Analytic Solution of the Boltzmann Equation in an Expanding System

D. Bazow,[1] G. S. Denicol,[2,3] U. Heinz,[1] M. Martinez,[1] and J. Noronha[4,5]

[1]*Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA*
[2]*Department of Physics, McGill University, 3600 University Street, Montreal, Quebec H3A 2T8, Canada*
[3]*Physics Department, Brookhaven National Lab, Building 510A, Upton, New York 11973, USA*
[4]*Instituto de Física, Universidade de São Paulo, C.P. 66318, 05315-970 São Paulo, São Paulo, Brazil*
[5]*Department of Physics, Columbia University, 538 West 120th Street, New York, New York 10027, USA*

For a massless gas with a constant cross section in a homogeneous, isotropically expanding spacetime we reformulate the relativistic Boltzmann equation as a set of nonlinear coupled moment equations. For a particular initial condition this set can be solved exactly, yielding the first analytical solution of the Boltzmann equation for an expanding system. The nonequilibrium behavior of this relativistic gas can be mapped onto that of a homogeneous, static nonrelativistic gas of Maxwell molecules.

Introduction.—The relativistic Boltzmann equation plays a prominent role in understanding the complex nonequilibrium dynamics displayed by dilute relativistic gases. It has applications in many areas of physics including, e.g., theoretical description of the quark-gluon plasma [1–6], neutrino transport in supernovae [7,8], and structure formation in cosmology [9–11]. While analytical solutions of the Boltzmann equation have been thoroughly studied for homogeneous systems [12–14], solutions for an expanding system remain to be found even in the nonrelativistic regime.

In the relativistic regime, progress in this direction was made recently [15–22] using the Anderson-Witting equation [23], an approximation of the Boltzmann equation that relies on the relaxation time approximation [24]. In this scheme, the nonlinear collision kernel of the Boltzmann equation is replaced by a linearized version that qualitatively describes the relaxation of the system to equilibrium on a microscopic time scale. These analyses served to improve our understanding of the domain of applicability of a number of extended hydrodynamic theories used in the description of rapidly expanding plasmas [15–20], with a focus on the description of ultrarelativistic heavy-ion collisions.

However, a complete description of dilute gases can be achieved only by solving the full Boltzmann equation. While this can be done numerically, simple yet physically motivated analytical solutions of the Boltzmann equation, if available, can lead to valuable insights into nonequilibrium phenomena. In this Letter we take a step in this direction and derive the first analytical solution of the full Boltzmann equation for an expanding dilute gas. This is done using the method of moments [6,25] to calculate the nonlinear collision term of the relativistic Boltzmann equation for a massless gas with a constant cross section in a homogeneous and isotropically expanding spacetime. The derived solution describes the expansion-driven deviation of the particles' momentum distribution from local thermal equilibrium as a function of the system's expansion rate.

We note that the nonexpanding limit of this solution, which describes the homogeneous relaxation of a *relativistic* gas towards equilibrium, is also new.

Boltzmann equation.—We consider a homogeneous and isotropically expanding system of massless particles in a Friedmann-Lemaître-Robertson-Walker (FLRW) metric [26,27] (the most general homogeneous and isotropic metric in flat space)

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \qquad (1)$$

We note that in this Letter we make the further restriction of zero global curvature. For a gas of massless particles, the energy density and the pressure are related as $\varepsilon = 3p$, and here the scale factor $a(t) > 0$ [27] is a free function that may be fixed by additional physics assumptions. For the FLRW metric the nonzero Christoffel symbols are $\Gamma^i_{0j} = \delta^i_j H(t)$ and $\Gamma^0_{ij} = a(t)\dot{a}(t)\delta_{ij}$, where i, j denote spatial indices, $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble parameter, and $\sqrt{-g} = a^3(t)$, with g being the determinant of the metric in (1). Even though the fluid flow of this system is locally static, $u^\mu = (1, 0, 0, 0)$, the expanding FLRW geometry induces a nonzero fluid expansion rate $\theta(t) \equiv \partial_\mu(\sqrt{-g}u^\mu)/\sqrt{-g} = 3H(t)$.

The dynamics of the single-particle distribution function, $f_{\boldsymbol{k}}(x)$, is given by the relativistic Boltzmann equation in curved space [18,19,28–30]

$$k^\mu \partial_\mu f_{\boldsymbol{k}} + \Gamma^\lambda_{\mu i} k_\lambda k^\mu \frac{\partial f_{\boldsymbol{k}}}{\partial k_i} = \mathcal{C}[f]. \qquad (2)$$

Since the FLRW geometry is based on the assumptions of spatial homogeneity and isotropy, $f_{\boldsymbol{k}}$ cannot depend on spatial position $\boldsymbol{x}$ and must be locally isotropic in momentum, depending only on $u \cdot k = k^0$ where for massless

particles $k^0 = k/a(t)$ with $k = |\mathbf{k}|$. We therefore write $f_{\mathbf{k}}(x) = f_k(t)$ from here on.

