

Magnetic Corrections to the Soft Photon Theorem

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(Received 30 September 2015; published 21 January 2016)

The soft photon theorem, in its standard form, requires corrections when the asymptotic particle states carry magnetic charges. These corrections are deduced using electromagnetic duality and the resulting soft formula conjectured to be exact for all Abelian gauge theories. Recent work has shown that the standard soft theorem implies an infinity of conserved electric charges. The associated symmetries are identified as “large” electric gauge transformations. Here the magnetic corrections to the soft theorem are shown to imply a second infinity of conserved magnetic charges. The associated symmetries are identified as large magnetic gauge transformations. The large magnetic symmetries are naturally subsumed in a complexification of the electric ones.

DOI: 10.1103/PhysRevLett.116.031602

Introduction.—The soft photon theorem [1–5] relates the leading infrared behavior of scattering amplitudes with and without single soft photon emission

$$\langle p_{m+1}, \dots | a_\alpha(q) \mathcal{S} | p_1, \dots \rangle = S_0 \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle + O(q^0), \quad (1)$$

where p_k is the momentum of the k th particle and a_α annihilates the momentum $q \rightarrow 0$ photon. The soft factor S_0 [Eq. (2) below] has a pole in q . The formula (1) is exact as long as there are no magnetic monopoles among the asymptotic particles. In this Letter we argue that the general form of the relation (1) remains valid in the presence of monopoles, but the formula for S_0 is corrected. Electromagnetic duality transformations are used to deduce the corrected form of S_0 [Eq. (6) below] which is conjectured to be exact [6]. The corrected soft formula should play an important role in understanding the infrared structure of Abelian gauge theories with magnetic monopoles.

Recently, it has been understood [7–10] that, in the absence of magnetic monopoles, the usual soft photon theorem is the Ward identity of an infinite-dimensional asymptotic symmetry group comprised of certain “large” Abelian gauge transformations that do not die off at infinity. The associated infinity of conservation laws equates arbitrary moments of the electric field measured at the past of future null infinity with the antipodal moments measured at the future of past null infinity. Here we spell out the analogous magnetic story. Abelian gauge theories also have magnetic gauge symmetries which shift the dual magnetic potentials. When magnetic charges are present, there is an infinite-dimensional large subgroup that acts nontrivially on the \mathcal{S} matrix. The infinity of associated conserved charges are comprised of moments of the magnetic field [11]. The large magnetic gauge symmetries are naturally contained in a complexification of the large electric gauge symmetries [12]. The corrected soft photon theorem is then the Ward identity

of the complexified large gauge transformations. All of these symmetries are spontaneously broken and the soft photons are the Goldstone bosons. The argument can also be run backwards: the corrected soft photon theorem implies the existence of an infinite number of conserved electric and magnetic charges, together with the associated symmetries.

This Letter is organized as follows. In Sec. 2 we derive the magnetically corrected soft photon theorem. Section 3 derives the associated asymptotic symmetries and conserved charges. Some conventions are given in the appendix.

Magnetically modified soft theorem.—The leading soft factor for an $(m \rightarrow n + 1 - m)$ -particle scattering process defined in Eq. (1) is [1–5]

$$S_0(q, \varepsilon_\alpha; p_k, e_k) = \sum_{k=m+1}^n \frac{e_k p_k \cdot \varepsilon_\alpha}{p_k \cdot q} - \sum_{k=1}^m \frac{e_k p_k \cdot \varepsilon_\alpha}{p_k \cdot q}. \quad (2)$$

