Scale-Invariant Resummed Perturbation at Finite Temperatures

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(Received 17 July 2015; revised manuscript received 22 October 2015; published 19 January 2016)

We use the scalar model with quartic interaction to illustrate how a nonperturbative variational technique combined with renormalization group (RG) properties efficiently resums perturbative expansions in thermal field theories. The resulting convergence and scale dependence of optimized thermodynamical quantities, here illustrated up to two-loop order, are drastically improved as compared to standard perturbative expansions, as well as to other related methods such as the screened perturbation or (resummed) hard-thermal-loop perturbation, that miss RG invariance (as we explain). Being very general and easy to implement, our method is a potential analytical alternative to dealing with the phase transitions of field theories such as thermal QCD.

DOI: 10.1103/PhysRevLett.116.031601

At sufficiently high temperature or density, one could naively hope that the asymptotic freedom property of quantum chromodynamics (QCD) would give a reliable perturbation theory (PT) handle on the quark-gluon plasma physics. However, it is well known that severe infrared divergences unavoidably spoil a standard PT approach in thermal QCD, and generically also for other thermal field theories, such that PT gives poorly convergent and, furthermore, badly scale-dependent results at successive orders (see, e.g., Ref. [1] for reviews). Nowadays, the development of powerful computers and numerical techniques offers the possibility of solving these nonperturbative problems *in silico*, employing lattice field theory (LFT). Thus far, LFT has been very successful in the description of the QCD phase transitions at finite temperatures and near vanishing baryonic densities, with results [2] which can be directly used for interpreting the experimental output from heavy ion experiments envisaged to scan over this particular region of the phase diagram. However, the well-known numerical sign problem [3], which plagues this method when one considers the possibility of a particle-antiparticle asymmetry (signaled by a finite chemical potential), prevents LFT from being successfully used to describe compressed baryonic matter. Therefore, at the present stage, one cannot rely on LFT to describe the physics of compact stellar objects or to explore the complete QCD phase diagram. In parallel, over the last decades many efforts have been devoted to trying to understand more analytically the bad convergence generically observed for thermal PT, even for moderate coupling values. Typically, the dynamical generation of a thermal screening mass $m_D \sim \sqrt{\lambda}T$ influences the relevant expansion of thermodynamical quantities, such as the pressure, involving powers of $\sqrt{\lambda}$ rather than only λ. Accordingly, the predictions are, *a priori*, less convergent than for the $T = 0$ case. A plethora of nonperturbative approximations attempting to resum thermal perturbative expansions have been developed and refined over the years [1,4–6]. The so-called optimized perturbation theory (OPT) is a variational approach in which a related solvable case is rewritten in terms of an unphysical parameter, allowing for optimized nonperturbative results. In recent decades this strategy has been recycled, appearing under different names [7–9]. At each successive order of such a modified perturbative expansion, the arbitrary variational mass is fixed by a stationary condition. This strategy has already been used in a variety of different physical situations, including, e.g., the determination of the critical temperature for homogeneous Bose gases [10,11], the phase diagram of magnetized planar fermionic systems [12,13], and the evaluation of quark susceptibilities within effective QCD inspired models [14]. The development of a similar method—known as screened perturbation theory (SPT) [15] or its version tailored to treat thermal gauge theories [16], hard-thermal-loop (resummed) perturbation theory (HTLpt) [17]—has been pushed to three-loop perturbative order [18–21]. Given the inherent technical difficulties of the (three-loop) evaluation of the QCD pressure for the case of hot and dense quark matter, the recent results in Ref. [21] represent an impressive achievement. Moreover, their agreement with LFT simulations is quite remarkable, down to about twice the critical temperature, for the scale choice $\mu = 2\pi T$ in the modified minimal subtraction ($\overline{\mathrm{MS}}$) renormalization scheme. However, the SPT or HTLpt presents several shortcomings overshadowing its potential as a reliable nonperturbative alternative to LFT. Perhaps the most embarrassing issue is the strongly *enhanced* scale dependence displayed at increasing two- and three-loop orders, at odds with intuitive expectations: at three-loop order, even moderate scale variations dramatically affect thermodynamical quantities by relative $O(1)$ variations [18,20,21]. Another issue with the standard variational methods such as OPT, SPT, and HTLpt is that beyond the lowest orders, optimization gives more and more solutions, with unphysical complex-valued ones, often leading to the use of alternative prescriptions such as replacing the variational mass with a purely perturbative mass [20], therefore losing valuable nonperturbative information.

