Converting Nonclassicality into Entanglement

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Quantum mechanics exhibits a wide range of nonclassical features, of which entanglement in multipartite systems takes a central place. In several specific settings, it is well known that nonclassicality (e.g., squeezing, spin squeezing, coherence) can be converted into entanglement. In this work, we present a general framework, based on superposition, for structurally connecting and converting nonclassicality to entanglement. In addition to capturing the previously known results, this framework also allows us to uncover new entanglement convertibility theorems in two broad scenarios, one which is discrete and one which is continuous. In the discrete setting, the classical states can be any finite linearly independent set. For the continuous setting, the pertinent classical states are "symmetric coherent states," connected with symmetric representations of the group $SU(K)$. These results generalize and link convertibility properties from the resource theory of coherence, spin coherent states, and optical coherent states, while also revealing important connections between local and nonlocal pictures of nonclassicality.

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Quantum mechanics currently provides our deepest description of nature. Despite this, much of our everyday experience can be accurately captured within a classical description. What is special about the nonclassical states of a physical system, and what distinguishes them from the more commonplace classical states? Certainly, one of the most important manifestations of nonclassicality is entanglement of multipartite systems. Schrödinger even viewed entanglement as "*the* characteristic trait of quantum mechanics" [1]. Yet, there are situations where entanglement has no natural role in describing nonclassicality, such as Fock states in optics [2]. Particularly for noncomposite systems, other notions of nonclassicality appear better suited for characterizing quantum states.

A key aspect where quantum mechanics departs from classical mechanics is the prominence of the superposition principle. This elementary tenet of quantum theory supplies a very general framework for categorizing classical and nonclassical states. Depending on the particular setting, we may specify some important subset of pure states $\{|c\rangle\}_{c\in\mathcal{I}}$ to be the "classical" pure states of the system. We can then directly associate nonclassicality with superposition: a state $|\psi\rangle$ is nonclassical if and only if it is a nontrivial superposition of classical states. In fact, entanglement fits naturally within this superposition framework, by specifying factorized states as the classical states.

One famous example of classical states is that of optical coherent states, $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ [2–4]. A completely different example is found in the resource theories of coherence [5] or reference frames [6], where the classical states are some fixed orthonormal basis $\{|k\rangle\}_{k=0}^{M}$. In both examples, there is no distinction of subsystems, and entanglement is not obviously relevant. Nevertheless, there are fundamental connections between these single-system concepts of classicality and the multipartite property of entanglement. It is well known that a beam splitter (with the second port in vacuum) transforms optical coherent states as $|\alpha\rangle\otimes|\mathrm{vac}\rangle\to|r\alpha\rangle\otimes|t\alpha\rangle$. Analogously, a generalized CONTROLLED-NOT (with the target in $|0\rangle$) has the effect $|k\rangle\otimes|0\rangle\to|k\rangle\otimes|k\rangle$. For both cases, the given operation transforms all the classical states into factorized states. Importantly, the same operation transforms all nonclassical states into entangled states [7–13]. Put another way, these transformations faithfully convert nonclassical resource states into entangled resource states. This connection, illustrated in two quite different settings, provokes intriguing questions. How general is this convertibility property? For a given notion of nonclassicality, can we always convert the nonclassical states into entangled states, while leaving the classical states unentangled?

In this Letter, we show that faithful unitary conversion is possible in two wide-ranging new scenarios—one discrete, one continuous. In the discrete setting, the classical pure states can be an arbitrary linearly independent set. This generalizes the notions of classicality and convertibility from the resource theory of coherence [5,13]. For the continuous setting, the applicable classical states are generalized coherent states associated with symmetric representations of the group $SU(K)$, for $2\le K<\infty$. Such states bridge the gap between the two-level spin coherent states [14–17] and the infinite dimensional optical coherent states. Furthermore, for both the discrete and continuous scenarios, we outline the operations which carry out the desired conversions. These results provide valuable new insights into the resource theory of coherence [5,13], the classical nature of coherent states [15,16,18], the nature of entanglement in identical particles [19–21], and the basic structure of quantum mechanics.