The symmetries of the FLRW metric strongly constrain the form of the energy-momentum tensor $T^{\mu\nu}$ and particle 4-current N^μ of the matter. Because of local momentum isotropy the viscous shear-stress tensor and particle diffusion current vanish exactly. Further, the bulk viscous pressure is zero for massless particles. Therefore, the conserved currents always take their equilibrium form, and the time evolution of the energy and particle densities, ε and n, is fully determined by the conservation laws

$$\partial_t n + 3nH(t) = 0, \qquad \partial_t \varepsilon + 4\varepsilon H(t) = 0. \qquad (3)$$

With the initial condition $a(t_0) = 1$, they are solved by $n(t) = n(t_0)/a^3(t)$ and $\varepsilon(t) = \varepsilon(t_0)/a^4(t)$.

Even though the conserved currents $T^{\mu\nu} = \varepsilon u^\mu u^\nu - p g^{\mu\nu}$ and $N^\mu = nu^\mu$ take the same form as in local thermal equilibrium, the system itself does not have to be in equilibrium. In fact, the local momentum distribution $f_k(t)$ is driven away from its local equilibrium form by an amount proportional to the expansion rate $\theta(t) = 3H(t)$ of the FLRW geometry. However, this process happens without disturbing the spatial homogeneity and isotropy of the system and, consequently, there are no out-of-equilibrium corrections to the conserved currents. A similar behavior was found in [21,22] in the context of the relaxation time approximation.

The contributions from the Christoffel symbols in Eq. (2) cancel exactly, and the equation reduces to

$$k^0 \partial_t f_k = \mathcal{C}[f]. \qquad (4)$$

Ignoring quantum statistics, the collision term reads

$$\mathcal{C}[f] = \frac{1}{2} \int_{k'pp'} W_{\mathbf{kk'} \to \mathbf{pp'}}(f_p f_{p'} - f_k f_{k'}), \qquad (5)$$

where $\int_k \equiv \int d^3k/[(2\pi)^3 \sqrt{-g} k^0]$ and $W_{\mathbf{kk'} \to \mathbf{pp'}}$ is the transition rate. To make progress on evaluating (5) we make the simplifying assumption of isotropic scattering with energy-independent total cross section σ. Then $W_{\mathbf{kk'} \to \mathbf{pp'}}$ takes the form [28–30]

$$W_{\mathbf{kk'} \to \mathbf{pp'}} = (2\pi)^5 \sqrt{-g} \sigma s \delta^4(k + k' - p - p'), \qquad (6)$$

where $s = (k^\mu + k'^\mu)(k_\mu + k'_\mu)$. Thus, the Boltzmann equation becomes

$$k^0 \partial_t f_k$$
$$= \frac{(2\pi)^5}{2} \sqrt{-g} \sigma \int_{k'pp'} s \delta^4(k + k' - p - p')(f_p f_{p'} - f_k f_{k'}). \qquad (7)$$

Note that even for the highly symmetric case considered here the relativistic Boltzmann equation is still a nonlinear integro-differential equation for f_k.

Moment equations.—We proceed to solve the Boltzmann equation using the method of moments [6]. Because of local momentum isotropy, the distribution function f_k can be fully described by scalar moments only [31]:

$$\rho_m(t) = \int_k (u \cdot k)^{m+1} f_k(t) = \int_k (k^0)^{m+1} f_k(t)$$
$$= \frac{1}{2\pi^2} \frac{1}{a^{m+3}(t)} \int_0^\infty dk\, k^{m+2} f_k(t) \quad (m \in \mathbb{N}_0). \qquad (8)$$

The time dependence of the two lowest moments, the number density $n(t) \equiv \rho_0(t)$ and the energy density $\varepsilon(t) \equiv \rho_1(t)$, is given by the conservation laws already discussed. For a classical gas of massless particles, they provide the time dependence of the temperature T and fugacity $\lambda = \exp(\mu/T)$ (where μ is the chemical potential) via the matching conditions [6] $T = \varepsilon/(3n)$ and $\nu_d \lambda = n\pi^2/T^3$, where ν_d is the number of massless degrees of freedom (including statistical degeneracy factors). In the following, we set $\nu_d = 1$ for simplicity.