Recently, the OPT method at vanishing temperatures and densities has been consistently combined with renormalization group (RG) properties [22–24]. The resulting RGOPT gives stable and precise results for the Gross-Neveu mass gap [22], and new independent determinations [24] of the basic QCD scale ($\Lambda_{\overline{\rm MS}}$) and the related coupling α_S, or the quark condensate [25]. Moreover, unique and real optimization solutions can often be obtained [24] by matching those solutions to the RG behavior for small couplings, and by using appropriate renormalization scheme changes.

Here, we take an important step forward by showing that the RGOPT is also compatible with the introduction of control parameters such as the temperature. To illustrate how to implement the procedure, we have chosen a simple, yet versatile, model so that one can easily grasp the basic ideas and follow the main steps when performing a particular application. More detailed results and formulas are given elsewhere [26]. Aside from purely calculation difficulties, the method described in this Letter can be directly extended to a large class of models.

We thus start by considering the Lagrangian for one neutral scalar field with a quartic interaction,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1}$$

where we have introduced a generic mass term m. The textbook result for the two-loop free energy (equivalently, minus the pressure) is [18,27]

$$\mathcal{F}_0 = \frac{T}{2}\sum\!\!\!\!\!\!\!\int_p \ln(\omega_n^2+\omega_{\mathbf{p}}^2) + \frac{\lambda T^2}{8}\left(\sum\!\!\!\!\!\!\!\int_p \frac{1}{\omega_n^2+\omega_{\mathbf{p}}^2}\right)^2 + \mathcal{F}_0^{\rm ct}, \tag{2}$$

where in the imaginary time formalism $\omega_n = 2\pi T n$ ($n = 0, 1, \cdots$) represents the bosonic Matsubara frequencies and $\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$ is the dispersion relation. The sum integral in Eq. (2), as usual, represents the sum over Matsubara frequencies and the remaining integration with measure $d^{3-2\epsilon}\mathbf{p}/(2\pi)^{3-2\epsilon}$, using dimensional regularization to perform the integration. The one-loop part of Eq. (2) is

$$(4\pi)^2\mathcal{F}_0 = -\frac{m^4}{8}\left[\frac{2}{\epsilon} + 3 + 2\ln\left(\frac{\mu^2}{m^2}\right)\right] + \mathcal{F}_0(T) + \mathcal{F}_0^{\rm ct}, \tag{3}$$

where μ is the arbitrary renormalization scale in the $\overline{\rm MS}$ renormalization scheme, and $\mathcal{F}_0^{\rm ct} = m^4/(4\epsilon)$ represents the vacuum energy counterterm [18]. We can already address a crucial point by considering the one-loop part of free energy (3). It is a trivial matter to check that the renormalized result spoils perturbative RG invariance. Acting on Eq. (3) with the standard RG operator,

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_m(\lambda)m\frac{\partial}{\partial m}, \tag{4}$$

and noting that the thermal contribution $\mathcal{F}_0(T)$ is scale independent, yields a remnant contribution: $-m^4/2$, not compensated for by lowest order terms from $\beta(\lambda)$ or $\gamma_m(\lambda)$ in Eq. (4), those being at least of next order $O(\lambda)$. This is a manifestation of the fact that perturbative RG invariance generally occurs from cancellations between terms from the RG equation at order λ^k and the explicit μ dependence at the next order λ^{k+1} [our normalization is $\beta(\lambda) \equiv d\lambda/d\ln\mu = b_0\lambda^2 + b_1\lambda^3 + \cdots$ for the β function and $\gamma_m(\lambda) \equiv d\ln m/d\ln\mu = \gamma_0\lambda + \gamma_1\lambda^2 + \cdots$ for the anomalous mass dimension, with [28] $(4\pi)^2 b_0 = 3$, $(4\pi)^2\gamma_0 = 1/2$, $(4\pi)^4 b_1 = -17/3$, $(4\pi)^4\gamma_1 = -5/12$]. Nevertheless, perturbative RG invariance can easily be restored by adding a finite vacuum energy term, $\mathcal{E}_0$, to the action without changing the dynamics. Although this term is usually ignored, minimally set to zero in the (thermal) literature [17,18,20], we stress that it is instrumental for perturbative RG invariance to be achieved. Not surprisingly, we claim it largely explains the degrading scale dependence at higher orders in other similar resummation methods like SPT and HTLpt, which ignore those finite vacuum energy terms. The subtraction in the $\overline{\rm MS}$ scheme is conveniently written as [24,25,29]

$$\mathcal{E}_0(\lambda, m) = -(m^4/\lambda)\sum_{k\geq 0} s_k\lambda^k, \tag{5}$$

