# Nonlinear Bell Inequalities Tailored for Quantum Networks

Denis Rosset,[1] Cyril Branciard,[2] Tomer Jack Barnea,[1] Gilles Pütz,[1] Nicolas Brunner,[3] and Nicolas Gisin[1]
[1]Group of Applied Physics, Université de Genève, 1211 Genève, Switzerland
[2]Institut Néel, CNRS and Université Grenoble Alpes, 38042 Grenoble Cedex 9, France
[3]Département de Physique Théorique, Université de Genève, 1211 Genève, Switzerland
(Received 8 July 2015; published 7 January 2016)

In a quantum network, distant observers sharing physical resources emitted by independent sources can establish strong correlations, which defy any classical explanation in terms of local variables. We discuss the characterization of nonlocal correlations in such a situation, when compared to those that can be generated in networks distributing independent local variables. We present an iterative procedure for constructing Bell inequalities tailored for networks: starting from a given network, and a corresponding Bell inequality, our technique provides new Bell inequalities for a more complex network, involving one additional source and one additional observer. We illustrate the relevance of our method on a variety of networks, demonstrating significant quantum violations, which could not have been detected using standard Bell inequalities.

DOI: 10.1103/PhysRevLett.116.010403

Distant observers performing local measurements on a shared entangled quantum state can observe strong correlations, which have no equivalent in classical physics. This phenomenon, termed quantum nonlocality [1,2], is at the core of quantum theory and represents a key resource for quantum information processing [3,4].

This remarkable feature is now relatively well understood in the case of observers sharing entangled states originating from a single common source, for which a solid theoretical framework has been established [2], and many classes of Bell inequalities have been derived; see, e.g., Ref. [5]. The situation is, however, very different in the case of quantum networks, which have been far less explored so far. A quantum network features distant observers, as well as several independent quantum sources distributing entangled states to different subsets of observers (see Fig. 1). Crucially, by performing joint measurements, observers can correlate distant (and initially fully independent) quantum systems, hence establishing strong correlations across the entire network. Characterizing and detecting the nonlocality of such correlations represents a fundamental challenge, which is also highly relevant to the implementation of quantum networks [6] and quantum repeaters [7].

Importantly, the methods of standard Bell locality (where a single source is considered) are not appropriate for discussing nonlocality in networks featuring independent sources. This was demonstrated in a few exploratory works discussing the simplest networks [8–11]. An illustrative example is the scenario of entanglement swapping, where two independent sources distribute entanglement to three distant observers [see Fig. 2(a)]. Here there exist genuine quantum correlations which cannot be detected using standard Bell inequalities, as they admit a local model. However, the nonlocality of these correlations can be detected via more powerful nonlinear Bell inequalities. Notably, these inequalities are derived under the assumption that the sources are independent[8,9], which represents a natural extension of Bell's original idea of "local beables" [1] to the case of a network. Recently, the framework of causal inference was also shown to be relevant in this context [12–17]. Nevertheless, it is fair to say that adequate methods are still currently lacking for discussing nonlocal correlations in networks beyond the simplest possible cases.

In this work, we present simple and efficient methods for detecting and characterizing nonlocal correlations in a wide class of networks. Specifically, we give an iterative procedure for constructing Bell inequalities tailored for networks. Starting from a given network, and a Bell inequality for it, we then construct inequalities for a more complex network, involving one additional source and one additional observer. Importantly, the inequalities we construct capture the independence of the sources. We illustrate the relevance of our approach considering a variety of networks and demonstrate significant violations in quantum theory, which cannot be detected using standard Bell inequalities.

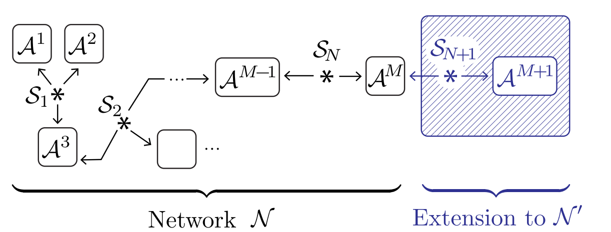


FIG. 1. We consider networks consisting of distant observers $\mathcal{A}^j$ sharing physical resources emitted from independent sources $\mathcal{S}_k$, and discuss nonlocality in such networks. In our approach, starting from a network $\mathcal{N}$ (in black) with $N$ sources and $M$ parties, we define a new network $\mathcal{N}'$ by adding a new independent source $\mathcal{S}_{N+1}$ connected to a single party $\mathcal{A}^M$ of $\mathcal{N}$ and to a new party $\mathcal{A}^{M+1}$ (in blue). We show how Bell inequalities for so-called $\mathcal{N}'$-local correlations can be derived starting from Bell inequalities for $\mathcal{N}$-local correlations.