Classical and nonclassical.—There are various concepts of nonclassicality, each relevant to a particular setting. Interestingly, the same state can be seen as classical in one setting and nonclassical in another. For instance, excluding the vacuum, Fock states $\{|n\rangle\}_{n=1}^{\infty}$ can be thought of as nonclassical because they are superpositions of classical optical coherent states. Alternatively, we can see them as classical, since, by orthogonality, mixtures of these states are in one-to-one correspondence with classical probability distributions. Because these different perspectives each have their uses, we must recognize that nonclassicality is a relative notion. For our purposes, we will allow the set of classical pure states to be arbitrarily specified, and assume there is some justification for the choice. Thus, we simply have a list of classical pure states $\mathcal{C}_P \coloneqq \{|c\rangle \in \mathcal{H}\}_{c\in\mathcal{I}}$, where $\mathcal{H}$ is a Hilbert space of dimension D and $\mathcal{I}$ is some indexing set. We permit an arbitrary number of classical states, even a continuous set (e.g., as with coherent states). Unfortunately, in the resource theory of coherence, the term "coherent" applies to *nonclassical* states, while in optics it is used for the *classical* states. To avoid confusion, we will reserve the name "coherent state" for the latter setting and its group-theoretic generalizations.

Suppose that the set $\mathcal{C}_P$ has been specified. To maintain the desired correspondence between nonclassicality and superposition, we take $\mathcal{H}$ as the span of the classical states. Every pure state $|\psi\rangle \in \mathcal{H}$ can thus be expanded using some superposition of classical states. We extend our framework to mixed states with one further requirement, namely, that convex combinations of classical states are classical [22]. Thus, the full set of classical states is given by the convex hull of the specified classical pure states $\mathcal{C} \coloneqq \mathrm{conv}(\mathcal{C}_P)$. Any state which cannot be written as a convex combination of classical pure states will be called nonclassical, and we denote the set of all such states as $\mathcal{NC}$. Together, this partitions the state space into two disjoint sets. When the classical pure states are finite, we can make the following definition. Out of all possible superpositions, there will be some *minimal* number $1 \leq r_C \leq D$ of nonzero terms that must be used. We will call this the *classical rank* r_C of $|\psi\rangle$:

$$r_C(|\psi\rangle) \coloneqq \min\left\{ r \,\middle|\, |\psi\rangle = \sum_{j=1}^{r} \psi_j |c_j^{(\psi)}\rangle \right\}, \tag{1}$$

where the states $|c_j^{(\psi)}\rangle$ are each classical (cf. Ref. [23]). All classical states have $r_C = 1$ and all nonclassical states necessarily have $r_C > 1$. Even if the classical states are overcomplete and different decompositions are possible, the classical rank is a well-defined quantity. We point out the conceptual similarity with the Schmidt rank from entanglement theory. For continuous $\mathcal{I}$, one might instead expand a state using an integral over classical states; however, the notion of classical rank for such systems is not so clear.

We can always convert from a single-system picture to a bipartite picture using the following procedure. The initial system is connected to an ancilla system (with Hilbert space $\mathcal{H}_{\mathrm{anc}} \cong \mathcal{H}$), which is in a fixed reference classical state $|\psi_{\mathrm{ref}}\rangle$ [24]. We apply some global operation Λ to the combined system. Since Λ is nonlocal, it has the potential to create entanglement where none existed before. The entanglement properties of the final state depend on both the chosen global operation and on the input state. The ancilla's role is passive; i.e., it should not contribute anything to the final state's entanglement. The goal is that Λ produces an entangled output state if and only if the input state is nonclassical. In other words, $\Lambda[\mathcal{C}] \subset \mathcal{S}$ and $\Lambda[\mathcal{NC}] \subset \mathcal{E}$, where $\mathcal{S}$ and $\mathcal{E}$ are, respectively, the separable and entangled states on the output space. A conversion Λ will be considered faithful when this property holds, since the partitionings on both the input and output spaces are respected [25]. We can picture the overall protocol not as the creation of entanglement out of nothing, but rather as the conversion of nonclassicality into entanglement. It has been recognized previously in setting-specific scenarios [9,12,13,26,27] that nonclassicality can be quantified using entanglement measures. Such methods fundamentally require that only nonclassical states have the potential to generate entanglement. Complementary results for discord-type quantum correlations have also been developed [28–32]. We explore here the qualitative aspects of nonclassicality conversion, postponing quantitative questions to future work.