where the coefficients s_k are perturbatively determined order by order from RG invariance. In the normalization of Eq. (3) we find $s_0 = [2(b_0 - 4\gamma_0)]^{-1} = 8\pi^2$, so that when augmented with $\mathcal{E}_0$ the renormalized free energy from Eq. (3) is RG invariant at the one-loop level. This can be carried out to higher orders to give $s_1 = -1$, $s_2 = (23 + 36\zeta[3])/(480\pi^2)$, etc. Note that the apparently singular behavior for $\lambda \to 0$ in Eq. (5) will actually disappear from the final optimized free energy. We stress that the previous construction, being dependent only on the renormalization procedure, does not depend on temperature-dependent parts: at arbitrary perturbative orders the s_k coefficients can be determined from the $T = 0$ contributions only. This is indeed well known, and the non-RG-invariant remnant part defines the so-called vacuum energy anomalous dimension, which has been calculated even to five-loop order for the general $O(N)$ scalar model [30]. Our independent results for the s_k's are fully consistent with Ref. [30]. A subtlety is that, according to Eq. (5), s_k is strictly required for RG invariance at order λ^k but contributes at order λ^{k-1}. So at order λ^k one may minimally choose to include only $s_0, \ldots, s_k$, or more completely to include $s_{k+1} \neq 0$, thus incorporating higher order RG information.

One can now proceed to apply the RGOPT resummation by first performing on the RG-invariant free energy the substitution which appropriately modifies its perturbative expansion:

$$m^2 \to m^2(1-\delta)^{2a}; \qquad \lambda \to \delta\lambda, \tag{6}$$

where now m is an arbitrary parameter, and the role of a is explained below. One then reexpands at successive orders δ^k, setting $\delta \to 1$ in the final results. This procedure is consistent with renormalizability [31–33] and gauge invariance [29], whenever the latter is relevant. The arbitrary mass parameter m is then most conveniently fixed by a variational optimization prescription [9],

$$\frac{\partial \mathcal{F}_0^{(k)}}{\partial m}(m, \lambda, \delta = 1)|_{m \equiv \tilde{m}} \equiv 0, \tag{7}$$

and $\tilde{m} \neq 0$ determines a nontrivial mass $\tilde{m}(\lambda)$ with nonperturbative λ dependence.

In most previous OPT [7] (similarly, SPT [15] and HTLpt [16]) applications, the linear δ expansion was used, i.e., assuming $a = 1/2$ in Eq. (6) mainly for simplicity and economy of parameters. However, to preserve RG invariance after performing Eq. (6), a is uniquely fixed [24] by the universal (renormalization scheme independent) first order RG coefficients, as we show below. Note that, once combined with Eq. (7), the RG equation (4) takes a reduced *massless* form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right]\mathcal{F}_0^{(k)}(m, \lambda, \delta = 1) = 0, \tag{8}$$

so Eq. (8) with the OPT equation (7) completely sets *optimized* $m \equiv \tilde{m}$ and $g \equiv \tilde{g}$ "variational fixed-point" values. Consider the one-loop equation (3), at $T = 0$, augmented by $\mathcal{E}_0 = -(m^4/\lambda)s_0$, where, as discussed above, $s_0 = 8\pi^2$. Performing Eq. (6), expanding to order δ^0 consistently, and taking *afterwards* $\delta \to 1$ yields

$$(4\pi)^2 \mathcal{F}_0^{\delta^0} = m^4\left[-\frac{s_0}{\lambda}(1-4a) - \left(\frac{3}{8} + \frac{1}{4}\ln\frac{\mu^2}{m^2}\right)\right]. \tag{9}$$

Then, to satisfy Eq. (8) implies $a = \gamma_0/b_0 = 1/6$. At this one-loop order, the RG equation (8) gives no further constraints, but at higher orders it fixes an optimized coupling, and $a = \gamma_0/b_0$ guarantees that among both the RG and OPT solutions, at least one (often unique) is consistent with the $T = 0$ standard perturbative behavior [24] for $\lambda \to 0$, i.e., infrared freedom in the present case: $\lambda(\mu \ll m) \simeq [b_0 \ln(m/\mu)]^{-1}$.

Switching on thermal effects, it is convenient to express our results in terms of the one-loop renormalized self-energy including all T dependence, Σ_R, explicitly [1,18]:

$$\Sigma_R = \gamma_0\lambda\left[m^2\left(\ln\frac{m^2}{\mu^2} - 1\right) + T^2 J_1\left(\frac{m}{T}\right)\right], \tag{10}$$

with the thermal integrals ($t = p/T$ and $x = m/T$)

$$J_n(x) = \frac{4\Gamma[1/2]}{\Gamma[5/2-n]}\int_0^\infty dt \frac{t^{4-2n}}{\sqrt{t^2+x^2}}\frac{1}{e^{\sqrt{t^2+x^2}} - 1}. \tag{11}$$

