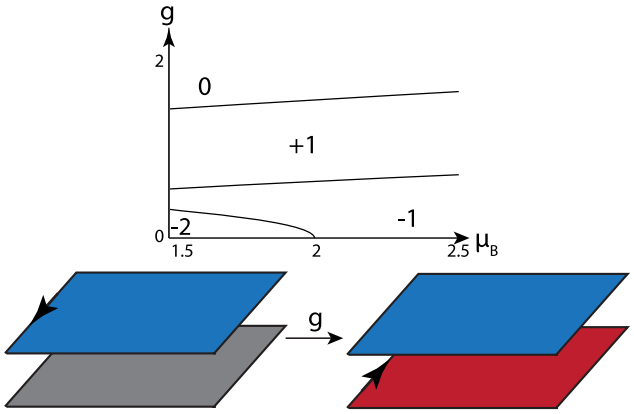

FIG. 3. Effect of nonzero Hamiltonian for system B. (a) Taking $H_A = H_{CI}(\mu = 1)$ (Chern number -1 ground state) and $H_B = 0.2H_{CI}(\mu)$, the phase diagram as a function of coupling g and μ is shown. The phases are labeled by the total Chern number of the composite system. (b) Intermediate coupling can completely reverse the direction of the edge state.

pressure-switchable diode where the pressure modifies the interlayer tunneling to switch the easy-current direction.

Interacting BTPE.—We expect the BTPE to occur generally in interacting systems as well. To illustrate, let us begin with a concrete model in which the effect can be explicitly demonstrated. For system A, we choose the "cluster state" [35–37] or "ZXZ" Hamiltonian of a spin-1/2 chain:

$$H_A = -\sum_i \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z. \tag{8}$$

This system possesses a $Z_2 \times Z_2$ symmetry generated by independent twofold rotations on the two alternating sublattices a and b: $g_1 = \prod_{j\in a} e^{i\pi\sigma_j^x/2}$, $g_2 = \prod_{j\in b} e^{i\pi\sigma_j^x/2}$. One-dimensional systems with this symmetry have a Z_2 topological classification, and the above Hamiltonian provides an example of the nontrivial phase. Now we add another spin-1/2 chain B, with $H_B = 0$, and introduce the coupling

$$H_{AB} = g\sum_i (\sigma_i^z \tilde{\sigma}_i^z + \sigma_i^x \tilde{\sigma}_i^x), \tag{9}$$

where the operators with tildes act on the B spins. Per our framework, such a coupling, when dominant, locks together the corresponding A and B spins into singlets.

Because of the fact that H_A is integrable (all eigenstates can be labeled by the conserved quantities $\sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z = \pm 1$), degenerate perturbation theory can be performed exactly. Since all one-body and two-body operators anticommute with at least one conserved quantity, the lowest order effect occurs at $O(g^3)$ and yields the effective Hamiltonian for B:

$$H_B^{\text{eff}} = g^3 \sum_i \tilde{\sigma}_i^z \tilde{\sigma}_{i+1}^x \tilde{\sigma}_{i+2}^z. \tag{10}$$

Hence, system B precisely inherits the topological order of system A (a Z_2 SPT is its own inverse).

This behavior generalizes to other interacting SPTs, such as a spin-1 chain in the Haldane phase [38] coupled to a Kondo lattice chain of isolated spin-1 sites via antiferromagnetic exchange. In this case, second order perturbation theory will imprint the exponentially decaying correlations of chain A onto the effective Hamiltonian of chain B, and we expect that the resulting effective nearest neighbor antiferromagnetic interaction will place chain B in the Haldane phase as well.

Beyond the framework.—While the assumptions specified and exemplified above are useful for understanding a large class of topological phases and their couplings to proximate systems, it is interesting to consider relaxing those assumptions. In particular, thus far we have considered identical Hilbert spaces for systems A and B, but allowing B to be a different lattice, dimension, or even particle type may give rise to new phenomena when coupled to A. In a similar vein, the current framework specializes to couplings which, when infinite, maximally entangle corresponding degrees of freedom of A and B, and the myriad ways of relaxing this constraint, especially when A and B are different Hilbert spaces, may prove fruitful. For example, one may imagine coupling a bosonic SPT state (system A) to a fermionic auxiliary system (system B) or vice versa [39].

Another interesting avenue involves invertible topological phases with nonunique inverses. For example, system A could be *stably* equivalent to a conventionally inequivalent state, i.e., only equivalent after being combined with additional trivial states. These systems might induce a "stable" proximity effect where they are screened not by their conventional inverse, but by a stably equivalent inverse. For example, a $\nu = 8$ fermionic integer quantum Hall state could be screened by a free-fermion $\nu = -8$ state, or the (antichiral) bosonic integer quantum Hall E_8 state to which its inverse is stably equivalent [40,41]. We expect that the choice of coupling terms between A and B would determine this outcome.

Finally, we note that systems with intrinsic topological order may also exhibit the BTPE, albeit of a different variety than that considered above. For example, consider two-dimensional Z_2 topological order, realized by the "toric code" Hamiltonian [42] for spin 1/2s on the links of a square lattice:

$$H_A = -\sum_s \prod_{l\in s} \sigma_l^z - \sum_p \prod_{l\in p} \sigma_l^x, \tag{11}$$

where s denotes stars of four links emerging from each vertex, and p denotes spins on the square plaquettes. Like the cluster state model, all of the terms in the Hamiltonian commute, and thus they provide conserved quantities to label eigenstates. However, given periodic boundary conditions, there are an additional two global conserved quantities which endow A with a fourfold ground state degeneracy.

Upon adding an identical Hilbert space B of free spins $\tilde{\sigma}$ and coupling A and B with the Hamiltonian in Eq. (9), the

lowest order effect [at $O(g^4)$] is to induce an effective Hamiltonian in B given by the same star and plaquette terms above. Hence, at small coupling, there is a proximity effect in which the ground state of the composite system is two copies of the toric code ground states. In contrast to the above framework for invertible topological order, these two copies are not smoothly connected to the trivial product state at infinite coupling—the ground state degeneracy at weak coupling is 16-fold. As a result, there is at least one topological phase transition at an intermediate g. How this occurs is nontrivial and would be an interesting subject for future study.

The general setup for BTPE requires a separation of energy scales for A and B, which is natural for many systems, including Kondo lattices. Several parameters may control the coupling g, including matrix elements between the states of A and B (which depend on their constituents), the separation between A and B (tunable by pressure or strain), and, potentially, tunneling barriers between A and B (as used in the superconducting proximity effect). Finally, while BTPE is most natural for one- and two-dimensional systems, it is possible that, in three dimensions, the roles of A and B can be played by sublattices or internal degrees of freedom.

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*thsieh@kitp.ucsb.edu

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