To construct the conversion operations, we will leverage a useful theorem from Refs. [33,34] which involves Gram matrices. Before stating it, we quickly review a few helpful definitions and properties. For a fixed set of states $\{|\psi_i\rangle\}_{i=1}^{N}$, we define an $N \times N$ Gram matrix $G^{(\psi)}$ by

$$[G^{(\psi)}]_{ij} = \langle \psi_i | \psi_j \rangle. \tag{2}$$

For any Gram matrix, we have that $G^{(\psi)} \geq 0$, and $\mathrm{rank}(G^{(\psi)})$ equals the number of linearly independent vectors in $\{|\psi_i\rangle\}_{i=1}^{N}$. Further, when the states are normalized, $\mathrm{diag}(G^{(\psi)}) = \mathrm{diag}(\mathbf{1})$, and Gram matrices for product states $\{|\psi_i\rangle \otimes |\phi_i\rangle\}_{i=1}^{N}$ necessarily have the form $G^{(\psi,\phi)} = G^{(\psi)} \circ G^{(\phi)}$, where "$\circ$" denotes the entrywise Hadamard product $[X \circ Y]_{ij} = X_{ij} Y_{ij}$. Finally, every N-dimensional matrix $M \geq 0$ with $\mathrm{diag}(M) = \mathrm{diag}(\mathbf{1})$ is the Gram matrix for some appropriate set of states $\{|\delta_i\rangle\}_{i=1}^{N}$ (determined from the columns of C in $M = C^{\dagger} C$). If $\{|\psi_i\rangle\}_{i\in\mathcal{I}}$ is a continuous set, we can consider a two variable function $G^{(\psi)}(i,j) = \langle \psi_i | \psi_j \rangle$ analogous to Eq. (2), which we will, for convenience, also call a Gram matrix.

Theorem 1 (unitary conversion) [33,34].—Let $\{|\psi_i\rangle\}_{i\in\mathcal{I}}$ and $\{|\phi_i\rangle\}_{i\in\mathcal{I}}$ be two sets of states. There exists a unitary operation Λ such that $\Lambda|\psi_i\rangle = |\phi_i\rangle$ for all $i \in \mathcal{I}$ if and only if $G^{(\psi)} = G^{(\phi)}$ [35].

Discrete case.—In the discrete setting, we fix the set of classical pure states to be finite, $\mathcal{C}_P = \{|c_i\rangle\}_{i=1}^D$. We can immediately state our first main result.

Theorem 2 (discrete convertibility).—If the classical pure states $\{|c_i\rangle\}_{i=1}^D$ are linearly independent, then there exists a unitary Λ such that, for all $|\psi\rangle \in \mathcal{H}$, the Schmidt rank of $\Lambda|\psi\rangle$ is equal to the classical rank of $|\psi\rangle$. For mixed states, we have $\Lambda\rho\Lambda^\dagger \in \mathcal{S}$ if and only if $\rho \in \mathcal{C}$.

Proof.—If $\{|c_i\rangle\}_{i=1}^D$ are linearly independent, then $G^{(c)}$ is full rank and hence $G^{(c)} > 0$. Using a construction of Ref. [36], define a $D \times D$ matrix $B(\lambda)$ with entries $B_{ij} = \lambda$ for $i \neq j$ and $\mathrm{diag}(B) = \mathrm{diag}(\mathbf{1})$. For $0 \le \lambda < 1$, we have $B(\lambda) > 0$. The matrix $M(\varepsilon) \coloneqq G^{(c)} \circ B(1+\varepsilon)$ is Hermitian and $M(\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$ since $\lim_{\varepsilon\to 0^+} M(\varepsilon) = G^{(\alpha)} > 0$. Choosing any valid ε, we have $G^{(c)} = B[1/(1+\varepsilon)] \circ M(\varepsilon)$, with $B[1/(1+\varepsilon)] > 0$ and $M(\varepsilon) > 0$. In fact, both $B[1/(1+\varepsilon)]$ and $M(\varepsilon)$ have only ones on their diagonals, so we actually have $B[1/(1+\varepsilon)] = G^{(d)}, M(\varepsilon) = G^{(e)}$ where $G^{(d)}$ and $G^{(e)}$ are Gram matrices for some linearly independent sets $\{|d_i\rangle\}_{i=1}^D$ and $\{|e_i\rangle\}_{i=1}^D$.

