Level Statistics and Localization Transitions of Lévy Matrices

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This work provides a thorough study of Lévy, or heavy-tailed, random matrices (LMs). By analyzing the self-consistent equation on the probability distribution of the diagonal elements of the resolvent we establish the equation determining the localization transition and obtain the phase diagram. Using arguments based on supersymmetric field theory and Dyson Brownian motion we show that the eigenvalue statistics is the same one as of the Gaussian orthogonal ensemble in the whole delocalized phase and is Poisson-like in the localized phase. Our numerics confirm these findings, valid in the limit of infinitely large LMs, but also reveal that the characteristic scale governing finite size effects diverges much faster than a power law approaching the transition and is already very large far from it. This leads to a very wide crossover region in which the system looks as if it were in a mixed phase. Our results, together with the ones obtained previously, now provide a complete theory of Lévy matrices.

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Since the well-known pioneering applications of Gaussian random matrices to nuclear spectra, random matrix theory (RMT) has found successful applications in many areas of physics [1] and also in other research fields such as wireless communications [2], financial risk [3], and biology [4]. The reason for such remarkable versatility is that RMT provides *universal* results which are independent of the specific probability distribution of the random entries: only a few features that determine the universality class matter. The most commonly studied RMs belong to the Gaussian ensembles [1]. They have been analyzed in great depth taking advantage of the symmetry under the orthogonal (or unitary/symplectic) group of the probability distribution. As an example of universality, $N \times N$ real symmetric RMs, although they belong to the Gaussian orthogonal ensemble (GOE) only if the elements are Gaussian variables, display a GOE-like level statistics also when the distribution of the elements is not Gaussian, provided that it decreases fast enough to infinity [1,5,6].

There exists, however, a large set of matrices that fall out of the universality classes based on the Gaussian paradigm [7]. These are obtained when the entries are heavy-tailed i.i.d. random variables (i.e., with infinite variance). The reference case for this different universality class corresponds to entries that are *Lévy* distributed. This is the natural generalization of the Gaussian case since the limiting distribution of the sum of a large number of heavy-tailed i.i.d. random variables is indeed a Lévy distribution, as is the Gaussian distribution for nonheavy tailed random variables. Understanding the statistical spectral properties of these, so called, Lévy matrices (LMs) is an exciting problem from the mathematical and the physical sides [7–14]. They represent a new (and very broad) universality class, with different and somehow unexpected properties with respect to the Gaussian case. Actually, a huge variety of distributions in physics and in other disciplines exhibit power-law behavior. Accordingly, LMs appear in several contexts: in models of spin glasses with dipolar RKKY interactions [15], in disordered electronic systems [16], in portfolio optimization [17], and in the study of correlations in big data sets [18], just to cite a few.

Contrary to the Gaussian case, the theory of random LMs is not yet well established. LMs were introduced in the pioneering work of Ref. [7] and further studied in Refs. [8–14]. By now, the behavior of their density of states is well understood (even rigorously) [7–10]. Instead, on finer observables, such as level and eigenfunction statistics, there are scarcer and even conflicting results. This is probably due to the fact that the behavior of LMs is richer, and hence more difficult to understand, than the one of GOE matrices. For instance, a mobility edge separating high energy localized states from low energy extended states appears within their spectrum [7]. It was also argued that they display a new intermediate mixed phase, characterized by a nonuniversal level statistics. Although some aspects of the scenario put forward in Ref. [7] are in contradiction with recent rigorous results [13], such mixed phase could indeed exist and actually be related to the one recently observed in the Anderson model on the Bethe lattice [19,20]. It may be the simplest case of the nonergodic delocalized phase advocated for quantum many body disordered systems in Ref. [21].

In the following we focus on $N \times N$ real symmetric matrices $\mathcal{H}$ with entries $h_{ij} = h_{ji}$ distributed independently according to a law, $P(h_{ij}) = N^{1/\mu} f(N^{1/\mu} h_{ij})$, characterized by heavy tails:

$$P(h_{ij}) \simeq \frac{\mu}{2N|h_{ij}|^{1+\mu}}, \qquad |h_{ij}| \to \infty; \qquad \mu < 2.$$

The specific form of $f(x)$ does not matter. For concreteness in numerical applications we will focus on a Student distribution with exponent $1+\mu$ and symmetric entries, $f(x) = f(-x)$. The scaling of the entries with N is such that almost all eigenvalues are $O(1)$ for $N \to \infty$.