Then, noting that $T(\partial/\partial m^2)\sum\!\!\!\!\!\!\int \ln(\omega_n^2 + \omega_{\mathbf{p}}^2) = 2\Sigma_R/\lambda$, the exact solution of the one-loop OPT equation (7) takes the form of a self-consistent "gap" equation,

$$\tilde{m}^2 = (4\pi)^2 b_0 \Sigma_R(\tilde{m}^2), \tag{12}$$

which is exactly scale invariant by construction. To illustrate this more explicitly, it is convenient to use the high-T expansion $m/T \equiv x \ll 1$ of $J_n(x)$, e.g., $J_0(x) \simeq 16\pi^4/45 - 4\pi^2x^2/3 + 8\pi x^3/3 + x^4[\ln x/(4\pi) + \gamma_E - 3/4] + O(x^6)$. This approximation is actually valid at the 0.1% level even for $x \lesssim 1$, sufficient for our purpose since the RGOPT one-loop solution $\tilde{m}/T$ always lies in this range. In this case the OPT equation (7) is a simple quadratic equation for x, with the unique physical ($x > 0$) solution

$$\tilde{x} = \frac{\tilde{m}^{(1)}}{T} = \pi\frac{\sqrt{1 + \frac{2}{3}(\frac{1}{b_0\lambda} + L_T)} - 1}{\frac{1}{b_0\lambda} + L_T}, \tag{13}$$

with $L_T \equiv \ln[\mu e^{\gamma_E}/(4\pi T)]$. We stress that the variational mass (13) is unrelated to the physical screening mass [34]. The corresponding one-loop RGOPT pressure reads [$P_0 = (\pi^2/90)T^4$ is the ideal gas pressure]

$$\frac{P^{(1)}}{P_0} = 1 - \frac{15}{4\pi^2}\tilde{x}^2 + \frac{15}{2\pi^3}\tilde{x}^3 + \frac{45}{16\pi^4}\left(\frac{1}{b_0\lambda} + L_T\right)\tilde{x}^4. \tag{14}$$

Equations (12)–(14) clearly have a nonperturbative dependence in λ, and they are *exactly* scale invariant upon using for $\lambda \equiv \lambda(\mu)$ the "exact" (one-loop) running, $1/\lambda(\mu') = 1/\lambda(\mu) - b_0 \ln\mu'/\mu$; then, $1/(b_0\lambda(\mu)) + L_T$ is explicitly μ independent. Thus, Eqs. (13) and (14) only depend on the single parameter $b_0\lambda(\mu_0)$, where μ_0 is some reference scale, typically $\mu_0 = 2\pi T$. This is a remarkable result, recalling that we started from Eq. (3) augmented by $-m^4(s_0/\lambda)$ being RG invariant up to the neglected higher order $O(\lambda)$, but not yet resummed, while Eqs. (13) and (14) are *all-order* RG invariant, showing the resummation efficiency after optimization. Equation (14), perturbatively expanded, gives for the first few orders $P^{(1)}/P_0 \simeq 1 - 5\alpha/4 + 5\sqrt{6}\alpha^{3/2}/3 + 5(L_T - 6)\alpha^2/4 + O(\alpha^{5/2})$, where $\alpha \equiv b_0\lambda$.

Equations (12)–(14) reproduce exactly at arbitrary orders the $O(N)$ scalar model large-N results [e.g., Eq. (5.7) of Ref. [35]], as can be checked upon identifying the correct large-N $b_0 = 1/(16\pi^2)$ value [35]. These results are also equivalent to those (at *two-loop* order) in Ref. [36]—*if* we replace $b_0 = 3/(16\pi^2)$ by $b_0/3$, as was argued in Ref. [36].

As we keep the correct b_0, Eq. (14) differs from standard perturbative pressure by $\lambda(\mu_0) \to \lambda_{\rm pert}(\mu_0)/3$: this is not a problem, simply a different calibration, as λ is yet arbitrary since the model is not fully specified by any data fixing a physical input scale, μ_0. Indeed, this apparent discrepancy disappears if expressing our results in terms of the *physical* mass: to see it, we solve Eq. (7) now for $\tilde\lambda(m)$ and replace it in Eq. (14); it gives, simply, $P^{(1)}/P_0 = 1–15x^2/(8\pi^2) + 15x^3/(8\pi^3) + O(10^{-4}x^6)$. But here $x = m/T$ is arbitrary, as we already used Eq. (7) to fix $\tilde\lambda(m)$. Now, taking for m the physical screening mass [34] $m^2 \simeq (\lambda/24)T^2[1 - \sqrt{6\lambda}/(4\pi) + \cdots]$ exactly reproduces the first two terms of the standard physical pressure [1].

Equation (14) is plotted in Fig. 1, compared with standard perturbative expansions at one- and two-loop orders with their notoriously bad scale dependence [1]. Note that at one loop, including, nonminimally, $s_1 \neq 0$ in Eq. (5) is actually equivalent to a simple scale redefinition, $\mu \to \mu e^{2s_1} = \mu e^{-2}$, in all of our results above.