From the matching conditions it follows that $T(t) = T(t_0)/a(t)$ and $\mu(t) = \mu(t_0)/a(t)$ such that the fugacity λ is time independent. $T(t)$ and λ define the local equilibrium distribution function $f_k^{\text{eq}}(t) = \lambda \exp[-u \cdot k/T(t)]$ and the equilibrium scalar moments

$$\rho_m^{\text{eq}}(t) = \frac{(m+2)!}{2\pi^2} \lambda T^{m+3}(t). \qquad (9)$$

To express the Boltzmann equation in terms of the scalar moments ρ_m we multiply (7) by $(u \cdot k)^m$ and integrate over k. This results in

$$\partial_t \rho_m(t) + (3+m)H(t)\rho_m(t) = \mathcal{C}_{\text{gain}}^{(m)}(t) - \mathcal{C}_{\text{loss}}^{(m)}(t), \qquad (10)$$

where the gain and loss terms are defined by

$$\mathcal{C}_{\text{gain}}^{(m)}(t)$$
$$= \frac{(2\pi)^5}{2} \sqrt{-g} \sigma \int_{kk'pp'} s(u \cdot p)^m \delta^4(k + k' - p - p') f_k f_{k'},$$
$$\mathcal{C}_{\text{loss}}^{(m)}(t)$$
$$= \frac{(2\pi)^5}{2} \sqrt{-g} \sigma \int_{kk'pp'} s(u \cdot k)^m \delta^4(k + k' - p - p') f_k f_{k'}. \qquad (11)$$

Here the gain term was simplified by using the symmetry of the transition rate under interchange of the incoming and outgoing momenta. With

$$\sqrt{-g}\int_{pp'}\delta^4(k+k'-p-p') = 1/(2\pi)^5, \quad (12)$$

the loss term reduces straightforwardly to [6]

$$C_{\text{loss}}^{(m)}(t) = \sigma\rho_m(t)n(t). \quad (13)$$

However, the gain term is more involved. One first writes

$$C_{\text{gain}}^{(m)}(t) = \frac{\sigma}{2}\int_{kk'} sf_k f_{k'}\mathcal{P}_m, \quad (14)$$

where the inner kernel is

$$\mathcal{P}_m \equiv (2\pi)^5\sqrt{-g}\int_{pp'}(u\cdot p)^m\delta^4(k+k'-p-p'). \quad (15)$$

This quantity is a scalar and can be calculated in any frame. In the center of momentum frame, the four-dimensional delta function can be integrated and the remaining angular integrals can be performed analytically [32,33]. The final result can be expressed as follows:

$$\mathcal{P}_m = \frac{m!}{[2a(t)]^m}\sum_{j \text{ odd}}^{m+1}\frac{(k+k')^{m+1-j}}{(m+1-j)!}\frac{|\boldsymbol{k}+\boldsymbol{k}'|^{j-1}}{j!}. \quad (16)$$

Plugging this back into (14) and performing some algebraic manipulations [32,33] one obtains

$$C_{\text{gain}}^{(m)}(t) = 2(m+2)m!\sigma\sum_{j=0}^{m}\frac{\rho_j(t)}{(j+2)!}\frac{\rho_{m-j}(t)}{(m-j+2)!}. \quad (17)$$

Using these results in Eq. (10) we derive the following exact evolution equation for the moments:

$$\partial_t\rho_m(t) + [(3+m)H(t) + \sigma n(t)]\rho_m(t)$$
$$= 2(m+2)m!\sigma\sum_{j=0}^{m}\frac{\rho_j(t)}{(j+2)!}\frac{\rho_{m-j}(t)}{(m-j+2)!}. \quad (18)$$

This infinite set of coupled nonlinear differential equations for the scalar moments $\rho_m(t)$ is equivalent to the original relativistic integro-differential Boltzmann equation. For $m=0$ and $m=1$, Eq. (18) reduces to Eqs. (3) for the particle and energy densities. Higher-order moments are more sensitive to the distribution function at higher momentum. We can further simplify the moment equations by introducing the scaled moments $M_m(t) \equiv \rho_m(t)/\rho_m^{\text{eq}}(t)$ [34] and the scaled time $\hat{t} = t/\ell_0$, where $\ell_0 = 1/[\sigma n(t_0)]$ is the (constant) mean free path at time t_0. This yields the surprisingly simple evolution equations

$$a^3(\hat{t})\frac{\partial M_m(\hat{t})}{\partial\hat{t}} + M_m(\hat{t}) = \frac{1}{m+1}\sum_{j=0}^{m}M_j(\hat{t})M_{m-j}(\hat{t}). \quad (19)$$

Equation (19) is the main result of this Letter. While the nonlinear coupling between different moments was expected from the nonlinearity of the collision kernel, the same cannot be said about another key feature of Eq. (19): it can be solved recursively; i.e., the solution of the evolution equation for M_n requires only previously solved moments $M_k(t)$ of lower order $k < n$. This property depends on our choice of an energy-independent cross section; it is essential for being able to solve (19) analytically.