From the above properties, $G^{(c)} = G^{(d)} \circ G^{(e)}$ is the Gram matrix for the product states $\{|d_i\rangle \otimes |e_i\rangle\}_{i=1}^D$. Therefore, there exists a unitary Λ such that $\Lambda|c_i\rangle = |d_i\rangle \otimes |e_i\rangle \ \forall\ i = 1, \ldots, D$. Finally, let $|\psi\rangle \in \mathcal{H}$ have classical rank r_C. Then $|\psi\rangle = \sum_{j=1}^{r_C} \psi_j |c_{\pi(j)}\rangle$, where π is some permutation of $\{1, \ldots, D\}$ depending on ψ. Thus, $\Lambda|\psi\rangle = \sum_{j=1}^{r_C} \psi_j |d_{\pi(j)}\rangle \otimes |e_{\pi(j)}\rangle$. Because the states $\{|d_{\pi(j)}\rangle\}_{j=1}^{r_C}$ and $\{|e_{\pi(j)}\rangle\}_{j=1}^{r_C}$ are locally linearly independent, it follows that the Schmidt rank of the output state $\Lambda|\psi\rangle$ will be exactly r_C. For mixed states, it is easily checked that $\rho \in \mathcal{C} \Rightarrow \Lambda\rho\Lambda^\dagger \in \mathcal{S}$. Conversely, observe that the only factorized (i.e., Schmidt rank 1) states in the image $\Lambda[\mathcal{C}_P]$ are exactly the states $\{|d_i\rangle \otimes |e_i\rangle\}_{i=1}^D$. Thus, we can conclude that $\Lambda\rho\Lambda^\dagger \in \mathcal{S} \Rightarrow \rho \in \mathcal{C}$. ∎

Any valid Λ from Theorem 2 is a faithful nonclassicality to entanglement conversion operation. Since $\mathrm{span}(\{|c_i\rangle\}) = \mathcal{H}$, a particular transformation Λ is completely specified by the corresponding Gram matrices, and hence by the continuous parameter ε. The infinitely many possibilities correspond to different possible ways of splitting the initial overlap structure between the two new subsystems. The splitting procedure can even be iterated to give multipartite output states. Of course, for more than two subsystems, one would have to consider generalizations of the Schmidt decomposition. We note that our framework also permits splitting of the overlaps in other (nonequally weighted) ways, though these may be more dependent on the particular classical states. We provide an example application of Theorem 2 in Sec. S1B of the Supplemental Material [37].

In the resource theory of coherence, the classical pure states are orthogonal. By Theorem 2, any superposition of these can be faithfully converted into an entangled state. A prototypical conversion operation is controlled displacement, which takes $|k\rangle \otimes |0\rangle \to |k\rangle \otimes |k\rangle$ [10,13]. This transformation arises in our proof in the limit $\varepsilon \to \infty$. But our result also applies to a more general notion of coherence, where the classical states are not orthogonal. Interestingly, the conversion transformations are close analogs: instead of controlled displacements, we have $|c_k\rangle \otimes |c_0\rangle \to |d_k\rangle \otimes |e_k\rangle$. At present, less is known about this more general notion of nonclassicality. Nonorthogonal states are important in quantum foundations [52], quantum key distribution [53], and quantum state estimation [54]. However, an abstract framework for linear independence, similar to the resource theories of coherence or entanglement, has not to our knowledge been constructed. Nevertheless, we now know that this form of nonclassicality is intimately connected to entanglement.

Continuous case.—In the introduction, we identified another notion of classical states: the *optical coherent states* $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$. Splitting these states is accomplished using a beam splitter [parametrized by (r, t), with $|r|^2 + |t|^2 = 1$]. In the Gram matrix formalism, we have

$$\langle\alpha|\beta\rangle = \langle\alpha|\beta\rangle^{|r|^2}\langle\alpha|\beta\rangle^{|t|^2} = \langle r\alpha|r\beta\rangle\langle t\alpha|t\beta\rangle. \tag{3}$$

Optical coherent states are strongly connected to representations of the *Heisenberg-Weyl* group [18]. The states $|r\alpha\rangle, |t\alpha\rangle$ can be thought of as belonging to separate "rescaled" representations of the coherent states, with displacement operators $\hat{D}_r(\alpha) \coloneqq \exp(r\alpha\hat{a}^\dagger - \mathrm{H.c.})$. We can thus view a beam splitter as a physical operation that transforms between different (bipartite) representations of the coherent states, while preserving the underlying group structure (mathematically, this is called an *equivariant map* or *intertwiner* [55]).

In fact, generalized coherent states can be constructed for any group [15,16,18]. We need the following ingredients: (i) an abstract group G; (ii) an irreducible representation (irrep) of the group as unitary operators $\hat{D}_q(g)$ on a Hilbert space $\mathcal{H}$ (where q labels the particular irrep); and (iii) a reference state $|\Phi_0\rangle \in \mathcal{H}$. The group coherent states are given by the set

$$\{|g; q\rangle \coloneqq \hat{D}_q(g)|\Phi_0\rangle | g \in G\}, \tag{4}$$

where states differing by a global phase are considered equivalent. An alternate way to generalize coherent states is explored in Sec. S4 of the Supplemental Material [37].