Here, $e_k = (1/e) \int *F$ is the electric charge of the k th particle. ε_α is the polarization vector of the soft photon whose annihilation operator a_α is defined in the Appendix. We wish to find the corrections to this formula required in the presence of asymptotic particles carrying magnetic charge $g_k = (1/e) \int F$. The form of the monopole-induced corrections are easily deduced via an electromagnetic duality transformation. This is simply a convenient field redefinition and we are not assuming any symmetry of the theory. Dual variables [13], denoted by a tilde, are defined by

$$\begin{aligned} \tilde{F} &= -\frac{4\pi}{e^2} *F, \\ \tilde{e} &= \frac{4\pi}{e}, \\ \tilde{e}_k &= \frac{1}{\tilde{e}} \int * \tilde{F} = g_k, \\ \tilde{g}_k &= \frac{1}{\tilde{e}} \int \tilde{F} = -e_k. \end{aligned} \quad (3)$$

The soft photon field strength is proportional to

$$F = dA, \quad A = e\varepsilon_\alpha e^{iq \cdot x}. \quad (4)$$

A dual soft photon vector potential and polarization is defined by

$$\tilde{F} = d\tilde{A} = -\frac{4\pi}{e^2} * dA, \quad \tilde{A} = \tilde{\varepsilon}_\alpha e^{iq \cdot x}. \quad (5)$$

This formula defines $\tilde{\varepsilon}_\alpha$ up to an irrelevant shift by q . $\tilde{\varepsilon}_\alpha$ is essentially the Hodge dual of ε_α in the spatial plane transverse to the spatial direction of photon propagation. The coupling of a magnetic monopole to a photon, at soft wavelengths much larger than the monopole size, can now be obtained by the replacement $A \rightarrow \tilde{A}$, $\varepsilon_\alpha \rightarrow \tilde{\varepsilon}_\alpha$ and $e_k \rightarrow \tilde{e}_k = g_k$. It follows that monopoles preserve a soft relation of the form (1) while correcting the leading soft factor to [14]

$$S_0(q, \varepsilon_\alpha; p_k, g_k, e_k) = \sum_{k=m+1}^n \frac{p_k \cdot (e_k \varepsilon_\alpha + g_k \tilde{\varepsilon}_\alpha)}{p_k \cdot q} - \sum_{k=1}^m \frac{p_k \cdot (e_k \varepsilon_\alpha + g_k \tilde{\varepsilon}_\alpha)}{p_k \cdot q}. \quad (6)$$

Electric and magnetic charge conservation imply that S_0 is separately invariant under electric and magnetic gauge transformations $\varepsilon_\alpha \rightarrow \varepsilon_\alpha + q$ and $\tilde{\varepsilon}_\alpha \rightarrow \tilde{\varepsilon}_\alpha + q$.

We conjecture the formula (6) is exact for all Abelian gauge theories.

Symmetries of the \mathcal{S} matrix.—In this section we describe the nontrivially acting electric and magnetic symmetries of the \mathcal{S} matrix and derive the associated Ward identities. For simplicity, we take all charged particles to be massive. Our analysis follows closely Ref. [7], to which we refer the interested reader for further details.

Preliminaries: The Minkowski metric in retarded coordinates reads

$$ds^2 = -dt^2 + (dx^i)^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z}, \quad (7)$$

where u is retarded time and $\gamma_{z\bar{z}}$ is the round metric on the unit radius S^2 with covariant derivative D_z . These are related to standard Cartesian coordinates by

$$r^2 = x_i x^i, \quad u = t - r, \quad x^i = r \hat{x}^i(z, \bar{z}). \quad (8)$$

Advanced coordinates (v, r, z, \bar{z}) near past null infinity (\mathcal{I}^-) are

$$ds^2 = -dv^2 + 2dvdr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z}; \quad r^2 = x_i x^i, \\ v = t + r, \quad x^i = -r \hat{x}^i(z, \bar{z}). \quad (9)$$

\mathcal{I}^+ (\mathcal{I}^-) is the null hypersurface $r = \infty$ in retarded (advanced) coordinates. Because of the last minus sign in (9) the angular coordinates on \mathcal{I}^+ are antipodally related to those on \mathcal{I}^- so that a light ray passing through the

interior of Minkowski space reaches the same value of z, \bar{z} at both \mathcal{I}^+ and \mathcal{I}^- . We denote the future (past) boundary of \mathcal{I}^+ by \mathcal{I}^+_{\pm} (\mathcal{I}^{\pm}), and the future (past) boundary of \mathcal{I}^- by \mathcal{I}^-_{\pm} (\mathcal{I}^-).