All information about the expansion appears in the factor $a^3(\hat{t})$ multiplying the time derivative in (19). Whether or not local equilibrium can be achieved thus depends on the state of expansion of the system. For example, when $a(\hat{t}) \sim \hat{t}^{1/2}$ the mean free path $\ell(\hat{t}) = 1/[\sigma n(\hat{t})]$ increases faster than the expansion rate, $\lim_{\hat{t}\to\infty}\ell(\hat{t})\theta(\hat{t}) \to \infty$, and local equilibrium cannot be reached even at asymptotically large times. Moreover, if the initial $f_k(\hat{t}_0)$ is positive definite, $M_n(\hat{t}_0) > 0$, and Eq. (19) then implies that all moments remain positive throughout the evolution, translating into positivity for f_k for all momenta at all times.

Equation (19) closely resembles the Bobylev-Krook-Wu (BKW) equation derived almost four decades ago [12,13] in a famous study about homogeneous and isotropic solutions of the *nonrelativistic* Boltzmann equation (for a review, see [14]) for Maxwell molecules. In fact, by defining the time variable $\tau = \int_{\hat{t}_0}^{\hat{t}}dt'/a^3(t')$ to take into account the expansion [35], our equation (19) for the moment becomes identical to the BKW equations [13]:

$$\partial_\tau M_m(\tau) + M_m(\tau) = \frac{1}{m+1}\sum_{j=0}^{m}M_j(\tau)M_{m-j}(\tau). \quad (20)$$

This indicates that even though the underlying symmetries of these physical systems are quite different (BKW's are based on Galilean invariance with static conditions while ours are embedded in an expanding system), these systems are actually equivalent from a dynamical perspective and evolve towards equilibrium in a universal manner. We note that our equations also reduce to BKW's when $a(\hat{t}) \equiv 1$, i.e., for a nonexpanding metric (though special relativistic effects are still fully taken into account).

Analytical solution of the moment equations.—Given the close relation of our Eq. (19) with the BKW equation (20) it is not surprising that it admits an exact analytic solution of the Krook-Wu type:

$$M_m(\tau) = \mathcal{K}(\tau)^{m-1}[m - (m-1)\mathcal{K}(\tau)] \quad (m \geq 0), \quad (21)$$

where $\mathcal{K}(\tau) = 1 - \frac{1}{4}\exp(-\tau/6)$. The time evolution of the moments is shown in Fig. 1(a). One sees that low-order
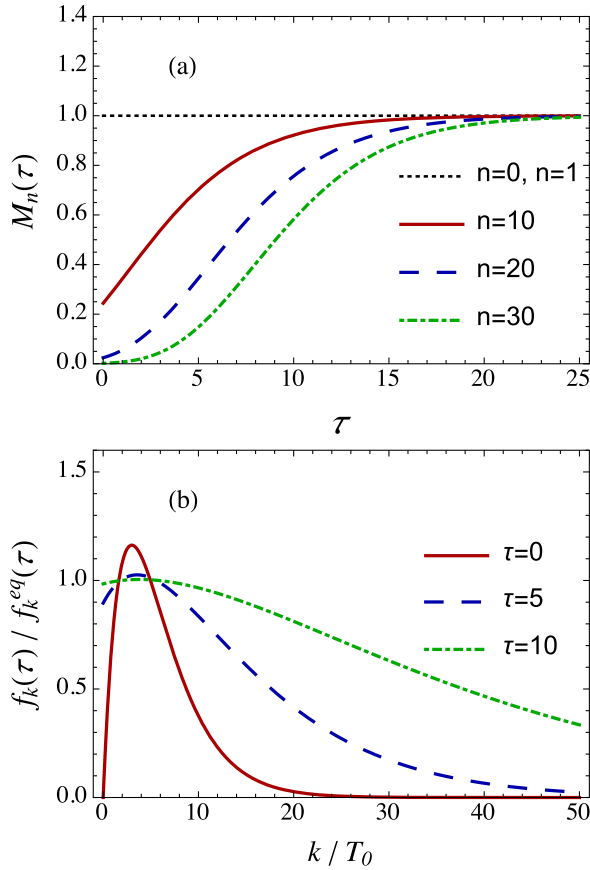
FIG. 1. Time evolution of the moments M_n (a) and of the ratio between the out-of-equilibrium solution (24) and the equilibrium distribution (b).

moments equilibrate more quickly than the higher-order ones that are needed to describe the high-momentum nonequilibrium tail of the distribution function.

