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Solution of the Dynamics of Liquids in the Large-Dimensional Limit

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We obtain analytic expressions for the time correlation functions of a liquid of spherical particles, exact in the limit of high dimensions d. The derivation is long but straightforward: a dynamic virial expansion for which only the first two terms survive, followed by a change to generalized spherical coordinates in the dynamic variables leading to saddle-point evaluation of integrals for large d. The problem is, thus, mapped onto a one-dimensional diffusion in a perturbed harmonic potential with colored noise. At high density, an ergodicity-breaking glass transition is found. In this regime, our results agree with thermodynamics, consistently with the general random first order transition scenario. The glass transition density is higher than the best known lower bound for hard sphere packings in large d. Because our calculation is, if not rigorous, elementary, an improvement in the bound for sphere packings in large dimensions is at hand.

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Introduction.—The physics of liquids and glasses belongs to the group of fields that are victims of the lack of a small parameter. Many approximations have been proposed over the years, but they suffer from the uncertainty about what is the limit in which they are supposed to become exact. This has been true both for equilibrium and for dynamic properties. From the point of view of dynamics, an extreme case is that of mode-coupling theory (MCT) [1–3]: it may be introduced by an (uncontrolled) resummation of an infinite subset of diagrams. The mode-coupling approximation yields mode-coupling dynamics: the phenomenology depends on the approximation itself [4], somewhat like a harmonic approximation is expected to predict oscillations.

An often used remedy for the absence of control parameters [5] is to promote the system to d dimensions, solve the large d limit, and (eventually) expand around. This strategy has been used with success for liquids [6–8], strongly coupled electrons [9], atomic physics [10], gauge field theory [11], and, most recently, the thermodynamics of amorphous systems [12,13]. In this Letter, we extend this procedure to the dynamics of liquids made of spherical particles. We restrict ourselves to equilibrium, although the extension to the glassy off-equilibrium "aging" regime is at hand. It has been a long-standing question whether MCT becomes exact in infinite dimensions [4,14–18], and the present computation gives an answer.

Statement of the main result.—We shall consider a system of N identical particles, interacting via a spherical potential $V(r)$ of typical interaction length σ in d dimensions, and obtain a solution for the equilibrium time correlations of the resulting liquid that becomes exact in the limit $d \to \infty$. We need to confine the particles in a finite volume $\mathcal{V}$. It is very convenient to do this in such a way that the "box" does not break rotational and translational invariances, which are crucial in our developments. A practical way to do this is to consider particles living on points x_i ($i = 1, \dots, N$) on the d-dimensional surface of a hypersphere with $x_i \cdot x_i = \sum_\mu [x_i^\mu]^2 = R^2 \equiv \sigma^2 \Delta_{\text{liq}}/(2d) \gg \sigma^2$ $(\mu = 1, \dots, d+1)$. The thermodynamic limit $R \to \infty$ with constant density $\rho = N/\mathcal{V}$, in which the flat space is recovered, will be taken before $d \to \infty$. Rotations and translations in d dimensions—with dimensions $d(d-1)/2$ and d, respectively—are encoded in the rotations in $d+1$ dimensions, with dimension $d(d+1)/2$. We consider a Langevin dynamics

$$m\ddot{x}_i(t) + \gamma \dot{x}_i(t) = -\nu_i(t) x_i(t) - \nabla_{x_i} H + \xi_i(t), \quad (1)$$

where ξ_i is a white noise with $\langle \xi_i^\mu(t) \xi_j^{\mu'}(t') \rangle = 2T\gamma \delta_{ij} \delta_{\mu\mu'} \delta(t-t')$, $H = \sum_{i<j} V(x_i - x_j)$, and $T = 1/\beta$ is the temperature. Here, and in what follows, $\langle \bullet \rangle$ denotes average over noise ξ_i and/or initial conditions. The ν_i are Lagrange multipliers, imposing the spherical constraints $x_i \cdot \dot{x}_i = 0$. For $d \to \infty$, they do not fluctuate and their value $\nu_i \sim \nu = dTp/R^2$ is proportional to the equilibrium reduced pressure $p = \beta P/\rho$ [19]. We shall, in what follows, treat the overdamped case $m = 0$, but the inertial term may be reinstalled at any stage (and in that case, the thermal bath may be disconnected setting $\gamma = 0$, to recover a purely Newtonian dynamics).