The first issue we address is determining the localization-delocalization transition line $E^\star(\mu)$ in the E-μ plane. In order to do so, we focus on the statistics of the diagonal elements of the resolvent matrix $\hat{G} = [(E - i\eta)\mathcal{I} - \mathcal{H}]^{-1}$, which allows one to compute in the $\eta \to 0^+$ limit spectral properties of $\mathcal{H}$ such as the global density of states $\rho(E) = (1/N)\sum_{n=1}^{N} \delta(E - \lambda_n) = \lim_{\eta\to 0^+}(1/N\pi)\sum_{i=1}^{N} \Im G_{ii}$ and the average inverse participation ratio (IPR) $\langle \Upsilon_{2,n} \rangle = \langle \sum_{i=1}^{N} |\langle i|n\rangle|^4 \rangle = \lim_{\eta\to 0^+}(1/N)\sum_{i=1}^{N} \eta |G_{ii}|^2$. As shown in Refs. [7–10], the probability distribution $Q(G)$ of a given G_{ii} is obtained in the large-N limit from the equation

$$G_{ii}^{-1} \stackrel{d}{=} E - i\eta - \sum_{j=1}^{N} h_{ij}^2 G_{jj}, \tag{1}$$

where all correlations between the terms on the rhs can be neglected and $\stackrel{d}{=}$ denotes the equality in distribution between random variables. This leads to a self-consistent equation on $Q(G)$, whose analysis yields the results on the density of states obtained in Refs. [7,10]: For $\mu < 2$, $\rho(E)$ is a μ-dependent symmetric distribution with support on the whole real axis and fat tails with exponent $1+\mu$ (the semicircle law is recovered for $\mu > 2$ only). There are several complementary ways to obtain the localization transition from the statistics of the G_{ii}s. We have followed the one more likely to receive a rigorous treatment, as it was shown for the Anderson transition on the Bethe lattice [22]. It consists in studying the stability of the localized phase, checking whether adding a small imaginary part to G_{ii} is an unstable perturbation [23]. Such stability is governed by an eigenvalue equation for the same integral operator found in Ref. [7], whose analysis can be considerably simplified, as shown in Ref. [24], and boils down to the following closed equation for the mobility edge $E^\star(\mu)$, which is one of the main results of this work:

$$K_\mu^2(s_\mu^2 - s_{1/2}^2)|\ell(E^\star)|^2 - 2 s_\mu K_\mu \Re\ell(E^\star) + 1 = 0, \tag{2}$$

where $K_\mu = \mu\Gamma(1/2 - \mu/2)^2/2$, $s_\mu = \sin(\pi\mu/2)$ and $\ell(E) = \int_0^{+\infty} k^{\mu-1} \hat{L}_{\mu/2}^{C(E),\beta(E)}(k) e^{ikE} \mathrm{d}k/\pi$. The function $\hat{L}_{\mu/2}^{C(E),\beta(E)}(k)$ is the Fourier transform of the probability distribution of the real part of the self-energy, that previous works have shown to be a Lévy stable distribution with exponent $1+\mu/2$ and parameters $C(E)$ and $\beta(E)$ determined self-consistently [7,8,10]. This equation has a solution for $\mu \in (0,1)$ only. For $\mu \to 1$ we find that $E^\star(\mu)$ diverges as $(1-\mu)^{-1}$. In Fig. 1 we show the numerical solution of Eq. (2) for several values of μ (we only consider $E > 0$ since the spectral properties are symmetric around zero). This quantitative phase diagram is in agreement with the sketch of Ref. [7] and the numerics of Ref. [9] (except for $\mu > 1$ where the results were likely inaccurate due to the very large values of E that had to be explored).

