What Determines the Static Force Chains in Stressed Granular Media?

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The determination of the normal and transverse (frictional) interparticle forces within a granular medium is a long-standing, daunting, and yet unresolved problem. We present a new formalism that employs the knowledge of the external forces and the orientations of contacts between particles (of any given size), to compute all the interparticle forces. Having solved this problem, we exemplify the efficacy of the formalism showing that the force chains in such systems are determined by an expansion in the eigenfunctions of a newly defined operator.

In a highly influential paper from 2005 Majmudar and Behringer [1] wrote, "Interparticle forces in granular media form an inhomogeneous distribution of filamentary force chains. Understanding such forces and their spatial correlations, specifically in response to forces at the system boundaries, represents a fundamental goal of granular mechanics. The problem is of relevance to civil engineering, geophysics and physics, being important for the understanding of jamming, shear-induced yielding and mechanical response." A visual example of such force chains in a system of plastic disks is provided in Fig. 1. In this Letter we present a solution of this goal.

To be precise, the problem that we solve is the following: consider a granular medium with known sizes of the granules, for example, the two-dimensional systems analyzed in Ref. [1] and shown in Fig. 1, of N disks of known diameters $\{\sigma_i\}_{i=1}^N$. Given the external forces, denoted below as $\boldsymbol{F}_i^{\text{ext}}$, and the external torques Γ_i^{ext} exerted on the granules, and given the angular orientations of the vectors connecting the center of masses of contacting granules (but not the distance between them), determine all the interparticle normal and tangential forces $\boldsymbol{f}_{ij}^n$ and $\boldsymbol{f}_{ij}^t$. The method presented below applies to granular systems in mechanical equilibrium; the issue of instabilities and abrupt changes in the force chains will be discussed elsewhere. For the sake of clarity and simplicity we will present here the two-dimensional case; the savvy reader will recognize that the formalism and the solution presented will go smoothly also for the three-dimensional case (as long as the system is in mechanical equilibrium). The full formalism will be presented in a longer publication in due course.

The obvious conditions for mechanical equilibrium are that the forces and the torques on each particle have to sum up to zero [2]. The condition of force balance is usefully presented in matrix form using the following notation. Denote the (signed) amplitudes of the interparticle forces f_{ij} as a vector $|f\rangle$, where the amplitudes f_{ij}^n appear first and then the amplitudes f_{ij}^t. The vector of the x and y components $F_{i,x}^{\text{ext}}$ and $F_{i,y}^{\text{ext}}$ is denoted as $|F^{\text{ext}}\rangle$, where all the x components are presented in $|F^{\text{ext}}\rangle$ first and then all the y components. The vector $|f\rangle$ has $2c$ entries, where c is the number of contacts between particles. The vector $|F^{\text{ext}}\rangle$ has $2N$ entries where N is the number of particles, with zero entries for all the particles on which there is no external force. We can then write the force balance condition as

$$M|f\rangle = -|F^{\text{ext}}\rangle, \qquad (1)$$

where M is a $2N \times 2c$ matrix. The entries in the matrix M contain the directional information; see the Supplemental Material [3] for an example of an M matrix. Denote the unit vector in the direction of the vector distance between the centers of mass of particles i and j by $\hat{\boldsymbol{n}}_{ij}$, and the tangential vector by $\hat{\boldsymbol{t}}_{ij}$ orthogonal to $\hat{\boldsymbol{n}}_{ij}$. Then the entries of M display the projections $\hat{n}_{ij,x}$ and $\hat{n}_{ij,y}$ or $\hat{t}_{ij,x}$ and $\hat{t}_{ij,y}$ as appropriate. We thus guarantee that Eq. (1) is equivalent to the mechanical equilibrium condition
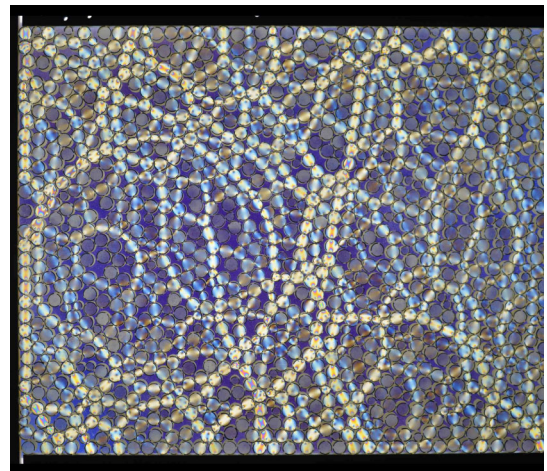