We define the adimensional scaled correlation and response functions (see Fig. 1 in [20])

$$C(t,t') \sim C_i(t,t') = \frac{2d}{\sigma^2} x_i(t) \cdot x_i(t'),$$

$$R(t,t') \sim R_i(t,t') = \frac{2d}{\sigma^2} \sum_\mu \frac{\delta x_i^\mu(t)}{\delta h_i^\mu(t')},$$

$$\Delta(t,t') \sim \frac{d}{\sigma^2} |x_i(t) - x_i(t')|^2 = \Delta_{\text{liq}} - C(t,t'), \quad (2)$$

where h_i is an external field conjugated to x_i, and we have also introduced the single particle mean square displacement Δ [19]. As we shall see, these quantities have a finite and nonfluctuating limit for $d \to \infty$. In the following, and in [20], we shall derive exact equations for the correlations (2). For simplicity, we will restrict to equilibrium where $C(t,t') = C(t-t')$, and $R(t) = -\beta\theta(t)\dot{C}(t)$; here, $\theta(t)$ is the Heaviside step function.

Let us define a scaled density $\hat{\varphi} = \rho\mathcal{V}_d(\sigma)/d$ where $\mathcal{V}_d(\sigma)$ is the volume of a d-dimensional sphere of radius σ, a scaled $\hat{\gamma} = \sigma^2\gamma/(2d^2)$, a scaled potential $\bar{V}(y) = V[\sigma(1+y/d)]$, and the interparticle force $F(y) = -\bar{V}'(y)$. The averages are obtained from a dynamics with a colored noise ζ

$$\begin{aligned}&\hat{\gamma}\,\dot{y}(t) = -w'[y(t)] - \beta\int_0^t duM(t-u)\dot{y}(u) + \zeta(t),\\ &\langle\zeta(t)\zeta(t')\rangle = 2\hat{\gamma}T\delta(t-t') + M(t-t'),\end{aligned} \tag{3}$$

whose memory kernel satisfies

$$M(t-t') = \frac{\hat{\varphi}}{2}\int dy_0 e^{-\beta w(y_0)}\langle F(t)F(t')\rangle_{M,y_0}. \tag{4}$$

The average in (4) is over the process (3) with $y(t=0) = y_0$. The total effective potential has the form (see Fig. 4 in [20])

$$w(y_0) = \bar{V}(y_0) - Ty_0 + \frac{Ty_0^2}{2\Delta_{\rm liq}}. \tag{5}$$

The quadratic part of the potential plays the role of the confining “box.” In fact, it is negligible for finite times and large $\Delta_{\rm liq}$, and the probability distribution is exponential in y near $y \sim 0$ (expressing the growth of entropy as a function of distance along the $(d+1)$-dimensional sphere), the region relevant for those times. Reassuringly, box details are irrelevant at short times.

Once M is determined by solving Eqs. (3) and (4), one can obtain the correlations from

$$\begin{aligned}&\hat{\gamma}\,\dot{C}(t) = -\frac{T}{\Delta_{\rm liq}}C(t) - \beta\int_0^t duM(t-u)\dot{C}(u),\\ &\hat{\gamma}\,\dot{\Delta}(t) = T - \beta\int_0^t duM(t-u)\dot{\Delta}(u),\end{aligned} \tag{6}$$

the second relation being valid for $\Delta \ll \Delta_{\rm liq}$.