The two-loop $O(\delta^1)$ contribution to the free energy, for $\delta = 1$, takes a compact form in terms of Σ_R in Eq. (10):

$$\mathcal{F}_0 = \frac{\mathcal{E}_0^{\delta^1}}{(4\pi)^2} + \frac{T}{2}\sum\!\!\!\!\!\!\int_{\mathbf{p}} \ln(\omega_n^2 + \omega_{\mathbf{p}}^2) - \left(\frac{2\gamma_0}{b_0}\right)\frac{m^2}{\lambda}\Sigma_R + \frac{\Sigma_R^2}{2\lambda}, \tag{15}$$

where $\mathcal{E}_0^{\delta^1} = -m^4/[1/(3b_0\lambda) + s_1/3]$ from Eq. (5), and by abuse of notation the finite part of this already renormalized expression is meant. The exact two-loop OPT and RG equations (7) and (8) can be written compactly as

$$f_{\rm OPT} = \frac{2}{3}h\left(1 - \frac{1}{b_0\lambda}\right) + \frac{2}{3}S + \Sigma_R'\left(S - \frac{1}{3\lambda}\right) \equiv 0,$$
$$f_{\rm RG} = h\left[\frac{1}{6} + \left(\frac{b_1}{3b_0} - S\right)\lambda\right] + \frac{1}{2}\beta^{(2)}(\lambda)S^2 \equiv 0, \tag{16}$$

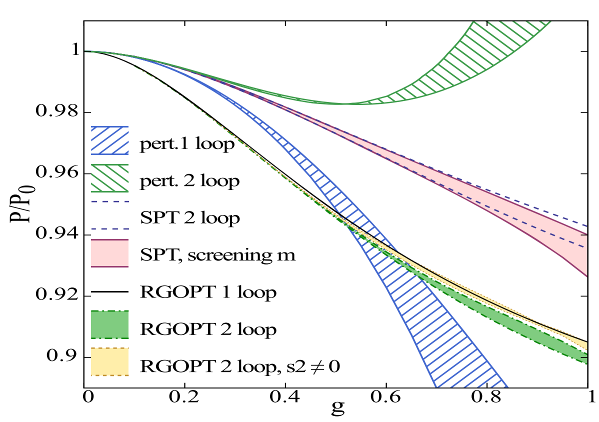

FIG. 1. RGOPT $P/P_0(g \equiv \sqrt{\lambda/24})$ at one and two loop vs standard perturbative and two-loop SPT pressures with scale dependence $\pi T < \mu < 4\pi T$.

with $h \equiv (4\pi)^{-2}$, $\beta^{(2)}(\lambda) = b_0\lambda^2 + b_1\lambda^3$, and the reduced (dimensionless) self-energy $S(m,\mu,T) \equiv \Sigma_R/(m^2\lambda)$. We also have, from Eq. (10), $\Sigma_R' \equiv \partial_{m^2}(\Sigma_R) = \lambda(S + m^2S') = \gamma_0\lambda[\ln(m^2/\mu^2) - J_2(m/T)]$. One may also solve the OPT and RG equations in the high-T expansion approximation, which is excellent up to large (rescaled) coupling $g \equiv \sqrt{\lambda/24} \sim O(1)$ values and gives exactly solvable cubic and quartic algebraic equations, respectively, with unique physical solutions ($\tilde m/T > 0$, etc.) easily identifiable. The resulting OPT and RG solutions for $\tilde m/T$ and P/P_0 are consistent with Eqs. (13) and (14) for the first two order terms perturbatively reexpanded, but they contain appropriate modifications at higher orders (detailed expressions are given elsewhere [26]).