FIG. 1. Force chains in a binary system of plastic disks of two diameters, stressed uniaxially at the boundaries. The force chains are made visual by the optical birefringence of the stressed disks. The image is courtesy of V. Sathish Akella and Mahesh Bandi, Okinawa Institute of Science and Technology.

$$\sum_j f_{ij} + F_i^{\text{ext}} = 0. \qquad (2)$$

As is well known, the frictionless granular system in the thermodynamic limit is jammed exactly at the isostatic condition $2N = c$ [4]. In the frictionless case M is a $2N \times c$ matrix and as long as $c = 2N$ one can solve the problem by multiplying Eq. (1) by the transpose M^T, getting

$$M^T M |f\rangle = -M^T |F^{\text{ext}}\rangle. \qquad (3)$$

In this case the matrix $M^T M$ has generically exactly three Goldstone modes (two for translation and one for rotation) [5], and since the external force vector is orthogonal to the Goldstone modes (otherwise the external forces will translate or rotate the system), Eq. (3) can be inverted with impunity by multiplying by $[M^T M]^{-1}$. In fact, even when $c < 2N$ but the system is small enough to be jammed, this method can be used since there are enough constraints to solve for the forces. This last comment is important for our developments below.

The problem becomes underdetermined above isostaticity in the frictionless case, when force chains begin to build up that span from one boundary to the other. With friction we anyway have twice as many unknowns and we need to add the constraints of torque balance. The condition of torque balance is $\sum_j r_i \times f_{ij}^t + \Gamma_i^{\text{ext}} = 0$ on every particle, where Γ_i^{ext} is the external torque exerted on the ith disk [5]. For disks, r_i is in the normal direction, and therefore the torque balance becomes a condition that the sum of tangential forces has to balance the external tangential force. This condition can be added to Eq. (1) using a new matrix B in the form

$$B|f\rangle \equiv \begin{pmatrix} M \\ 0 \quad T \end{pmatrix} \left| \begin{pmatrix} f^n \\ f^t \end{pmatrix} \right\rangle = - \left| \begin{pmatrix} F_x^{\text{ext}} \\ F_y^{\text{ext}} \\ \Gamma^{\text{ext}} \end{pmatrix} \right\rangle. \qquad (4)$$

The order of the extended matrix B is $3N \times 2c$; see Supplemental Material [3] for an example of T. Above jamming, when the number of contacts increases $2c \gg 3N$. The matrix B is not square, and the matrix $B^T B$, which is of size $2c \times 2c$, has at least $2c - 3N$ zero modes [6]. Accordingly, it cannot be inverted and one can conclude that the conditions of mechanical equilibrium are not sufficient to determine all the forces.

Obviously, what is missing are additional constraints to remove the host of zero modes. These additional constraints are *geometrical* constraints [7,8], which can be read from those disks that describe connected polygons. In other words, since we know the orientation $\hat{n}_{ij}$ of each contact in our system, we can determine which granules are stressed in a triangular arrangement, and which in a square or pentagonal, etc.; see Fig. 2. Each such arrangement is a
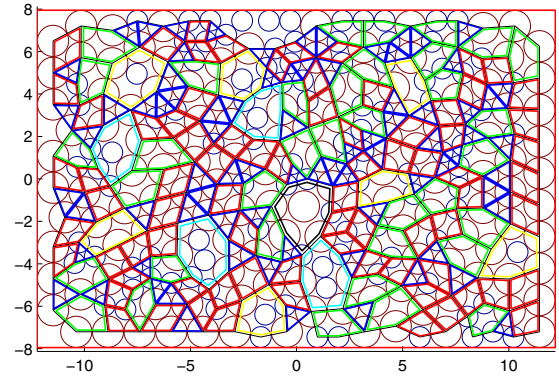