Near \mathcal{I}^+ , we assume the asymptotic expansion

$$A_u = \sum_{n=1}^{\infty} \frac{A_u^{(n)}(u, z, \bar{z})}{r^n}, \\ A_r = \sum_{n=2}^{\infty} \frac{A_r^{(n)}(u, z, \bar{z})}{r^n}, \\ A_z = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u, z, \bar{z})}{r^n} \quad (10)$$

along with similar expansions near \mathcal{I}^- and for \tilde{A}_μ . Near spatial infinity the field strengths are taken to obey the usual \mathcal{PT} - and Lorentz-invariant antipodal continuity condition

$$F_{ru}^{(2)}(z, \bar{z})|_{\mathcal{I}^+_{\pm}} = F_{ru}^{(2)}(z, \bar{z})|_{\mathcal{I}^-_{\pm}}, \quad (11)$$

$$F_{z\bar{z}}^{(0)}(z, \bar{z})|_{\mathcal{I}^+_{\pm}} = -F_{z\bar{z}}^{(0)}(z, \bar{z})|_{\mathcal{I}^-_{\pm}}. \quad (12)$$

The minus sign (12) arises because our advanced and retarded coordinate systems differ by a parity transformation.

Electric charges and symmetries: Abelian theories have an infinite number of local electric gauge symmetries under which

$$\delta A_\mu(x) = \partial_\mu \varepsilon(x), \quad \delta \Psi_k(x) = i\varepsilon \frac{e_k}{e} \Psi_k(x), \quad (13)$$

where Ψ_k is any field or wave function of charge e_k . One may attempt to define associated charges as various two-surface integrals of the field strength weighted by the gauge parameters. Many such charges are either trivial or not conserved. However, an infinite number of nontrivial outgoing (incoming) charges on \mathcal{I}^+ (\mathcal{I}^-) can be associated with large gauge transformations on \mathcal{I}^+ (\mathcal{I}^-) that approach the time-independent function $\varepsilon(z, \bar{z})$ on \mathcal{I} [16]. Explicitly, these are

$$Q_\varepsilon^+ = \frac{1}{e^2} \int_{\mathcal{I}^+_{\pm}} d^2 z \gamma_{z\bar{z}} \varepsilon F_{ru}^{(2)}, \\ Q_\varepsilon^- = \frac{1}{e^2} \int_{\mathcal{I}^-_{\pm}} d^2 z \gamma_{z\bar{z}} \varepsilon F_{rv}^{(2)}, \quad (14)$$

where $\varepsilon(z, \bar{z})$ is any function on S^2 . It follows immediately from (11) that these charges are conserved:

$$Q_\varepsilon^+ = Q_\varepsilon^-. \quad (15)$$

Under the associated symmetry the gauge field on \mathcal{I}^+ transforms as

$$\delta_\varepsilon A_z^{(0)}(u, z, \bar{z}) = \partial_z \varepsilon(z, \bar{z}). \quad (16)$$

It follows that the large gauge symmetry is spontaneously broken and the zero modes of $A_z^{(0)}$ —the soft photons—were the goldstone bosons.

It is useful to write the charges as integrals over \mathcal{I}^\pm . Defining the outgoing, positive-helicity soft photon operator

$$F_z^+ \equiv \int du F_{uz}^{(0)}, \quad (17)$$

integrating by parts and using the \mathcal{I}^+ constraint equation

$$\partial_u F_{ru}^{(2)} + D^z F_{uz}^{(0)} + D^{\bar{z}} F_{u\bar{z}}^{(0)} = 0 \quad (18)$$

one finds

$$\begin{aligned} Q_\varepsilon^+ &= \frac{1}{e^2} \int_{S^2} d^2 z \varepsilon (\partial_{\bar{z}} F_z^+ + \partial_z F_{\bar{z}}^+) + \frac{1}{e^2} \int_{\mathcal{I}^+} d^2 z \gamma_{z\bar{z}} \varepsilon F_{ru}^{(2)} \\ &\equiv Q_S^+ + Q_H^+. \end{aligned} \quad (19)$$