Finally, Eq. (4) has a microscopic counterpart [20]; in fact, the function $M(t)$ is, in the large d limit, the autocorrelation of the interparticle forces $F_{ij}(t) = -\nabla V[x_i(t) - x_j(t)]$

$$M(t) \propto \frac{1}{N}\sum_{i\neq j}\langle F_{ij}(t)\cdot F_{ij}(0)\rangle, \tag{7}$$

a result that provides a physical interpretation of $M(t)$.

Relation with MCT.—If M were a simple function of C, $M = \mathsf{F}(C)$, then Eq. (6) would be in the schematic MCT form. As is well known, schematic MCT is obtained as the exact dynamics of a system of spherical spins $\sum_i s_i^2 = N$ with p-spin random interactions [25–27], for which

$$M = \mathsf{F}(C) = \frac{p}{2}C^{p-1}. \tag{8}$$

However, as soon as one considers nonspherical variables, e.g., soft spins with a potential $V(s) = a(s^2-1)^2$, one obtains an equation like (3) with this $V(s)$ [26,28,29]

$$\dot{s}(t) = -V'(s) - \beta\int_0^t duM(t-u)\dot{s}(u) + \zeta(t). \tag{9}$$

Here again, Eq. (8) holds, and the system is closed by $C(t-t') = \langle s(t)s(t')\rangle$. Within the liquid phase, this more general form of dynamic equation has essentially the same phenomenology as schematic MCT. Our system of equations belongs to this more general class, and thus, they show exactly the same MCT phenomenology for what concerns universal quantities that are independent of details of the memory kernel (e.g., the dynamical scaling forms and the relations between critical exponents) [3,30,31].

However, important quantitative differences are observed with respect to applying the MCT approximation to the intermediate scattering functions, which leads to the “standard” formulation of MCT for liquids [1,2]. Standard MCT has the same qualitative structure as schematic MCT but also provides quantitative results for the self and collective scattering functions in all dimensions, in particular in $d = 3$ [32,33]; its $d \to \infty$ limit was discussed in Refs. [15,16]. Our result in $d \to \infty$ is formulated in terms of $\Delta(t,t')$, and most of the other natural observables are functionals of $\Delta(t,t')$. For example, for $q\sigma/d^{3/2} \ll 1$, we have for the self intermediate scattering function [20]:

$$\phi_q^{\rm s}(t,t') = \exp\left[-\frac{q^2\sigma^2}{2d^2}\Delta(t,t')\right], \tag{10}$$

in contrast to the non-Gaussianity in q one finds within MCT close to the plateau [15,16]. One could then write our equations in terms of $\phi_q^{\rm s}$. The result, however, is different from standard MCT, and in particular, our kernel M is not an analytic function of $\phi_q^{\rm s}$.

Because our equations fall in the same universality class as schematic MCT but provide different quantitative results with respect to standard MCT in $d \to \infty$, it remains a matter of taste whether one wishes to call them with the same name or, more generally, “dynamic random first order transition (RFOT)” [14].

Sketch of the derivation.—Let us outline the main steps in the derivation; more details are given in [20]. In order to construct the high-dimensional limit, a virial expansion is a reliable method. Following [34], we exploit the well-known analogy between trajectories and polymers. The dynamics are generated by a sum of trajectories in d-dimensional space, in our case, the surface of a $(d+1)$-dimensional sphere. To each trajectory is associated an Onsager-Machlup probability weight which is the exponential of an action. The sum over all trajectories of this quantity is analogous to a partition function and is, thus, used to generate averages over the Langevin process (1). The action is expressed in terms of trajectories of the x_i and auxiliary "response" variables $\hat{x}_i$ (Martin–Siggia–Rose–De Dominicis–Janssen generating path integral) [25,27]. The result reads, in the Itô convention

$$\begin{aligned} Z_N &= \int \prod_{i=1}^{N} Dx_i D\hat{x}_i e^{-\sum_{i=1}^{N}\Phi[x_i,\hat{x}_i]-\sum_{i<j}^{1,N}W[x_i-x_j,\hat{x}_i-\hat{x}_j]}, \\ \Phi[x,\hat{x}] &= \gamma\int dt(T\hat{x}^2 + i\dot{x}\cdot\hat{x} + \nu\hat{x}\cdot x), \\ W[x_1-x_2,\hat{x}_1-\hat{x}_2] &= i\int dt(\hat{x}_1-\hat{x}_2)\cdot\nabla V(|x_1-x_2|). \end{aligned} \tag{11}$$