Given the exact form (21) of the moments $M_m(\tau)$, the distribution function can be reconstructed as follows. One defines the function $\mathcal{F}(\tau, u \cdot k) = (u \cdot k)^2 \theta(u \cdot k) f_k$ where $u \cdot k = k/a(\tau)$, expands its Fourier transform with respect to k in terms of the moments $M_m(\tau)$, and then transforms it back. This yields the exact result [32]

$$f_k(\tau) = \lambda \exp\left(-\frac{u \cdot k}{\mathcal{K}(\tau)T(\tau)}\right)$$
$$\times \left[\frac{4\mathcal{K}(\tau) - 3}{\mathcal{K}^4(\tau)} + \frac{u \cdot k}{T(\tau)}\left(\frac{1 - \mathcal{K}(\tau)}{\mathcal{K}^5(\tau)}\right)\right]. \quad (22)$$

To the best of our knowledge, this is the first analytic solution of the full Boltzmann equation for an expanding interacting gas. At the initial time $\hat{t}_0$ [corresponding to $\tau = 0$ and $T(\tau) = T_0$] one finds the initial condition $f_k(0) = (256/243)(k/T_0)\lambda \exp[-4k/(3T_0)] > 0$. In Fig. 1(b) we plot the analytical solution for the ratio $f_k(\tau)/f_k^{\text{eq}}(\tau)$ as a function of $u \cdot k/T(\tau) = k/T_0$ for $\tau = 0$, 5, and 10. At $\tau = 0$ the high momentum tail is

largely underpopulated whereas momentum modes in the range $k/T_0 \sim 1.6$–5 are overoccupied relative to local equilibrium. As time evolves, high momentum modes are populated at the expense of the overpopulated moderate momentum region, in a process resembling an energy cascade. Note, however, that when $a(\hat{t}) \sim \hat{t}^{1/2}$ one finds that $\lim_{\hat{t}\to\infty}\tau(\hat{t})$ is finite and (22) never assumes the equilibrium form.

The moment equations (19) can also be studied numerically for a variety of initial conditions for which no analytic solution is known. Here, we briefly address the question of whether or not the direct energy cascade seen in Fig. 1 is related to self-similar behavior characteristic of turbulence [36]. While a detailed study of self-similarity in this system is beyond the scope of this short paper, we note here that expressing the Boltzmann equation in terms of its moments may be quite useful in this context. For instance, one can show that nonthermal fixed points corresponding to exact self-similar solutions of the kind

$$f_k(\hat{t}) = a^\gamma(\hat{t})f_S[a^\beta(\hat{t})u \cdot k], \quad (23)$$

where γ and β are the scaling exponents and f_S is the fixed point distribution [36], are not compatible with Eq. (19). In fact, using Eq. (3) one can see that the scaling exponents are necessarily $\gamma = 0$ and $\beta = 1$, and that the ansatz (23) for the distribution function (if assumed to be valid for all momenta) leads to moments

$$\rho_m^S(\hat{t}) = \frac{1}{2\pi^2}\frac{c_m^S}{a^{m+3}(\hat{t})} \quad (24)$$

[where $c_m^S = \int_0^\infty d\xi \xi^{m+2} f_S(\xi)$] that have the same time dependence as the equilibrium moments. This implies that the corresponding normalized moments $M_m^S = \rho_m^S/\rho_m^{\text{eq}}$ are time independent. Since the conservation laws require that $M_0^S = M_1^S = 1$, one can then use (19) to show that $M_m^S \to 1$ for all m. This shows that there are no other true fixed points of the dynamics besides local thermal equilibrium.

Conclusions.—We derived from the relativistic Boltzmann equation a nonlinear set of coupled moment equations for a massless gas with constant cross section in a homogeneous, isotropically expanding spacetime. For a particular initial condition, we found that the moment equations can be solved exactly, thereby obtaining the first analytical solution of the Boltzmann equation with full nonlinear collision term for an expanding system.

The normalized moments of this expanding relativistic gas can be directly mapped onto the corresponding moments for a homogeneous, static nonrelativistic gas of Maxwell molecules. This happens even though the distribution functions of these systems are not the same. This nontrivial correspondence suggests that exact self-similar nonequilibrium solutions may not exist even in rapidly expanding systems.

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[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.022301 for more details about this derivation.

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