FIG. 2. A generic situation in a stressed arrangement of $N = 242$ binary disks of diameters $\sigma_1 = 1$, $\sigma_2 = 1.5$. Here $c = 432$. Shown are the $P = 205$ polygons: 80 triangles in blue, 77 squares in red, 36 pentagons in green, 7 hexagons in yellow, 4 heptagons in cyan, and 1 octagon in black. The Euler characteristic of the system is as expected $N - c + (P + 1) = 2 + R$ since we have 14 rattlers.

constraint on the radius vectors adjoining the centers of mass. For example, if particles i, j, and k are in a triangular arrangement then $r_{ij} + r_{jk} + r_{ki} = 0$, with the analogous constraint on squares, pentagons, etc. These constraints can be written in a matrix form by denoting the *amplitudes* of interparticle vector distances as $|r\rangle$, where we again arrange the x components first and the y components second:

$$Q|r\rangle = 0, \qquad (5)$$

where the matrix Q again has entries $\hat{n}_{ij,x}$ or $\hat{n}_{ij,y}$ as appropriate to represent the vectorial geometric constraints; see Supplemental Material [3] for an example of Q. Denoting the total number of polygons by P, the dimension of the matrix Q is $2P \times c$. Of course, $|r\rangle$ has c entries while $|f\rangle$ had $2c$ entries. Note that in generic situations there can be also disks which are not stressed at all. These are referred to as "rattlers." For example, in the configuration shown in Fig. 2 there exist 14 rattlers.

At this point we specialize the treatment to Hookean normal forces with a given force constant κ [9]. Non-Hookean forces result in a nonlinear theory that can still be solved but much less elegantly. For the present case

$$f_{ij}^n = \kappa[(\sigma_i + \sigma_j)\hat{n}_{ij}/2 - r_{ij}]. \qquad (6)$$

Denoting the amplitudes of the vectors $(\sigma_i + \sigma_j)\hat{n}_{ij}/2$ as the vector $|\sigma\rangle$ (again with first the x and then the y components), we can rewrite Eq. (5) in the form

$$Q|r\rangle = Q|\sigma - f^n/\kappa\rangle = 0, \qquad (7)$$

$$Q|f^n\rangle = Q|\kappa\sigma\rangle. \qquad (8)$$

Having this result at hand we can formulate the final problem to be solved. Arrange now a new matrix, say G,

operating on a vector $|f\rangle$, with a right-hand side being a vector, say $|t\rangle$, made of a stacking of $|F^{\text{ext}}\rangle$, $|\Gamma^{\text{ext}}\rangle$, and $Q|\kappa\sigma\rangle$, as before with x and then y components:

$$G|f\rangle \equiv \begin{pmatrix} M \\ 0 & T \\ Q & 0 \end{pmatrix} \left| \begin{pmatrix} f^n \\ f^t \end{pmatrix} \right\rangle = \left| \begin{pmatrix} -F_x^{\text{ext}} \\ -F_y^{\text{ext}} \\ -\Gamma^{\text{ext}} \\ Q|\kappa\sigma\rangle \end{pmatrix} \right\rangle . \quad (9)$$

Using these objects our problem is now

$$G|f\rangle = |t\rangle. \quad (10)$$

The dimension of the matrix G is $(3N + 2P) \times 2c$ and the matrix $G^T G$ has the dimension $2c \times 2c$. We can use now the Euler characteristic [10] to show that the situation has been returned here to the analog of the invertible matrix $M^T M$ when $c \leq 2N$: the Euler characteristic in two dimensions requires that

$$N - c + (P + 1) = 2 + R, \quad (11)$$

where R is the number of rattlers, i.e., disks on which there is no force. Accordingly, we find that

$$2c = 2N + 2P - 2 - 2R \ll 3N + 2P. \quad (12)$$

Consequently, the matrix $G^T G$ has no zero eigenmodes. Thus, the final solution for the forces can be obtained as

$$|f\rangle = (G^T G)^{-1} G^T |t\rangle = \sum_i \frac{\langle \Psi_i | G^T | t \rangle}{\lambda_i} |\Psi_i\rangle, \quad (13)$$

where Ψ_i is the set of eigenfunctions of $G^T G$ associated with eigenvalues λ_i. We compared the interparticle forces obtained from direct numerical simulations (see below for details) to those computed from Eq. (13). Both normal and tangential forces are of course identical to machine accuracy. We reiterate that we did not need to know the distances between particles. This is important in applying the formalism to experiments since it is very difficult to measure with precision the degree of compression of hard particles like, say, metal balls or sand particles. Note also the remarkable fact that we never had to provide the frictional (tangential) force law in the formalism to obtain the correct forces.