Following standard liquid theory [19,34], we introduce the density function for trajectories

$$\rho[x,\hat{x}] \equiv \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta[x-x_i]\delta[\hat{x}-\hat{x}_i]\right\rangle, \tag{12}$$

where $\delta[x]$ is the functional Dirac δ (a product of deltas over all times), and we construct a virial (Mayer) expansion as a power series in $\rho[x,\hat{x}]$. One can show that all terms involving a product of more than two density fields are subleading for $d\to\infty$ [8]: they are exponentially suppressed because of the requirement that the three trajectories overlap (see Fig. 1 in [20]), which is exponentially unlikely in large d. Truncating the virial expansion accordingly, we get [20]

$$\begin{aligned} \mathcal{S} \equiv \frac{\ln Z_N}{N} &= -\int D\rho[x,\hat{x}](\Phi[x,\hat{x}] + \ln\rho[x,\hat{x}]) \\ &+ \frac{N}{2}\int D\rho[x_1,\hat{x}_1]D\rho[x_2,\hat{x}_2]f[x_1-x_2,\hat{x}_1-\hat{x}_2], \end{aligned} \tag{13}$$

where $D\rho[x,\hat{x}] = D[x,\hat{x}]\rho[x,\hat{x}]$ and $f[x_1-x_2,\hat{x}_1-\hat{x}_2] = e^{-W[x_1-x_2,\hat{x}_1-\hat{x}_2]} - 1$. The physical $\rho[x,\hat{x}]$ is determined by $\delta\mathcal{S}/\delta\rho[x,\hat{x}] = 0$ and the normalization $\int D\rho[x,\hat{x}] = 1$. The first term in Eq. (13) is an ideal gas contribution and the second accounts for interactions.

Following the thermodynamic treatment [35], we may now argue that due to rotational invariance on the hypersphere, $\rho[x(t),\hat{x}(t)] = \rho[C(t,t'),R(t,t'),D(t,t')]$ where $R(t,t')\equiv(2d/\sigma^2)x(t)\cdot\hat{x}(t')$ and $D(t,t')\equiv(2d/\sigma^2)\hat{x}(t)\cdot\hat{x}(t')$. Thus, we can make a change of variables in the functional integration over $x(t),\hat{x}(t)$ to $Q(t,t')\equiv\{C(t,t'),R(t,t'),D(t,t')\}$. The change of variables gives for density averages [20]

$$\int D[x,\hat{x}]\bullet\rho \to \int DQ\bullet e^{d\,\mathrm{str}\ln Q - d\Omega(Q)}, \tag{14}$$

where $e^{d\,\mathrm{str}\ln Q}$ is the Jacobian of the transformation [20], and we defined $\rho(Q) = e^{-d\Omega(Q)}$. The appearance of the dimension in the exponent leads to a narrowing of fluctuations of correlations, and saddle-point evaluation becomes exact [20]. In this way, we can compute the ideal gas term in Eq. (13). For the interaction term, that involves two ρ functions, we need the variables corresponding to $(x_1,\hat{x}_1)$, $(x_2,\hat{x}_2)$ and also $\omega = |x_1-x_2|^2$, $\hat{\omega} = (x_1-x_2)\cdot(\hat{x}_1-\hat{x}_2)$. The Jacobian may be calculated with the same methods [20], and the crucial result is that, at the saddle-point level, $Q_1 = Q_2 = Q$ with Q determined by the same saddle-point as in Eq. (14), while the remaining integration over $\omega(t),\hat{\omega}(t)$ is effectively one dimensional. Changing variables with $\omega(t) = \sigma^2[1+y(t)/d]$ leads to a finite integration over $y(t)$ that eventually gives Eq. (3). Hence, the typical distance between two trajectories turns out to be $\sigma + O(1/d)$. This scaling physically tells us that a particle vibrates inside a cage with $1/d$ amplitude and interacts with $O(d)$ neighbours. Finally, equilibrium and causality at the saddle-point level imply $D(t,t') = 0$, and the fluctuation-dissipation relation $R(t-t') = \beta\theta(t-t')\partial_{t'}C(t-t')$. The fact that saddle-point evaluation is made for the two-time variables is the mathematical justification of the fact mentioned above, that C, R, D do not fluctuate. The saddle-point equation for Q gives Eqs. (4) and (6) [20].