We now address more subtle issues related to the level and eigenfunction statistics. We present first, two analytical arguments which show that the statistics is a GOE in the whole delocalized phase and Poisson-like in the localized phase for $N \to \infty$. The former is based on the supersymmetric zero-dimensional field theory introduced for random GOE matrices [31]. Since we follow closely the techniques developed in Refs. [32,33] we just discuss the main steps and refer to Ref. [24] and a longer paper [34] for more details. The starting point is the field theory $Z = \int \prod_i \mathrm{d}\Phi_i e^{S[\Phi_i]}$, with the action

$$S = \frac{i}{2}\left(\sum_{l,m} \Phi_l^\dagger \mathcal{L}(E\delta_{lm} - h_{lm})\Phi_m + \sum_l \Phi_l^\dagger \Phi_l \frac{r + i0^+}{N}\right).$$

The field Φ_i is a eight-component super-vector $(\Phi_i^{(1)}, \Phi_i^{(2)}) = (S_i^a, S_i^b, \chi_i, \chi_i^*, P_i^a, P_i^b, \eta_i, \eta_i^*)$, where each of the four components $\Phi_i^{(1,2)}$ of the supervectors $\Phi^{(1,2)}$ is formed by two real and two Grassman variables. The matrix $\mathcal{L}$ is diagonal with elements $(1,1,1,1,-1,-1,-1,-1)$. The level statistics, in particular the density of states and the correlation between two levels at distance $r/2N$, can be obtained from correlation functions of the fields [31]. Averaging over the matrix elements and introducing the function $\rho(\Phi) = (1/N)\sum_i \delta(\Phi - \Phi_i)$ one can rewrite Z as $\int \mathcal{D}\rho(\Phi) e^{S[\rho]}$ with the action reading

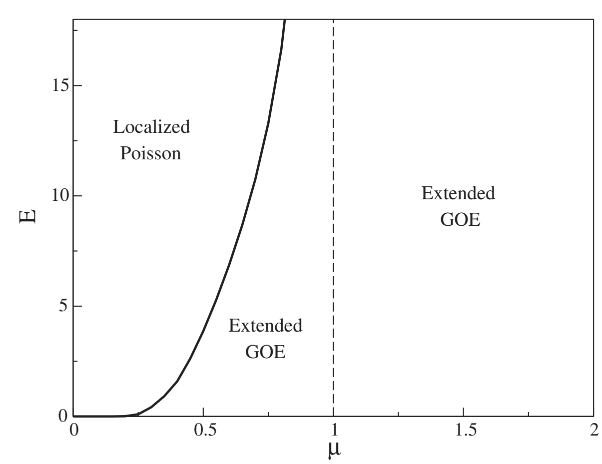

FIG. 1. Phase diagram of LMs in the μ-E plane.

$$S = \frac{i}{2}NE\int \mathrm{d}\Phi\rho(\Phi)\Phi^\dagger\mathcal{L}\Phi + \frac{i}{2}(r+i0^+)\int \mathrm{d}\Phi\rho(\Phi)\Phi^\dagger\Phi - N\int \mathrm{d}\Phi\rho(\Phi)\log\rho(\Phi) + \frac{i}{2}N\int \mathrm{d}\Phi\mathrm{d}\Psi\rho(\Phi)C(\Phi^\dagger\mathcal{L}\Psi)\rho(\Psi),$$

where $C(y) = \mu\int(\mathrm{d}x/2|x|^{1+\mu})[\exp(-ixy)-1]$. Since the second term is subleading compared to the other three that are $O(N)$, one can neglect it at first and perform a saddle point. The solution of the corresponding equation reads

$$\rho(\Phi) = \int d\Sigma R(\Sigma)\exp\left(\frac{i}{2}\Phi^\dagger\mathcal{L}\Phi(E-\Re\Sigma) + \frac{1}{2}\Phi^\dagger\Phi\Im\Sigma\right),$$

where, as it can be shown in full generality [24,32], $R(\Sigma)$ is the probability distribution of the local self-energy, which coincides with the complex Lévy stable law rigorously proven in Ref. [10] (see Ref. [24]). Note that the saddle point equation is invariant under the symmetry $\Phi \to \mathcal{T}\Phi$ where the super matrix $\mathcal{T}$ verifies the equation $\mathcal{T}^\dagger\mathcal{L}\mathcal{T} = 1$. Thus given a solution $\rho(\Phi)$, $\rho_{\mathcal{T}}(\Phi) = \rho(\mathcal{T}\Phi)$ is also a solution. The localization transition corresponds to the breaking of this symmetry [31,33]: in the localized phase the typical value of the imaginary part of the self-energy is zero, whereas it is finite in the delocalized phase. In consequence, in the former case $\rho(\Phi)$ is a function of $\Phi^\dagger\mathcal{L}\Phi$ only, invariant under the symmetry generated by $\mathcal{T}$, whereas in the latter it depends also on $\Phi^\dagger\Phi$. Since this dependence breaks the symmetry there is a manifold of solutions $\rho_{\mathcal{T}}(\Phi)$. It is the integration over this manifold that leads to GOE statistics for the level correlations. The derivation is identical to the one presented in Ref. [33] since the only term in the action that depends on $\mathcal{T}$, i.e., that breaks the symmetry, is the r one as it happens for Erdös-Rényi graphs [35] and GOE RMs [31]. In the localized phase, the saddle point solution is instead unique. Therefore no integration over $\mathcal{T}$ has to be performed and this leads to uncorrelated levels, i.e., Poisson statistics [33].