The exact two-loop pressure P/P_0 obtained from the RG equation (4), as a function of $g \equiv \sqrt{\lambda/24}$, is plotted in Fig. 1, with scale dependence from exact two-loop running, compared with one-loop RGOPT and standard perturbative one- and two-loop pressure. The RGOPT improvement on convergence and scale dependence as compared to standard perturbative results is drastic, although a moderate residual scale dependence appears at two loop, visible on the figure for (rescaled) coupling values $g \gtrsim 0.6$. This is not surprising since the construction relies on a two-loop truncated basic free energy. At one-loop RGOPT the exact scale invariance is due to the peculiar form of the exact running coupling perfectly matching Eq. (13). At two-loop RGOPT the residual scale dependence reappears first at order λ^3: $\Delta P^{(2)}_{\rm RGOPT}(\mu) \simeq (0.075\ln\mu/\mu_0 - 1.92)g^6$, i.e., one order higher than the normally expected λ^2 from standard RG properties. Moreover, including, nonminimally, $s_2 \neq 0$ (thus catching a RG part of the three-loop contributions) modifies the perturbative pressure only at order λ^3, but it further improves slightly the (nonperturbative) scale dependence, as intuitively expected and seen in Fig. 1. More remarkably, with $s_2 \neq 0$ the two-loop pressure almost coincides with the one-loop result up to a relatively large $g \sim 1$. In Fig. 1 we also compare the RGOPT with the SPT two-loop results—i.e., discarding $\mathcal{E}_0$ in Eq. (5), taking $a = 1/2$ in Eq. (6), and using Eq. (7)—and another prescription using instead the screening mass [34], similar to the QCD HTLpt prescription [20]. Note that the missing one-loop RG invariance from the unmatched $m^4\ln\mu$ terms in Eq. (3) remains somewhat hidden at one- and two-loop thermal expansion order since, perturbatively, $m^4 \sim \lambda^2$, explaining why it plainly resurfaces at three-loop λ^2 order in SPT [18] or, similarly, HTLpt [20]. In contrast, the RGOPT scale dependence should further improve at higher orders: built on perturbative RG invariance at order k for *arbitrary* m, the mass gap will exhibit remnant scale dependence as $\tilde m^2 \sim \lambda T^2[1 + \cdots + O(\lambda^{k+1}\ln\mu)]$; thus, the dominant scale dependence in the free energy, coming from the leading term $-s_0m^4/\lambda$, should be $O(\lambda^{k+2})$.

Finally, we can combine the OPT and RG equations (16) to obtain the *full* two-loop RGOPT solution, fixing $\tilde{m}/T$ *and* $\tilde{\lambda} = 24\tilde{g}^2$ for a given input scale μ. For $\mu = 2\pi T$, we find $\tilde{m}/T \simeq 0.912$, $\tilde{g} \simeq 0.825$, $P^{(2)}_{\mathrm{RGOPT}}/P_0 \simeq 0.907$, and the scale variation for $\pi T < \mu < 4\pi T$ is consistent with the one above shown.

In conclusion, we have shown how resummations of thermal perturbative expansions based on a variational mass should be appropriately modified to restore perturbative RG invariance, missed by previous OPT, SPT, and HTLpt analogous methods. The resulting RGOPT has a different interpolation prescription, Eq. (6), uniquely dictated by universal first order RG coefficients $a = \gamma_0/b_0$. The RG equation gives us an alternative constraint to determine the nonperturbative variational mass and coupling, instead of solely using the optimization (7). The RGOPT pressure has exact one-loop RG or scale invariance, and a scale dependence and stability at two-loop order that is drastically reduced up to relatively large coupling values as compared with most other resummation approaches. For thermal QCD we anticipate a similarly improved scale dependence and stability from appropriate RGOPT adaptations of HTLpt.

M. B. P. is supported by CNPq.

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[1] J. P. Blaizot, E. Iancu, and A. Rebhan, in *Quark-Gluon Plasma*, edited by R. C. Hwa *et al.*, Advanced Series on Directions in High Energy Physics Vol. 6 (World Scientific, Singapore, 1990), p. 60; U. Kraemmer and A. Rebhan, Advances in perturbative thermal field theory, Rep. Prog. Phys. **67**, 351 (2004).

[2] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, The order of the quantum chromodynamics transition predicted by the standard model of particle physics, Nature (London) **443**, 675 (2006); Y. Aoki, Z. Fodor, S. D. Katz, and K. K. Szabo, The QCD transition temperature: Results with physical masses in the continuum limit, Phys. Lett. B **643**, 46 (2006); M. Cheng *et al.*, The transition temperature in QCD, Phys. Rev. D **74**, 054507 (2006).

[3] M. Cheng *et al.*, The QCD equation of state with almost physical quark masses, Phys. Rev. D **77**, 014511 (2008).

[4] R. R. Parwani, Resummation in a hot scalar field theory, Phys. Rev. D **45**, 4695 (1992); **48**, 5965(E) (1993).

[5] J. P. Blaizot, A. Ipp, and N. Wschebor, Calculation of the pressure of a hot scalar theory within the non-perturbative renormalization group, Nucl. Phys. **A849**, 165 (2011).

[6] J. P. Blaizot and N. Wschebor, Massive renormalization scheme and perturbation theory at finite temperature, Phys. Lett. B **741**, 310 (2015).