In this Letter, we are treating an equilibrium situation. Within the liquid phase, this may be achieved by starting from any configuration in the distant past. A more practical way, however, is to assume equilibrium at a convenient time t_0 (e.g., $t_0 = 0$). How does one deal with a non-Markovian equation of motion like (3)? The answer is simple: either one makes the memory kernel extend to the remote past, or, alternatively, one may assume equilibrium at t_0, in other words, summing all the past histories passing through $y(t_0)$ at t_0. It turns out [36] that this is implemented simply by cutting the memory at a lower limit t_0, as in Eq. (3) [20]. This completes the derivation of our basic dynamical equations.

Dynamic transition and time scale separation.—We now apply the standard MCT methodology to locate the density or temperature at which a dynamic transition occurs with freezing in a cage, corresponding to the development of a plateau in $\Delta(t)$ [1–3,27].

Consider the case when $M(t)$ falls from $M(0)$ to a plateau value M_{EA}, and then, at much larger times, to zero. Concomitantly, $\Delta(t)$ grows to a plateau value Δ_{EA}, and then continues to grow at a slower (diffusive) pace. Denote the fast part $\delta M(t) = M(t) - M_{EA}$. In the limit in which the plateau times are much larger than the microscopic times, and much smaller than the final relaxation times, the noise breaks into a fast variable $\delta\zeta(t)$ and a slow random variable $\bar{\zeta}$, as does the friction term. Their sum acts as an adiabatically slow field $Y(t)$ at those times. We may, thus, split the equilibration in two steps [37]: $P(y|Y)$ and $P_{\mathrm{slow}}(Y)$. When $t-t'$ is in the plateau region, we may write the expectations

$$\langle A(t)B(t')\rangle = \int dY P_{\mathrm{slow}}(Y)\langle A\rangle_Y \langle B\rangle_Y, \tag{15}$$

where $\langle\bullet\rangle_Y = \int dY \bullet P(y|Y)$, $P_{\mathrm{eq}}(y) \propto e^{-\beta w_0(y)}$, and

$$P(y|Y) = \frac{P_{\mathrm{eq}}(y)e^{-(\beta^2 M_{EA}/2)(y-\frac{T}{M_{EA}}Y)^2}}{\int dy' P_{\mathrm{eq}}(y')e^{-(\beta^2 M_{EA}/2)(y'-\frac{T}{M_{EA}}Y)^2}},$$
$$P_{\mathrm{slow}}(Y) = \frac{\int dy P_{\mathrm{eq}}(y)e^{-(\beta^2 M_{EA}/2)(y-\frac{T}{M_{EA}}Y)^2}}{\sqrt{2\pi M_{EA}}}. \tag{16}$$

Obviously, $P_{\mathrm{eq}}(y) = \int dY P(y|Y)P_{\mathrm{slow}}(Y)$. We, therefore, obtain the self-consistent equation for M_{EA}

$$M_{EA} = \frac{\hat{\varphi}}{2}\int dY P_{\mathrm{slow}}(Y)\langle F\rangle_Y^2 \equiv \mathcal{M}(M_{EA}). \tag{17}$$

From Eq. (6), we obtain

$$\beta^2 M_{EA} = \frac{1}{\Delta_{EA}} - \frac{1}{\Delta_{\mathrm{liq}}} \sim \frac{1}{\Delta_{EA}}. \tag{18}$$

The dynamical transition point, at which the plateau becomes infinite, happens when Eq. (17) first has a nonzero solution for $M_{EA} = T^2/\Delta_{EA}$. This point happens as a bifurcation, and may be quickly obtained by solving Eq. (17) together with $\mathcal{M}'(M_{EA}) = 0$ [20].

