Erratum: Vanishing Quantum Discord is Necessary and Sufficient for Completely Positive Maps [Phys. Rev. Lett. 102, 100402 (2009)]

Alireza Shabani and Daniel A. Lidar

(Received 15 October 2015; published 29 January 2016)

DOI: 10.1103/PhysRevLett.116.049901

In our Letter we claimed that the condition of vanishing quantum discord is not just sufficient but also necessary for complete positivity. In particular, we proved necessity in our Letter for subspaces of the form given in Eq. (1) below. However, subspaces of this form amount to a constraint. Namely, while they can capture almost any given initial state (i.e., the union of such subspaces contains all possible initial states except for a set of measure zero), they are not general enough to allow for all possible linear subspaces of initial states. Yet, the nature of the subsystem dynamical map depends on the chosen subspace of initial states. This was unfortunately not recognized at the time and weakens the conclusions and claims we made in our Letter. Indeed, more recently, Brodutch *et al.* [1] and Buscemi [2] demonstrated that the connection between complete positivity and discord does not generalize to all cases. Brodutch *et al.* did so by offering a counterexample in the form of a set of initial system-bath states, almost all of which are discordant, which nevertheless exhibit completely positive subdynamics. Buscemi described a general method for constructing examples yielding completely positive subdynamics, even though they may feature highly entangled states.

To rigorously and critically address the issue, we developed a complete and consistent mathematical framework for the discussion and analysis of linear subsystem dynamics, including the question of complete positivity for correlated initial states, in Ref. [3]. We refer the reader to this work for complete details. Here, we provide a brief summary of the key ideas pertinent to the results in our Letter. Our analysis builds on the notion of what we call "$\mathcal{G}$-consistent operator spaces," which represent the set of admissible initial system-bath states. $\mathcal{G}$ consistency is necessary to ensure that initial system-bath states with the same reduced state on the system evolve under all admissible unitary operators to system-bath states with the same reduced state on the system, ensuring that the dynamical maps on the system are well defined.

Let $\mathcal{D}_{SB} \subset \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ denote the convex set of density matrices on $\mathcal{H}_S \otimes \mathcal{H}_B$, where S and B denote system and bath, respectively. Let $\mathcal{G} \subset U(\mathcal{H}_S \otimes \mathcal{H}_B)$ be a subgroup of the unitary group acting on the joint system-bath Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. A subset of system-bath states $\mathcal{S} \subset \mathcal{D}_{SB}$ will be called $\mathcal{G}$ consistent if it is U consistent for all $U \in \mathcal{G}$, i.e., if, whenever $\rho_1, \rho_2 \in \mathcal{S}$ are such that $\mathrm{Tr}_B\rho_1 = \mathrm{Tr}_B\rho_2$, $\mathrm{Tr}_B(U\rho_1U^\dagger) = \mathrm{Tr}_B(U\rho_2U^\dagger)$ for all $U \in \mathcal{G}$.

The subspace $\mathcal{V} = \mathrm{Span}_\mathbb{C}\mathcal{S} \subset \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ constructed as the complex span of a convex, $\mathcal{G}$-consistent $\mathcal{S}$ exhibits the following properties. (i) $\mathcal{V} \subset \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ is a $\mathbb{C}$-linear subspace. (ii) $\mathcal{V}$ is spanned by states, i.e., $\mathrm{Span}_\mathbb{C}(\mathcal{D}_{SB} \cap \mathcal{V}) = \mathcal{V}$. (iii) $\mathcal{V}$ is $\mathcal{G}$ consistent, i.e., if $X, Y \in \mathcal{V}$ are such that $\mathrm{Tr}_B X = \mathrm{Tr}_B Y$, and if $U \in \mathcal{G}$, then $\mathrm{Tr}_B(UXU^\dagger) = \mathrm{Tr}_B(UYU^\dagger)$. We call any $\mathbb{C}$-linear subspace $\mathcal{V} \subset \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ a *$\mathcal{G}$-consistent subspace* if it satisfies these three properties.

In our Letter we expanded the class of consistent subspaces from the zero-discord subspaces considered in Ref. [4] to the class of all valid $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspaces of the form

$$\mathcal{V} = \mathrm{Span}_\mathbb{C}\{|i\rangle\langle j| \otimes \phi_{ij}\}, \tag{1}$$

where $\{|i\rangle\}$ is an orthonormal basis for $\mathcal{H}_S$ and $\{\phi_{ij}\} \subset \mathcal{B}(\mathcal{H}_B)$ are bath operators. The correct interpretation of the main result in our Letter is that, within this class of consistent subspaces, the dynamical maps obtained from the standard subdynamics prescription (of a unitary evolution followed by the partial trace over the bath) are completely positive for all unitaries $U \in U(\mathcal{H}_S \otimes \mathcal{H}_B)$ if and only if $\mathcal{V} \cap \mathcal{D}_{SB}$ comprises only zero-discord states. Thus, the results of our Letter do not amount, in general, to necessary conditions for complete positivity, but they are instead restricted to the class of consistent subspaces of the form (1). The issue is that, while the form (1) is general enough to describe almost any given state ρ_{SB}, it is not general enough to describe every $\mathcal{G}$-consistent subspace (examples are given in Ref. [3]). Thus, we hereby retract the claim made in our Letter that vanishing quantum discord is always necessary for complete positivity. It remains an open problem to fully elucidate the relationship between structural features of the set of initial system-bath states and the behavior of the resulting dynamics, including whether or not the dynamics are completely positive.

We are grateful to the authors of [1,2] for the valuable discussions that spurred this erratum, and especially to Jason Dominy for his key role in elucidating the issue.

[1] A. Brodutch, A. Datta, K. Modi, Á. Rivas, and C. A. Rodríguez-Rosario, Phys. Rev. A **87**, 042301 (2013).
[2] F. Buscemi, Phys. Rev. Lett. **113**, 140502 (2014).
[3] J. M. Dominy, A. Shabani, and D. A. Lidar, Quantum Inf. Process. **15**, 465 (2016).
[4] C. A. Rodríguez-Rosario, K. Modi, A.-m. Kuah, A. Shaji, and E. C. G. Sudarshan, J. Phys. A **41**, 205301 (2008).