We consider the group $SU(K)$, for arbitrary $2 \le K < \infty$, i.e., all possible unitary transformations on a K-level system. The irreps of $SU(K)$ are strongly connected with permutation symmetry, coming in symmetric, antisymmetric, and mixed symmetry types (see, e.g., Ref. [56]). Our results focus on the symmetric irreps (labeled by natural numbers N), where an element $U \in SU(K)$ is represented as the unitary operator $\hat{D}_N(U) \coloneqq U^{\otimes N}$. With the reference state $|0\rangle^{\otimes N}$, our coherent states take the form $|U;N\rangle \coloneqq [U|0\rangle]^{\otimes N}$. Importantly, the representing Hilbert space, $\mathcal{H}_{SU(K);N} \coloneqq \mathrm{span}(\{|U;N\rangle\})$, is also the symmetric subspace of the larger space $\bigotimes_{p=1}^{N} \mathbb{C}^K$ [57], so its vectors are invariant under any permutation of the label p. Bosonic particles (and quasiparticles) have such permutation symmetry, but the symmetric subspace is also important for state estimation, optimal cloning, and the de Finetti theorem [57]. For convenience, we refer to $\{|U;N\rangle\}$ as symmetric coherent states.

Symmetric coherent states also have the structure of Eq. (3), except with natural number labels $N_X + N_Y = N$:

$$\begin{aligned}\langle U;N|V;N\rangle &= \langle 0|U^\dagger V|0\rangle^{N_X}\langle 0|U^\dagger V|0\rangle^{N_Y} \\ &= \langle U;N_X|V;N_X\rangle\langle U;N_Y|V;N_Y\rangle. \qquad (5)\end{aligned}$$

Using this property, we can give our next main result.

Theorem 3 (continuous convertibility).—Let $\mathcal{C}_P$ be the symmetric coherent states for some fixed $2 \le K < \infty$ and $2 \le N < \infty$. For every pair of positive integers (N_X, N_Y) with $N_X + N_Y = N$, there is a unitary Λ such that $\Lambda|U;N\rangle = |U;N_X\rangle \otimes |U;N_Y\rangle$ for all $U \in SU(K)$. For mixed states, we have $\Lambda\rho\Lambda^\dagger \in \mathcal{S}$ if and only if $\rho \in \mathcal{C}$.

Proof.—Consider the Gram matrix of the coherent states $\{|U;N\rangle\}$, denoted by $G^{(N)}(U,V)$. From Eq. (5), $G^{(N)}(U,V) = G^{(N_X,N_Y)}(U,V)$ for all positive integers (N_X, N_Y) such that $N_X + N_Y = N$ and $\forall U,V \in SU(K)$. Here, $G^{(N_X,N_Y)}$ is the Gram matrix of the set $\{|U;N_X\rangle \otimes |U;N_Y\rangle\}$. Fix any valid pair (N_X, N_Y). By Theorem 1, there exists a unitary Λ such that $\Lambda|U;N\rangle = |U;N_X\rangle \otimes |U;N_Y\rangle$, independent of U. Denote the output spaces $\mathcal{H}_{N_{X/Y}} \coloneqq (\mathbb{C}^K)^{\otimes N_{X/Y}}$ and let $|\Omega\rangle = |\Omega_{N_X}\rangle \otimes |\Omega_{N_Y}\rangle$, with $|\Omega_{N_{X/Y}}\rangle \in \mathcal{H}_{N_{X/Y}}$, be any factorized state in the image of Λ. From permutation symmetry, it must have the form $|\Omega\rangle = |\omega\rangle^{\otimes N}$ for some $|\omega\rangle \in \mathbb{C}^K$ [58,59]. Taking $U_\omega \in SU(K)$ where $U_\omega|0\rangle = |\omega\rangle$, we have $|\Omega\rangle = |U_\omega;N_X\rangle \otimes |U_\omega;N_Y\rangle = \Lambda|U_\omega;N\rangle$. For mixed states, clearly $\rho \in \mathcal{C} \Rightarrow \Lambda\rho\Lambda^\dagger \in \mathcal{S}$. Conversely, let $\sigma = \Lambda\rho\Lambda^\dagger$ be separable with respect to $\mathcal{H}_{N_X} \otimes \mathcal{H}_{N_Y}$. We expand $\sigma = \sum_k p_k|\Omega^k\rangle\langle\Omega^k|$, where $|\Omega^k\rangle = |\Omega^k_{N_X}\rangle \otimes |\Omega^k_{N_Y}\rangle$. Each term in this mixture must be supported only on the symmetric subspace (otherwise σ would not be), so by the above argument, $|\Omega^k\rangle = |U_{\omega_k};N_X\rangle \otimes |U_{\omega_k};N_Y\rangle = \Lambda|U_{\omega_k};N\rangle$ for some $U_{\omega_k} \in SU(K)$. Thus $\sigma = \Lambda\rho\Lambda^\dagger$ for classical $\rho \coloneqq \sum_k p_k|U_{\omega_k};N\rangle\langle U_{\omega_k};N|$, and hence $\Lambda\rho\Lambda^\dagger \in \mathcal{S} \Rightarrow \rho \in \mathcal{C}$. ■