For hard spheres, the result is $\hat{\varphi}_{\mathrm{d}} = 4.807$. This result is fully consistent with the one based on thermodynamics [12], and, in fact, Eq. (17) is exactly identical to the one that can be derived using the replica method [20] consistently with the general RFOT scenario [14,38–43]. This is a particular instance of a general correspondence between thermodynamic and dynamic results that is verified by the infinite-d solution, and can be extended to critical MCT exponents [44] and to correlation functions [45–47]. In fact, expanding around Δ_{EA} [1–3], one can show that $\Delta_{EA} - \Delta(t) \sim t^{-a}$ upon approaching the plateau, while $\Delta(t) - \Delta_{EA} \sim t^b$ upon leaving the plateau. The exponents a, b satisfy the famous relation [1–3,44]

$$\frac{\Gamma(1-a)^2}{\Gamma(1-2a)} = \frac{\Gamma(1+b)^2}{\Gamma(1+2b)} = \lambda. \tag{19}$$

For hard spheres, we obtain $\lambda = 0.707$ which implies $a = 0.324$ and $b = 0.629$ [48]. Finally, from Eq. (10), one can show that the factorization property of MCT [1,2] still holds in $d \to \infty$, namely, that close to the plateau, $\phi_q^{\mathrm{s}}(t) - \phi_{q,\mathrm{EA}}^{\mathrm{s}} \sim -(q^2\sigma^2/2d^2)\phi_{q,\mathrm{EA}}^{\mathrm{s}}[\Delta(t) - \Delta_{\mathrm{EA}}]$ factorizes in a function of q and a function of t [20].

Diffusion, viscosity, Stokes-Einstein relation.—At long times, in the liquid phase $\hat{\varphi} < \hat{\varphi}_d$, the motion is diffusive and $\Delta(t) \sim (2d^2 D/\sigma^2)t$, where D is the diffusion coefficient. Plugging this form in Eq. (6), and recalling that $M(t)$ decays to zero over a finite time, we obtain an exact result for D

$$\frac{2d^2}{\sigma^2}D = \frac{T}{\hat{\gamma} + \beta\int_0^\infty dt M(t)}. \tag{20}$$

At low density $M(t) = 0$, and we recover the bare diffusion coefficient $D = T/\gamma$. Upon increasing density, $M(t)$ increases and D decreases. For $\hat{\varphi} \to \hat{\varphi}_{\mathrm{d}}^-$, where a finite plateau of $M(t)$ emerges, $\int_0^\infty dt M(t)$ diverges and the diffusion coefficient vanishes as $D \sim (\hat{\varphi}_{\mathrm{d}} - \hat{\varphi})^\gamma$ with the exponent $\gamma = 1/(2a) + 1/(2b) = 2.34$, which is consistent with the numerical results of [49].

Within linear response theory, the shear viscosity η_S of the liquid is given by [19,50]

$$\eta_S = \frac{\beta}{\mathcal{V}}\int_0^\infty dt \langle\sigma_{\mu\nu}(t)\sigma_{\mu\nu}(0)\rangle,$$
$$\sigma_{\mu\nu}(t) = \sum_{i<j}(x_i^\mu - x_j^\mu)\nabla^\nu V(x_i - x_j), \tag{21}$$

where $\mu \neq \nu$ are two arbitrary components of the stress tensor $\sigma_{\mu\nu}$. In [20], we show that $\langle\sigma_{\mu\nu}(t)\sigma_{\mu\nu}(0)\rangle = dNM(t)$, and thus,

$$\eta_S = \beta\rho d\int_0^\infty dt M(t). \tag{22}$$

This relation shows that, for $\hat{\varphi} \to \hat{\varphi}_{\mathrm{d}}^-$, $\eta_S \sim 1/D \sim (\hat{\varphi}_{\mathrm{d}} - \hat{\varphi})^{-\gamma}$, as it is found in MCT.