The first piece of the charge is written in terms of the soft photon operator and will be referred to as the soft charge Q_S^+ . The second piece is proportional to the electric fields produced by the asymptotic outgoing hard particles and will be referred to as the hard charge Q_H^+ .

Similar observations apply to \mathcal{I}^- . The charge is given by

$$\begin{aligned} Q_\varepsilon^- &= \frac{1}{e^2} \int_{S^2} d^2 z \varepsilon (\partial_z F_{\bar{z}}^- + \partial_{\bar{z}} F_z^-) + \frac{1}{e^2} \int_{\mathcal{I}^-} d^2 z \gamma_{z\bar{z}} \varepsilon F_{ru}^{(2)} \\ &\equiv Q_S^- + Q_H^-. \end{aligned} \quad (20)$$

where $F_z^- \equiv \int dv F_{vz}^{(0)}$ creates and annihilates incoming soft photons.

The conservation law (15) is equivalent to the \mathcal{S} -matrix Ward identity:

$$\langle \text{out} | (Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^-) | \text{in} \rangle = 0. \quad (21)$$

In order to facilitate later comparison with the soft theorem, we rewrite this in the form

$$\langle \text{out} | (Q_S^+ \mathcal{S} - \mathcal{S} Q_S^-) | \text{in} \rangle = -\langle \text{out} | (Q_H^+ \mathcal{S} - \mathcal{S} Q_H^-) | \text{in} \rangle. \quad (22)$$

In Refs. [7,8] it was shown that, in the absence of magnetic charges, Q_ε^+ (Q_ε^-) generates large gauge symmetries on \mathcal{I}^+ (\mathcal{I}^-) via commutators. Q_S^+ , which is linear in the soft photon operator, generates the inhomogeneous transformation $[Q_S^+, A_z^{(0)}] = i \partial_z \varepsilon$ in (16). Q_H^+ generates the large gauge action on the hard particles, whose charge densities are (in the massive case) generally distributed over the asymptotic S^2 . However, the analyses in Refs. [7,8] assumed the absence of magnetic fields at \mathcal{I}_\pm^\pm and do not directly apply to the present context. Moreover, we expect the value of the θ angle to affect zero mode commutators

and be important for such an analysis. We expect it remains true that the charges Q_ε^\pm generate the large electric symmetries, but we will not show it here.

Magnetic charges and symmetries: Abelian theories also have an infinite number of local magnetic gauge symmetries under which

$$\tilde{\delta}_\varepsilon \tilde{A}_\mu(x) = \partial_\mu \varepsilon(x), \quad \tilde{\delta}_\varepsilon \Psi_k(x) = i \varepsilon \frac{e g_k}{4\pi} \Psi_k(x), \quad (23)$$

where Ψ_k is any field or wave function of magnetic charge g_k . We consider large magnetic gauge transformations, under which the dual gauge field on \mathcal{I}^+ transforms as

$$\tilde{\delta}_\varepsilon \tilde{A}_z^{(0)}(u, z, \bar{z}) = \partial_z \varepsilon(z, \bar{z}). \quad (24)$$

Conclusions parallel to those of the previous subsection apply to the magnetic case. This is obvious by working in the terms of the dual variables, which simply amounts to putting a tilde on every variable of the previous subsection. It is also useful to describe the magnetic charges in the original variables without using duality as follows.