[7] V. I. Yukalov, Model of a hybrid crystal, Theor. Math. Phys. **28**, 652 (1976); W. E. Caswell, Accurate energy levels for the anharmonic oscillator and a summable series for the double-well potential in perturbation theory, Ann. Phys. (N.Y.) **123**, 153 (1979); I. G. Halliday and P. Suranyi, Convergent perturbation series for the anharmonic oscillator, Phys. Lett. **85B**, 421 (1979); R. P. Feynman and H. Kleinert, Effective classical partition functions, Phys. Rev. A **34**, 5080 (1986); H. F. Jones and M. Moshe, Renormalization of the linear σ expansion. The gross-neveu model, Phys. Lett. B **234**, 492 (1990); A. Neveu, Variational improvement of perturbation theory and/or perturbative improvement of variational calculations, Nucl. Phys. B, Proc. Suppl. **18**, 242 (1991); V. Yukalov, Method of self-similar approximations, J. Math. Phys. (N.Y.) **32**, 1235 (1991); S. Gandhi, H. F. Jones, and M. Pinto, The δ-expansion in the large-N limit, Nucl. Phys. **B359**, 429 (1991); C. M. Bender, F. Cooper, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., δ expansion for local gauge theories. I. A one-dimensional model, Phys. Rev. D **45**, 1248 (1992); H. Yamada, Spontaneous symmetry breaking in QCD, Z. Phys. C **59**, 67 (1993); A. N. Sisakian, I. L. Solovtsov, and O. P. Solovtsova, ß-function for the φ^4-model in variational perturbation theory, Phys. Lett. B **321**, 381 (1994); H. Kleinert, Strong-coupling behavior of φ^4 theories and critical exponents, Phys. Rev. D **57**, 2264 (1998); Strong-coupling φ^4-theory in $4-\epsilon$ dimensions, and critical exponents, Phys. Lett. B **434**, 74 (1998).

[8] R. Seznec and J. Zinn-Justin, Summation of divergent series by order dependent mappings: Application to the anharmonic oscillator and critical exponents in field theory, J. Math. Phys. (N.Y.) **20**, 1398 (1979); J. C. Le Guillou and J. Zinn-Justin, The hydrogen atom in strong magnetic fields: Summation of the weak field series expansion, Ann. Phys. (N.Y.) **147**, 57 (1983).

[9] P. M. Stevenson, Optimized perturbation theory, Phys. Rev. D **23**, 2916 (1981); Sense and nonsense in the renormalization-scheme-dependence problem, Nucl. Phys. **B203**, 472 (1982).

[10] J.-L. Kneur, A. Neveu, and M. B. Pinto, Improved optimization of perturbation theory: Applications to the oscillator energy levels and Bose-Einstein condensate critical temperature, Phys. Rev. A **69**, 053624 (2004).

[11] H. Kleinert, Five-loop critical temperature shift in weakly interacting homogeneous Bose–Einstein condensate, Mod. Phys. Lett. B **17**, 1011 (2003); B. Kastening, Bose-Einstein condensation temperature of a homogeneous weakly interacting Bose gas in variational perturbation theory through six loops, Phys. Rev. A **68**, 061601 (2003); Bose-Einstein condensation temperature of a homogenous weakly interacting Bose gas in variational perturbation theory through seven loops, Phys. Rev. A**69**, 043613 (2004).

[12] J. L. Kneur, M. B. Pinto, R. O. Ramos, and E. Staudt, Emergence of tricritical point and liquid-gas phase in the massless $2+1$ dimensional Gross-Neveu model, Phys. Rev. D **76**, 045020 (2007); J. L. Kneur, M. B. Pinto, and R. O. Ramos, Phase diagram of the magnetized planar Gross-Neveu model beyond the large-N approximation, Phys. Rev. D **88**, 045005 (2013).

[13] T. E. Restrepo, J. C. Macias, M. B. Pinto, and G. N. Ferrari, Dynamical generation of a repulsive vector contribution to the quark pressure, Phys. Rev. D **91**, 065017 (2015).

[14] J. L. Kneur, M. B. Pinto, and R. O. Ramos, Thermodynamics and phase structure of the two-flavor Nambu–Jona-Lasinio model beyond large-N_c, Phys. Rev. C **81**, 065205

(2010); L. Ferroni, V. Koch, and M. B. Pinto, Multiple critical points in effective quark models, Phys. Rev. C **82**, 055205 (2010).

[15] F. Karsch, A. Patkos, and P. Petreczky, Screened perturbation theory, Phys. Lett. B **401**, 69 (1997).

[16] E. Braaten and R. D. Pisarski, Soft amplitudes in hot gauge theories: A general analysis, Nucl. Phys. **B337**, 569 (1990).

[17] J. O. Andersen, E. Braaten, and M. Strickland, Hard-Thermal-Loop Resummation of the Free Energy of a Hot Gluon Plasma, Phys. Rev. Lett. **83**, 2139 (1999); J. O. Andersen, E. Braaten, and M. Strickland, Hard thermal loop resummation of the free energy of a hot quark-gluon plasma, Phys. Rev. D **61**, 074016 (2000).