Let us now turn to the other analytical argument, which is very straightforward but limited to $\mu > 1$ only. Taking inspiration from the recent mathematical breakthrough on RMT [5], we slightly modify the distribution $P(h_{ij})$ into $(1-\epsilon)P(h_{ij}) + \epsilon N^{1/\mu}W(N^{1/\mu}h_{ij})$ where $W(x)$ is a Gaussian distribution with unit variance. This is equivalent to modifying $\mathcal{H}$ into $\mathcal{H}_\epsilon = (1-\epsilon)\mathcal{H} + \epsilon\mathcal{W}$ where $\mathcal{H}$ is a LM and $\mathcal{W}$ a very small GOE matrix whose elements have exactly the same scaling with N than the ones of $\mathcal{H}$. Since this change does not alter the fat tails of the matrix elements, one naturally expects $\mathcal{H}_\epsilon$ and $\mathcal{H}$ to be in the same universality class for any $\epsilon < 1$ and in particular for $\epsilon \to 0$. The statistics of the modified LM—and, by the previous argument, of $\mathcal{H}$—can be obtained using the Dyson Brownian motion (DBM): $\mathcal{H}_\epsilon$ can be interpreted, in the basis that diagonalizes $\mathcal{H}$, as a diagonal matrix to which an infinite number of infinitesimal GOE matrices have been added. The probability of the eigenvalues of $\mathcal{H}_\epsilon$ is therefore given by the DBM starting from the eigenvalues of $\mathcal{H}$, and evolving over a fictive time of the order $N^{-1/\mu}$. Recent rigorous results [5] guarantee that the DBM has enough "time" to reach its stationary distribution, which is the GOE distribution, if $N^{-1/\mu} \gg N^{-1}$ and the typical level spacing of $\mathcal{H}$ is $O(1/N)$—a very reasonable assumption that agrees well with the numerics. This implies that for $\mu > 1$ the level statistics of the modified LM, and hence of the original LM too, is indeed GOE-like in the bulk of the spectrum [36] (see Refs. [24,34] for more details).

We now present several numerical results with the aim of backing up our previous analytical arguments and also of studying the behavior of large *but finite* LMs. In applications N is never truly infinite, actually in several cases it can be just a few thousand. Thus, it is of paramount importance to study finite size effects and determining the characteristic value of N above which the $N \to \infty$ limit is recovered. We performed exact diagonalization of LMs for several system sizes $N = 2^n$, from $n = 8$ to $n = 15$ and averaging over 2^{22-n} realizations of the disorder. We have resolved the energy spectrum in 64 small intervals ν, centered around the energies $E_\nu = \langle\lambda_n\rangle_{n\in\nu}$, and analyzed the statistics of eigenvalues and eigenfunctions in each one of them. We have focused on several observables that display different universal behaviors in the GOE and Poisson regimes: The first probe, introduced in Ref. [37], is the ratio of adjacent gaps $r_n = \min\{\delta_n, \delta_{n+1}\}/\max\{\delta_n, \delta_{n+1}\}$ where $\delta_n = \lambda_{n+1} - \lambda_n \geq 0$ denotes the level spacings between neighboring eigenvalues. It has different universal distributions in the GOE and Poisson cases encoding, respectively, the repulsion or the independence of levels. The second one is the overlap between eigenvectors corresponding to subsequent eigenvalues, defined as $q_n = \sum_{i=1}^N |\langle i|n\rangle||\langle i|n+1\rangle|$. Its typical value $q_\nu^{\mathrm{typ}} = e^{\langle\ln q_n\rangle_{n\in\nu}}$ allows us to make the difference between the localized phase, in which subsequent eigenvectors do not overlap ($q^{\mathrm{typ}} = 0$), and the delocalized GOE one in which they do ($q^{\mathrm{typ}} = 2/\pi$). Finally, the wave function support set, recently introduced in Ref. [38], is defined for an eigenvector n with sites ordered according to $|\langle i|n\rangle| > |\langle i+1|n\rangle|$ as the sets of sites $i < S_\epsilon^{(n)}$ such that $\sum_{i=1}^{S_\epsilon^{(n)}} |\langle i|n\rangle|^2 \leq 1-\epsilon < \sum_{i=1}^{S_\epsilon^{(n)}+1} |\langle i|n\rangle|^2$. The scaling of $S_\epsilon^{(n)}(N)$ for $N \to \infty$ and ϵ arbitrary small but finite allows us to discriminate between a localized and extended phase. The analysis of all these probes clearly shows that for $\mu > 1$ the eigenvalues and eigenvectors statistics is a GOE in the limit of large N, in agreement with our previous arguments (and also with the rigorous results on the delocalized nature of wave functions [13]). As an example we show in the top panel of Fig. 2 the behavior of the average value of r_n for $\mu = 1.5$. It clearly converges in the limit $N \to \infty$ and for any