The infinity of outgoing and incoming magnetic charges associated with (24) are

$$\begin{aligned} \tilde{Q}_\varepsilon^+ &= \frac{i}{4\pi} \int_{\mathcal{I}^+} d^2 z \varepsilon F_{z\bar{z}}^{(0)}, \\ \tilde{Q}_\varepsilon^- &= -\frac{i}{4\pi} \int_{\mathcal{I}^-} d^2 z \varepsilon F_{z\bar{z}}^{(0)}, \end{aligned} \quad (25)$$

where $\varepsilon(z, \bar{z})$ is any function on S^2 . It follows immediately from (12) that these charges are conserved:

$$\tilde{Q}_\varepsilon^+ = \tilde{Q}_\varepsilon^-. \quad (26)$$

Integrating by parts and using the Bianchi identity (instead of the constraint equation) one finds

$$\begin{aligned} \tilde{Q}_\varepsilon^+ &= \frac{i}{4\pi} \int_{S^2} d^2 z \varepsilon (\partial_{\bar{z}} F_z^+ - \partial_z F_{\bar{z}}^+) + \frac{i}{4\pi} \int_{\mathcal{I}^+} d^2 z \varepsilon F_{z\bar{z}}^{(0)} \\ &\equiv \tilde{Q}_S^+ + \tilde{Q}_H^+, \\ \tilde{Q}_\varepsilon^- &= \frac{i}{4\pi} \int_{S^2} d^2 z \varepsilon (\partial_z F_{\bar{z}}^- - \partial_{\bar{z}} F_z^-) - \frac{i}{4\pi} \int_{\mathcal{I}^-} d^2 z \varepsilon F_{z\bar{z}}^{(0)} \\ &\equiv \tilde{Q}_S^- + \tilde{Q}_H^-. \end{aligned} \quad (27)$$

The first term creates and annihilates soft photons, while the second acts on the hard asymptotic magnetically charged particles, much as in the electric case. The magnetic Ward identity is

$$\langle \text{out} | (\tilde{Q}_S^+ \mathcal{S} - \mathcal{S} \tilde{Q}_S^-) | \text{in} \rangle = -\langle \text{out} | (\tilde{Q}_H^+ \mathcal{S} - \mathcal{S} \tilde{Q}_H^-) | \text{in} \rangle. \quad (28)$$

Electromagnetic charges and symmetries: It turns out that the electric and magnetic charges and symmetries combine simply into a single complexified charge and

symmetry. To see this, consider the action of duality on the fields at \mathcal{I}^+ :

$$\begin{aligned}\tilde{F}_{z\bar{z}}^{(0)} &= \frac{4\pi i}{e^2} \gamma_{z\bar{z}} F_{ru}^{(2)}, \\ \tilde{F}_{ru}^{(2)} &= \frac{4\pi i}{e^2} \gamma^{z\bar{z}} F_{z\bar{z}}^{(0)}, \\ \tilde{F}_{uz}^{(0)} &= \frac{4\pi i}{e^2} F_{uz}^{(0)}, \\ \tilde{F}_{u\bar{z}}^{(0)} &= -\frac{4\pi i}{e^2} F_{u\bar{z}}^{(0)},\end{aligned}\quad (29)$$

and \mathcal{I}^- :

$$\begin{aligned}\tilde{F}_{z\bar{z}}^{(0)} &= -\frac{4\pi i}{e^2} \gamma_{z\bar{z}} F_{rv}^{(2)}, \\ \tilde{F}_{rv}^{(2)} &= -\frac{4\pi i}{e^2} \gamma^{z\bar{z}} F_{z\bar{z}}^{(0)}, \\ \tilde{F}_{vz}^{(0)} &= \frac{4\pi i}{e^2} F_{vz}^{(0)}, \\ \tilde{F}_{v\bar{z}}^{(0)} &= -\frac{4\pi i}{e^2} F_{v\bar{z}}^{(0)}.\end{aligned}\quad (30)$$