[18] J. O. Andersen, E. Braaten, and M. Strickland, Screened perturbation theory to three loops, Phys. Rev. D **63**, 105008 (2001); J. O. Andersen and M. Strickland, Mass expansions of screened perturbation theory, Phys. Rev. D **64**, 105012 (2001); Resummation in hot field theories, Ann. Phys. (Amsterdam) **317**, 281 (2005).

[19] R. L. S. Farias, G. Krein, and R. O. Ramos, Applicability of the linear δ expansion for the $\lambda\phi^4$ field theory at finite temperature in the symmetric and broken phases, Phys. Rev. D **78**, 065046 (2008).

[20] J. O. Andersen, M. Strickland, and N. Su, Gluon Thermodynamics at Intermediate Coupling, Phys. Rev. Lett. **104**, 122003 (2010); Three-loop HTL gluon thermodynamics at intermediate coupling, J. High Energy Phys. 08 (2010) 113; J. O. Andersen, L. E. Leganger, M. Strickland, and N. Su, Three-loop HTL QCD thermodynamics, J. High Energy Phys. 08 (2011) 053.

[21] S. Mogliacci, J. O. Andersen, M. Strickland, N. Su, and A. Vuorinen, Equation of state of hot and dense QCD: resummed perturbation theory confronts lattice data, J. High Energy Phys. 12 (2013) 055; N. Haque, J. O. Andersen, M. G. Mustafa, M. Strickland, and N. Su, Three-loop pressure and susceptibility at finite temperature and density from hard-thermal-loop perturbation theory, Phys. Rev. D **89**, 061701 (2014); N. Haque, A. Bandyopadhyay, J. O. Andersen, M. G. Mustafa, M. Strickland, and N. Su, Three-loop HTLpt thermodynamics at finite temperature and chemical potential, J. High Energy Phys. 05 (2014) 027.

[22] J.-L. Kneur and A. Neveu, Renormalization group improved optimized perturbation theory: Revisiting the mass gap of the O(2N) Gross-Neveu model, Phys. Rev. D **81**, 125012 (2010).

[23] J.-L. Kneur and A. Neveu, $\Lambda^{\rm QCD}_{\overline{\rm MS}}$ from renormalization group optimized perturbation, Phys. Rev. D **85**, 014005 (2012).

[24] J.-L. Kneur and A. Neveu, α_S from F_π and renormalization group optimized perturbation theory, Phys. Rev. D **88**, 074025 (2013).

[25] J.-L. Kneur and A. Neveu, The chiral condensate from renormalization group optimized perturbation, Phys. Rev. D **92**, 074027 (2015).

[26] J.-L. Kneur and M. B. Pinto, Renormalization group optimized perturbation theory at finite temperatures, Phys. Rev. D **92**, 116008 (2015).

[27] J. I. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1985); M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).

[28] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Addendum to Wilson's theory of critical phenomena and Callan-Symanzik equations in $4-\epsilon$ dimensions, Phys. Rev. D **9**, 1121 (1974).

[29] J.-L. Kneur, Variational quark mass expansion and the order parameters of chiral symmetry breaking, Phys. Rev. D **57**, 2785 (1998).

[30] B. M. Kastening, Four-loop vacuum energy β function in $O(N)$ symmetric scalar theory, Phys. Rev. D **54**, 3965 (1996); Five-loop vacuum energy β function in φ^4 theory with $O(N)$-symmetric and cubic interactions, Phys. Rev. D**57**, 3567 (1998).

[31] C. Arvanitis, F. Geniet, M. Iacomi, J.-L. Kneur, and A. Neveu, Variational solution of the Gross–Neveu model: Finite N and renormalization, Int. J. Mod. Phys. A **12**, 3307 (1997).

[32] S. Chiku and T. Hatsuda, Optimized perturbation theory at finite temperature, Phys. Rev. D **58**, 076001 (1998).

[33] M. B. Pinto and R. O. Ramos, High temperature resummation in the linear δ expansion, Phys. Rev. D **60**, 105005 (1999); Nonperturbative study of inverse symmetry breaking at high temperatures, Phys. Rev. D**61**, 125016 (2000).

[34] E. Braaten and A. Nieto, Effective field theory approach to high-temperature thermodynamics, Phys. Rev. D **51**, 6990 (1995).

[35] I. T. Drummond, R. R. Horgan, P. V. Landshoff, and A. Rebhan, Foam diagram summation at finite temperature, Nucl. Phys. **B524**, 579 (1998).

[36] J. P. Blaizot, E. Iancu, and A. Rebhan, Approximately self-consistent resummations for the thermodynamics of the quark gluon plasma. 1. Entropy and density, Phys. Rev. D **63**, 065003 (2001).