As earlier, we can picture the conversion in Theorem 3 as the bipartite transformation $|U;N\rangle \otimes |\mathrm{vac}\rangle \to |U;N_X\rangle \otimes |U;N_Y\rangle$, where the reference state $|\mathrm{vac}\rangle$ is the vacuum state. At first, the existence of this transformation could seem obvious, since we explicitly defined symmetric coherent states with a factorized structure. However, it is important to recognize that the *physical encoding* of the coherent states may change during conversion. In particular, we can convert from a setting where the "subsystems" defined by the tensor product are inaccessible into one where they are accessible. This issue of inaccessible subsystems is encountered frequently with systems of identical bosons. In Sec. S1A of the Supplemental Material [37], we present in detail a conversion example connected to this setting.

Other implications.—The most direct implication of the above results is to suggest new methods and resources for the physical generation of entanglement. Beyond this, because our nonclassicality framework and associated convertibility theorems are quite general, they also lead to a variety of other interesting consequences. We give here a broad overview of these; interested readers can find technical details in the Supplemental Material [37].

First, knowing that nonclassicality and entanglement are so closely related allows us to import and export theoretical concepts and tools between the two pictures. For example, given a conversion operator Λ and an entanglement witness W, we can define a nonclassicality witness $\tilde{W}$ by inverting the conversion and dropping the ancilla, $\tilde{W} = (\mathbf{1} \otimes \langle\psi_{\mathrm{ref}}|)\Lambda^\dagger W \Lambda(\mathbf{1} \otimes |\psi_{\mathrm{ref}}\rangle)$. If W detects entanglement after conversion, then $\tilde{W}$ detects nonclassicality without needing to convert. Entanglement conversion can also enhance our capabilities in settings where constraints or superselection rules limit our available measurements. For example, in condensed spin systems, we are limited to only collective observables, e.g., the total spin operators $\hat{J}_k$. But any single-mode nonclassical state (e.g., a spin-squeezed state) can be faithfully converted into its equivalent two-mode entangled form. The bipartite setting then allows us to break the collective symmetry, and measure spin operators on separate subcomponents, $\hat{J}^A_k$, $\hat{J}^B_k$, in addition to the total system. This extra measurement information can help us detect more nonclassicality than in the original setting. These ideas are laid out in more detail in section S2 of the Supplemental Material [37].

Another advantage of the general nonclassicality framework is that it suggests connections between seemingly unrelated physical settings. On the face of it, the resource theory of coherence and the setting of quantum optics are quite different, since their classical states are completely orthogonal and nonorthogonal, respectively. However, any finite collection of optical coherent states $\{|\alpha_i\rangle\}_{i=1}^N$ is linearly independent [12]. These states can therefore be split not only using a beam splitter, but also using the methods of Theorem 2, with any nontrivial superposition becoming entangled. Thus, the notion of nonclassicality based on linear independence provides a kind of intermediary setting between its counterparts in the resource theory of coherence and quantum optics. A more specific example of this connection is presented in section S3 of the Supplemental Material [37].

Conclusion.— Observing that many distinct physical settings share similar fundamental structures, we investigated the question of when single-system nonclassicality can be faithfully converted to entanglement. We introduced a general Gram matrix framework which provides a platform linking all previous setting-specific results. Further, we prove that entanglement conversion is possible in two broad new scenarios. Though convertibility is now established in a wide variety of settings, we still do not have a set of universal necessary and sufficient conditions for it. Our results suggest that superposition, long known as a distinguishing feature of quantum mechanics, may be the underlying ingredient connecting quantum resources in so many seemingly different physical settings.

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