It is convenient to define $Q_\varepsilon^\pm \equiv eQ_\varepsilon^\pm + (4\pi i/e)\tilde{Q}_\varepsilon^\pm$, which may be expressed

$$\begin{aligned}Q_\varepsilon^+ &= \frac{2}{e} \int_{S^2} d^2z \varepsilon \partial_z F_{\bar{z}}^+ + \frac{1}{e} \int_{\mathcal{I}^+} d^2z \varepsilon (\gamma_{z\bar{z}} F_{ru}^{(2)} - F_{z\bar{z}}^{(0)}) \\ &\equiv Q_S^+ + Q_H^+ \\ Q_\varepsilon^- &= \frac{2}{e} \int_{S^2} d^2z \varepsilon \partial_{\bar{z}} F_z^- + \frac{1}{e} \int_{\mathcal{I}^-} d^2z \varepsilon (\gamma_{z\bar{z}} F_{rv}^{(2)} + F_{z\bar{z}}^{(0)}) \\ &\equiv Q_S^- + Q_H^-\end{aligned}\quad (31)$$

These complexified charges are natural because they transform simply under duality

$$\tilde{Q}_\varepsilon^\pm = -iQ_\varepsilon^\pm. \quad (32)$$

The complexified Ward identity

$$\langle \text{out} | (Q_S^+ \mathcal{S} - S Q_S^-) | \text{in} \rangle = -\langle \text{out} | (Q_H^+ \mathcal{S} - S Q_H^-) | \text{in} \rangle \quad (33)$$

then implies both (22) and (28).

The structure of (31) suggests that the magnetic symmetries are contained in a complexification of the electric ones. This can be made more precise. It follows from the expansion (10) that $F_{uz}^{(0)} = \partial_u A_z^{(0)} = (e^2/4\pi i) \partial_u \tilde{A}_z^{(0)}$, so that the field strengths determine the asymptotic electric and magnetic vector potentials up to a (z, \bar{z}) -dependent integration constant. It is natural to choose this constant so that

$$\tilde{A}_z^{(0)} = \frac{4\pi i}{e^2} A_z^{(0)}, \quad \tilde{A}_{\bar{z}}^{(0)} = -\frac{4\pi i}{e^2} A_{\bar{z}}^{(0)}. \quad (34)$$

In this framework electromagnetic duality acts locally on the vector potential at \mathcal{I} as a $(\pi/2)$ rotation and a rescaling. (34)

requires $A_z^{(0)}$ must transform under magnetic gauge transformations as

$$\tilde{\delta}_\varepsilon A_z^{(0)} = -\frac{ie^2}{4\pi} \partial_z \varepsilon, \quad \tilde{\delta}_\varepsilon A_{\bar{z}}^{(0)} = \frac{ie^2}{4\pi} \partial_{\bar{z}} \varepsilon. \quad (35)$$

Q_ε^+ is associated with a real electric transformation proportional to $e\varepsilon$ and an imaginary magnetic one proportional to $(4\pi i/e)\varepsilon$. The associated transformation of the vector potential is

$$\begin{aligned}\left(e\delta_\varepsilon + \frac{4\pi i}{e} \tilde{\delta}_\varepsilon\right) A_z^{(0)} &= 2e\partial_z \varepsilon, \\ \left(e\delta_\varepsilon + \frac{4\pi i}{e} \tilde{\delta}_\varepsilon\right) A_{\bar{z}}^{(0)} &= 0.\end{aligned}\quad (36)$$

The complexified transformation acts only on the holomorphic vector potential $A_z^{(0)}$. This natural complexification of the large gauge group was encountered previously [9,10] in recasting it as a U(1) Kac-Moody symmetry acting on the conformal S^2 at \mathcal{I} , and is closely related to the complexification used to set $A_{\bar{z}} = 0$ on the boundary when recasting 3D Chern-Simons theory as a WZW model [17,18].

Soft theorem \rightarrow Ward identity: In this section we close the loop by showing that the magnetically modified soft photon theorem (1) implies the general Ward identity (33).