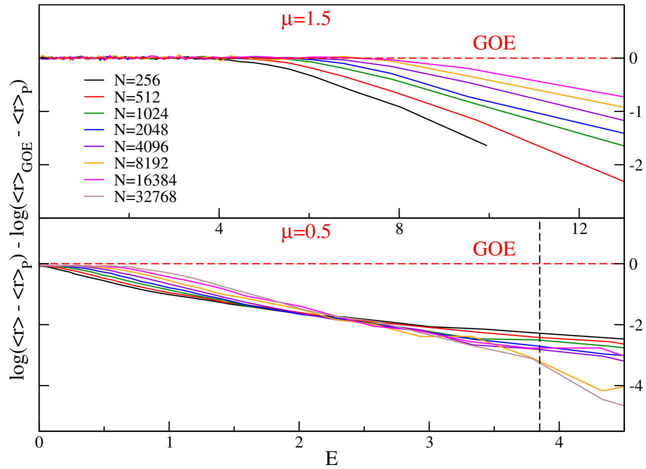

FIG. 2. $\ln[(\langle r\rangle - \langle r\rangle_P)/(\langle r\rangle_{\mathrm{GOE}} - \langle r\rangle_P)]$ as a function of E for different system sizes and for $\mu = 1.5$ (top) and $\mu = 0.5$ (bottom). The dashed line represents the position of the mobility edge, $E^\star \simeq 3.85$.

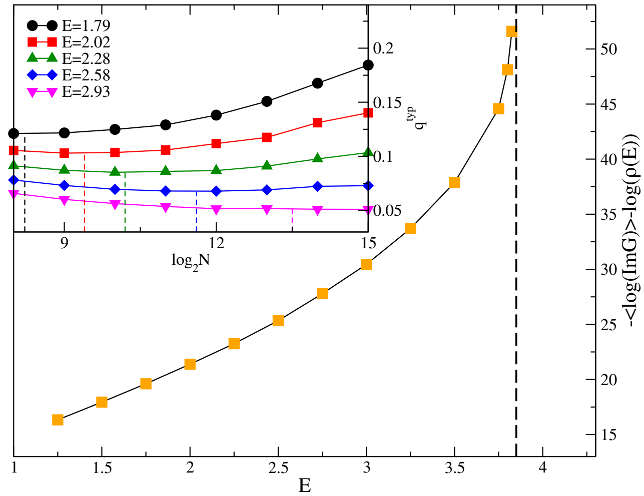

FIG. 3. Main panel: $\log N'_m(E) = -\log(\Im G_{ii}^{\mathrm{typ}} \rho(E)) = -\langle \log \Im G_{ii}\rangle - \log(\langle \Im G\rangle/\pi)$ as a function of E for $\mu = 0.5$. Inset: q_{typ} as a function of N for different energies and $\mu = 0.5$, showing the position of $N_m(E)$.