Combining Eqs. (20) and (22), we obtain a relation similar to the Stokes-Einstein relation (SER)

$$D = \frac{T}{\gamma + \frac{2d}{\rho\sigma^2}\eta_S} \approx \frac{T\rho\sigma^2}{2d}\frac{1}{\eta_S}, \tag{23}$$

where the second expression holds close to $\hat{\varphi}_d$. This relation is interesting because it predicts that the SER is not exactly satisfied in dense liquids: the quantity $D\eta_S/T$, which is constant in SER, has, instead, a small variation proportional to ρ. This is in agreement with results of Fig. 7(b) in [51] that show a linear variation of $D\eta_S$ with ρ in the dense regime for large enough dimension.

Conclusions.—In this Letter, we obtained an exact solution of the dynamics of liquids in the limit of infinite spatial dimension. The picture that emerges has a relevant "caging" length scale that is smaller than the particle radius by $O(1/d)$ (it is about $1/5$ for real colloids [52]). The physics of diffusion stems from interactions at that small scale, while all particle motion beyond that scale consists of uncorrelated steps of displacement, and memory of what happens at distances $\gg 1/d$ is lost. Diffusion coefficients and viscosity are, thus, decided at distances much smaller than the particle radius. This strongly suggests that the wave vectors q that matter for this transition are not the ones associated with the first neighbor distance $1/\sigma$, but rather the much larger ones $\sim d/\sigma$ corresponding to the cage size. Note that, at the transition, cages are correlated over large distances [38,53–55]. Also, an expansion in the number of collisions [7] seems difficult to reconcile with our results, because we expect multiple collisions within a cage.

Is the high-dimensional dynamics related to MCT? The answer is that the result is not the one obtained from the usual procedure for building up a MCT equation, which, for example, gives a different scaling of $\hat{\varphi}_d$ with dimension [15,16]. Instead, the system we obtain is formally quite close to the slightly more general case of soft-spin mean-field dynamics, Eq. (9), because we have mapped the system into a one-dimensional dynamics in the presence of a colored noise and friction, that have to be determined self-consistently through Eqs. (3) and (4).

In the dense glassy regime, we obtain predictions for the scaling of the cage radius, of the dynamical transition density $\hat{\varphi}_{\rm d}$, and of the parameter λ, that differ from the ones of usual MCT [15,16,18]. Our results are fully consistent with those obtained from the thermodynamic approach [12,13,48], which proves the exactness of the RFOT scenario [38–43] for statics and dynamics in $d \to \infty$, as conjectured in [14]. Our results are also in agreement with numerical simulations of hard spheres in large spatial dimension [49,51].

Interestingly, we find that an ergodic liquid phase of hard spheres exists for densities $\hat{\varphi} \leq \hat{\varphi}_d = 4.807$. This implies that hard sphere packings exist (at least) up to $\hat{\varphi}_d$, and they can be constructed easily through a sufficiently slow compression of the liquid [56,57]. Note that the value of $\hat{\varphi}_d$ is larger than the best known lower bound for the existence of sphere packings, $\hat{\varphi} \geq 6/e$ [58], and that it took 20 years to improve the previous best lower bound $\hat{\varphi} \geq 2$ [59] by a small factor $3/e$. Our calculation is simple enough that there is hope for transforming it into a rigorous proof, along the lines of [60]. This would result in an improved constructive lower bound for sphere packings.

Future extensions of this work include the investigation of the effect of dissipation [61–64], the study of out-of-equilibrium aging dynamics [65], and the study of non-perturbative processes in $1/d$ through an instantonic expansion. The thermodynamic partition function of quantum systems is formally very similar to Eq. (11) and could also be studied along these lines.

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