At this point we can discuss the force chains. By definition these are the large forces in the system that provide the tenuous network that keeps the system rigid. Observing Eq. (13) we should focus on the eigenfunction Ψ_i of $G^T G$ that have the smallest eigenvalues and the largest overlaps with $G^T |t\rangle$. These can be found and arranged in order of the magnitude of $\langle \Psi_i | G^T | t \rangle / \lambda_i$ independently of the calculation of $|f\rangle$. In Fig. 3 we show the contribution to the total energy $\langle f | f \rangle / \kappa$, learning that
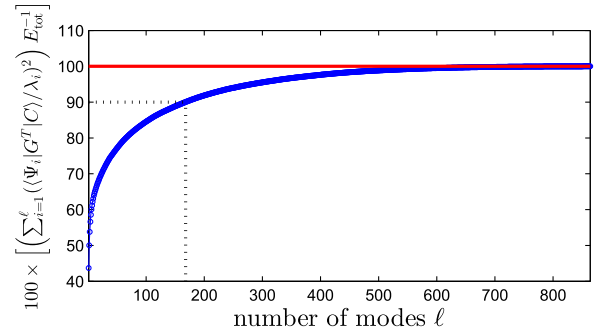

FIG. 3.   The cumulative percentage contribution to the energy of the eigenfunctions Ψ_i of $G^T G$, ordered according to the magnitude of $\langle \Psi_i | G^T | t \rangle / \lambda_i$. The convergence is relatively fast with the first 168 leading eigenfunctions (out of 864 modes) contributing 90% of the total energy.

about 20% of the leading eigenfunctions are responsible for 90% of the energy. We can therefore hope that the force chains will be determined by the same relatively small number of eigenfunctions. This is not guaranteed; due to contributions to the forces that oscillate in sign, the convergence can be much slower than in the case of the energy where the sum is of positive contributions. In Fig. 4 we show in the upper left panel the force chains in the configuration of Fig. 2. In the other panels we show the prediction of the force chains using 100, 200, and 300 of the (energy) leading modes. We learn that with 100 out of the 864 modes the main force chains begin to be visible. With 200 out of the 864 modes the full structure of the force chains is already apparent, although with 300 it is represented even better.
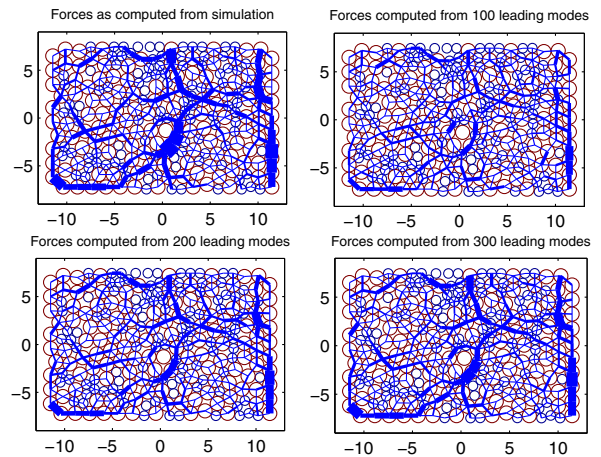

FIG. 4.   Upper panel: the force chains in the configuration shown in Fig. 2. Lower panel: the force chains as predicted from 100, 200, and 300 (energy) leading modes. While the main contributions to the force chains are visible already with 100 out of the 864 modes, the full structure is apparent only with 200. Using 300 modes, we see a reasonably faithful reproduction of the force chains.