energy E to the value $\langle r_n\rangle_{\mathrm{GOE}} \simeq 0.53$ characteristic of GOE statistics. All other probes show a similar convergence to the values expected for GOE statistics (see Ref. [24] for the corresponding plots). This is no longer true for $\mu < 1$, where the situation is more involved. In the lower panel of Fig. 2 we show again the behavior of the average of r_n but now for $\mu = 0.5$. For small and large energies we find the values $\langle r\rangle_{\mathrm{GOE}} \simeq 0.53$ and $\langle r\rangle_P \simeq 0.39$ corresponding respectively to GOE and Poisson statistics. Moreover, the curves corresponding to different values of N seem to cross much before the localization transition, that our previous analytical results located at $E^\star \simeq 3.85$ for $\mu = 0.5$. If this were representative of the truly asymptotic large-N behavior then it would possibly signal the existence of a mixed phase which could be *delocalized but nonergodic*, i.e., not displaying GOE statistics. However, analyzing carefully the data—thanks to the large number of samples used to average over the disorder—we find that the crossing point is in fact very slowly drifting towards higher energies as N is increased. The same behavior is found for *all* the probes we studied. As an example, in the inset of Fig. 3 we plot q_ν^{typ} as a function of the system size for energies belonging to the crossing region. This indeed shows that q_{typ} is a nonmonotonic function of N. We can then define a characteristic matrix size, $N_m(E)$, such that for $N \ll N_m(E)$ the statistics appears to be intermediate between Poisson and GOE (see Ref. [24]), whereas for $N \gg N_m(E)$ it tends again toward GOE.

The existence of a crossover size can be understood from the properties of the distribution $Q(G)$. What characterizes the delocalized phase is that, at any site i, the imaginary part of G_{ii} receives an infinitesimal contribution from an infinite number of eigenfunctions. This leads to a typical value of G_{ii} (defined as $\Im G_{ii}^{\mathrm{typ}} = e^{\langle \log \Im G_{ii}\rangle}$) which is finite for $N \to \infty$ and $\eta \to 0$. Instead, $\Im G_{ii}^{\mathrm{typ}} = 0$ in the localized phase. Approaching the transition from the delocalized side, $\Im G_{ii}^{\mathrm{typ}}$ becomes extremely small. Thus, one needs to take large enough systems in order to realize that it is different from zero, and hence that the system is in the delocalized and GOE-like phase. The argument, which is based on the interpretation of $\Im G_{ii}$ as the local density of states, is as follows. The number of states per unit of energy close to E is $N\rho(E)$. This number, multiplied by the typical value of the local density of states, has to be larger than one in order to be in a regime representative of the large-N limit. This defines the crossover scale $N'_m(E) \propto 1/(\Im G_{ii}^{\mathrm{typ}}\rho(E))$. We have compared numerically $\ln N'_m(E)$ and $\ln N_m(E)$ and found that they are indeed proportional (see Ref. [24] for a plot), thus showing that our argument correctly captures the origin of the finite size effects. We plot the crossover scale [actually $N'_m(E)$] as a function of E in Fig. 3 for $\mu = 0.5$: it diverges very fast approaching $E^\star(\mu)$. A good fit is provided by an essential singularity. These results therefore unveil what is the mechanism responsible for the non-GOE statistics observed for finite LMs in a wide regime before the localization transition.

In conclusion, we have presented a thorough analysis of the eigenvalues and eigenvectors statistics of random Lévy matrices. We have shown that the localization and the level statistics transitions coincide but also unveil the existence of a crossover scale which is very large even far from the transition. Thus, many practical cases are expected to be in the $N \ll N_m(E)$ regime. In consequence, the mixed behavior proposed in Ref. [7] will be often present in practice even though it is absent in the large-N limit. Our work, together with the results obtained previously, now provides a complete theory of LMs.

There are several directions worth pursuing more. It would be interesting to determine analytically the form of the divergence of $N_m(E)$. On the basis of our numerics and in analogy with previous works [31,33,35] we expect $N_m(E) \propto e^{c/(E^\star - E)^a}$. Most probably, the emergence of the crossover scale producing an apparent mixed phase takes place in several other related situations (e.g., Ref. [19]) that are, therefore, to be reanalyzed. Finally, our results provide a guideline for mathematicians working on RMT. Thanks to the recent advances in the mathematical analysis of random matrices [5] and localization phenomena [22] our findings are likely to be rigorously proven in a not too distant future.

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