The outgoing soft photon theorem can be written

$$\begin{aligned}\lim_{\omega \rightarrow 0} \omega \langle p_{m+1}, \dots | a_\alpha(q) \mathcal{S} | p_1, \dots \rangle \\ = \omega S_0 \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle,\end{aligned}\quad (37)$$

with S_0 given in (6) and conventions in the appendix. We parametrize the photon momentum as

$$q^\mu = \omega(1, \hat{x}(z, \bar{z})) \equiv \omega \hat{q}^\mu(z, \bar{z}), \quad (38)$$

with $\hat{x}^2 = 1$ and z a complex coordinate on S^2 as in (8). The left-hand side of (37) can be written in terms of the zero mode (17),

$$\begin{aligned}F_{\bar{z}}^+(z, \bar{z}) &= \int du F_{u\bar{z}}^{(0)} \\ &\equiv -\frac{e}{8\pi} \partial_{\bar{z}} \hat{x}^i \lim_{\omega \rightarrow 0} \sum_\alpha [\omega \varepsilon_i^{*\alpha} a_\alpha(\omega \hat{x}) + \omega \varepsilon_i^\alpha a_\alpha^\dagger(\omega \hat{x})],\end{aligned}\quad (39)$$

by taking a weighted sum over polarizations. Using the identity [7]

$$\partial_{\bar{z}} \hat{x}^i(z, \bar{z}) \sum_\alpha \varepsilon_i^{*\alpha} \frac{p_k \cdot \varepsilon_\alpha}{p_k \cdot \hat{q}(z, \bar{z})} = \partial_{\bar{z}} \log(p_k \cdot \hat{q}), \quad (40)$$

the soft theorem becomes

$$\begin{aligned} \langle p_{m+1}, \dots | F_{\bar{z}}^+ \mathcal{S} | p_1, \dots \rangle &= -\frac{e}{8\pi} \left(\sum_{k=m+1}^n (e_k + ig_k) \partial_{\bar{z}} \log(p_k \cdot \hat{q}) \right. \\ &\quad \left. - \sum_{k=1}^m (e_k + ig_k) \partial_{\bar{z}} \log(p_k \cdot \hat{q}) \right) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle. \end{aligned} \quad (41)$$

For incoming soft photons, one has

$$\lim_{\omega \rightarrow 0} \omega \langle p_{m+1}, \dots | \mathcal{S} a_\alpha^\dagger(q) | p_1, \dots \rangle = -\omega S_0^* \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle, \quad (42)$$

where S_0 is not real in a complex basis of polarizations. A nearly identical calculation then yields

$$\begin{aligned} \langle p_{m+1}, \dots | \mathcal{S} F_{\bar{z}}^- | p_1, \dots \rangle &= \frac{e}{8\pi} \left(\sum_{k=m+1}^n (e_k + ig_k) \partial_{\bar{z}} \log(p_k \cdot \hat{q}') \right. \\ &\quad \left. - \sum_{k=1}^m (e_k + ig_k) \partial_{\bar{z}} \log(p_k \cdot \hat{q}') \right) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle, \end{aligned} \quad (43)$$

where $\hat{q}' = (1, -\hat{x}^i(z, \bar{z}))$. Taking the divergence of (41) and (43), multiplying by ε and integrating over the sphere, we find [19]

$$\begin{aligned} \langle p_{m+1}, \dots | \int_{S^2} d^2z \varepsilon \partial_z F_{\bar{z}}^+ \mathcal{S} | p_1, \dots \rangle &= \frac{1}{4} \int_{S^2} d^2z \varepsilon ([\gamma_{z\bar{z}} F_{rv}^{(2)} + F_{z\bar{z}}^{(0)}]_{\mathcal{I}_+^-} \\ &\quad - [\gamma_{z\bar{z}} F_{ru}^{(2)} - F_{z\bar{z}}^{(0)}]_{\mathcal{I}_+^+}) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle, \end{aligned} \quad (44)$$

$$\begin{aligned} \langle p_{m+1}, \dots | \mathcal{S} \int_{S^2} d^2z \varepsilon \partial_z F_{\bar{z}}^- | p_1, \dots \rangle &= \frac{1}{4} \int_{S^2} d^2z \varepsilon ([\gamma_{z\bar{z}} F_{ru}^{(2)} - F_{z\bar{z}}^{(0)}]_{\mathcal{I}_+^+} \\ &\quad - [\gamma_{z\bar{z}} F_{rv}^{(2)} + F_{z\bar{z}}^{(0)}]_{\mathcal{I}_+^-}) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle. \end{aligned} \quad (45)$$