Since the number of geometric constraints is very large, one can ask whether all the geometric constraints are necessary, as Eq. (11) shows that $2c \ll 3N + 2P$. The answer is no, we could leave out constraints as long as we have enough conditions to determine the solution. There is the obvious question then why do we have a unique solution when the number of equations is larger than the number of unknowns. The answer to this question lies in the properties of the vector $|t\rangle$ and the matrix GG^T which does have many zero modes. A condition for the existence of a solution is that $|t\rangle$ is orthogonal to all the zero modes of GG^T, as can be easily checked. We have ascertained in our simulations that this condition is always met.

In the near future we will present an extension of this formalism to three dimensions and the use of the formalism to study the instabilities of the force networks to changes in the external forces. As a final comment we should note that in fact only *one* external force is necessary to determine *all* the interdisk forces. This single external force is necessary to remove the rescaling freedom that this problem has by definition.

Simulations.—For the numerical experiments in two dimensions we use disks of two diameters, a "small" one with diameter $\sigma_s = 1.0$ and a "large" one with diameter $\sigma_\ell = 1.5$. Such N disks are put between virtual walls at $x = \pm a$ and $y = \pm b$. These walls exert external forces on the disks. The external forces are taken as Hookean for simplicity. For disks near the wall at $x = \pm a$ we write

$$
\begin{aligned}
F^{\text{ext}}(r_{i,x}) &= -(r_{i,x} - a) \quad \text{if } r_{i,x} > a \\
&= -(r_{i,x} + a) \quad \text{if } r_{i,x} < -a \\
&= 0 \quad \text{otherwise.}
\end{aligned}
\tag{14}
$$

Here, $r_{i,x}$ denotes the x component of the position vector $\mathbf{r}_i$ of the center of mass of the ith disk, and we have a similar equation for the y components with a replaced by b. When two disks, say disk i and disk j are pressed against each other we define their amount of compression as δr_{ij}:

$$
\delta r_{ij} = \sigma_{ij} - r_{ij}, \qquad \sigma_{ij} \equiv (\sigma_i + \sigma_j)/2,
\tag{15}
$$

where r_{ij} is the actual distance between the centers of mass of the disks i and j.

In our simulations the normal force between the disks acts along the radius vector connecting the centers of mass. We employ a Hookean force $\mathbf{f}_{ij}^n = \kappa \delta r_{ij} \hat{\mathbf{n}}_{ij}$.

To define the tangential force between the disks we consider (an imaginary) tangential spring at every contact which is put at rest whenever a contact between the two disks is formed. During the simulation we implement memory such that for each contact we store the signed

distance δt_{ij} to the initial rest state. For small deviations we require a linear relationship between the displacement and the acting tangential force. This relationship breaks when the magnitude of the tangential force reaches μf_{ij}^n, where, due to Coulomb's law, the tangential loading can no longer be stored and is thus dissipated. When this limit is reached the bond breaks and after a slipping event the bond is restored with a the tangential spring being loaded to its full capacity (equal to the Coulomb limit).

The numerical results reported above were obtained by starting with $N = 242$ particles on a rectangular grid (ratio 1∶2) with small random deviations in space and no contacts. We implement a large box that contains all the particles. The box acts on the system by exerting a restoring harmonic normal force as described in Eq. (14). The experiment is an iterative process in which we first shrink the containing box infinitesimally (conserving the ratio). The second step is to annul all the forces and torques, to bring the system back to a state of mechanical equilibrium. We therefore annul the forces using a conjugate gradient minimizer acting to minimize the resulting forces and torque on all particles. We iterate these two steps until the system is compressed to the desired state.

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[2] S. Alexander, Amorphous solids: Their structure, lattice dynamics and elasticity, Phys. Rep. **296**, 65 (1998).

[3] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.078001 for examples of the M, T and Q matrices.

[4] M. Wyart, S. R. Nagel, and T. A. Witten, Geometric origin of excess low-frequency vibrational modes in weakly connected amorphous solids, Europhys. Lett. **72**, 486 (2005).

[5] The existence of rattlers that are not in close contact with other particles may increase the number of zero modes.

[6] To see this note that the matrices $\mathbf{B}^T\mathbf{B}$ and $\mathbf{B}\mathbf{B}^T$ have the same rank, but $\mathbf{B}\mathbf{B}^T$ cannot have more than $3N$ nonzero eigenmodes. Therefore, $\mathbf{B}^T\mathbf{B}$ must have at least $2c - 3N$ zero modes.

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