Taking the difference of (44) and (45) and using the continuity conditions (11)–(12), we reproduce (33)

$$\begin{aligned} \langle p_{m+1}, \dots | \mathcal{Q}_S^+ \mathcal{S} - \mathcal{S} \mathcal{Q}_S^- | p_1, \dots \rangle &= -\frac{1}{e} \int_{S^2} d^2z \varepsilon ([\gamma_{z\bar{z}} F_{ru}^{(2)} - F_{z\bar{z}}^{(0)}]_{\mathcal{I}_+^+} \\ &\quad - [\gamma_{z\bar{z}} F_{rv}^{(2)} + F_{z\bar{z}}^{(0)}]_{\mathcal{I}_+^-}) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle \end{aligned} \quad (46)$$

$$= -\langle p_{m+1}, \dots | \mathcal{Q}_H^+ \mathcal{S} - \mathcal{S} \mathcal{Q}_H^- | p_1, \dots \rangle. \quad (47)$$

In conclusion, the magnetically modified soft photon theorem is the Ward identity of complexified large electromagnetic gauge transformations.

I am grateful to T. Dumitrescu, D. Kapec, M. Pate, M. Schwartz, and N. Seiberg for useful discussions, and also to

N. Seiberg for asking an important question, and to D. Kapec and M. Pate for help with some computations. This work was supported in part by DOE Grant No. DE-SC0007870 and the Fundamental Laws Initiative at Harvard.

APPENDIX: MODE EXPANSIONS

We normalize the gauge field so that the action is $S = -(1/4e^2) \int F^2$ with e the gauge coupling. The standard mode expansion for the vector field is then

$$\begin{aligned} A_\mu(u, r, z, \bar{z}) &= e \sum_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} [\varepsilon_\mu^{*\alpha}(\vec{q}) a_\alpha(\vec{q}) e^{iq \cdot x} \\ &\quad + \varepsilon_\mu^\alpha(\vec{q}) a_\alpha^\dagger(\vec{q}) e^{-iq \cdot x}], \end{aligned} \quad (A1)$$

with

$$[a_\alpha(\vec{q}), a_\beta^\dagger(\vec{q}')] = \delta_{\alpha\beta} 16\pi^3 \omega \delta^3(\vec{q} - \vec{q}'). \quad (A2)$$

The large- r saddle point approximation gives

$$\begin{aligned} A_z^{(0)}(u, z, \bar{z}) &= -\frac{ie}{2(2\pi)^2} \partial_z \hat{x}^i \sum_\alpha \int_0^\infty d\omega_q [\varepsilon_i^{*\alpha} a_\alpha(\omega_q \hat{x}) e^{-i\omega_q u} \\ &\quad - \varepsilon_i^\alpha a_\alpha(\omega_q \hat{x})^\dagger e^{i\omega_q u}]. \end{aligned} \quad (A3)$$

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- [12] This comes about because, in appropriate conventions, electromagnetic duality can be locally implemented on the vector potential at null infinity by rescaling and multiplication by i . See Sec. 3.
- [13] Appropriate to $\theta = 0$ when monopoles carry no electric charge.
- [14] This formula also applies to incoming photons which have negative q^0 . The only consequence of the nonzero θ -term $(\theta/32\pi^2) \int F * F$ is that the spectrum constrains the electric charges to be of the form $e_k = en_k - (\theta e^2/8\pi^2)g_k$ for integer n_k [15].
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