 

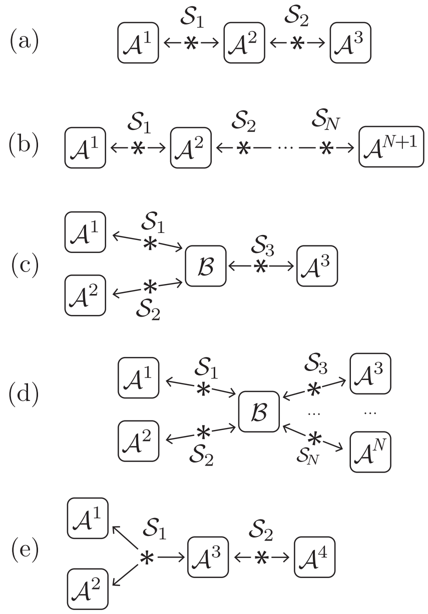

FIG. 2. In the main text, we discuss a variety of networks: (a) the scenario of bilocality, (b) a general chain network featuring $N$ sources and $M = N + 1$ observers, (c) a three-branch star network, (d) a general star-shaped network with $N$ branches, and (e) a network featuring a different topology, illustrating the versatility of our method.

We believe that the simplicity and versatility of our method makes it adequate for starting a systematic exploration of quantum nonlocality in networks.

*Scenario of $\mathcal{N}$-locality.*—Consider a network $\mathcal{N}$ consisting of $N$ independent sources $\mathcal{S}_1, \ldots, \mathcal{S}_N$ sending physical systems to $M$ parties $\mathcal{A}^1, \ldots, \mathcal{A}^M$ (see Fig. 1). Each party thus holds a number of systems, and performs a measurement on them (assumed here to be binary). Specifically, we denote by $x^j$ the input received by party $\mathcal{A}^j$, and by $a^j_{x^j} = \pm 1$, its corresponding output.

Our goal is to capture the strength of correlations that can be established in a network $\mathcal{N}$ for different types of resources. In particular, we want to compare the correlations established in the case of a quantum network (i.e., with quantum sources, and with the parties performing quantum measurements) to those that can arise in local (hidden) variable models. Importantly, the latter should feature the same network structure as $\mathcal{N}$, with independent sources of local variables, and are thus referred to as $\mathcal{N}$-local models. This represents the natural generalization of the notions of Bell locality [1,2] (tailored for the case of a single source), and "bilocality" [8,9] (tailored for the scenario of entanglement swapping with two independent sources), to arbitrary networks.

More formally, we associate with each source $\mathcal{S}_i$ a random local variable $\lambda_i$, which is sent to all parties connected to $\mathcal{S}_i$ in the network $\mathcal{N}$. The crucial assumption of $\mathcal{N}$-locality is that all $\lambda_i$'s are independent from one another, that is, $\rho(\lambda_1, \ldots \lambda_N) = \prod_i \rho_i(\lambda_i)$, for some (nonnegative and normalized) distributions $\rho_i(\lambda_i)$ over some sets $\Lambda_i$. We denote by $\vec{\lambda}_{\mathcal{A}^j}$ the list of random variables $\lambda_i$'s "received" by party $\mathcal{A}^j$. Then the ($M$-partite) joint probability distribution $P(a^1, \ldots, a^M | x^1, \ldots, x^M)$ (where we have omitted redundant subscripts) is $\mathcal{N}$-local if and only if it can be decomposed as

$$\begin{aligned} P(a^1, \ldots, a^M | x^1, \ldots, x^M) &= \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \ldots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \\ &\quad \times P(a^1 | x^1, \vec{\lambda}_{\mathcal{A}^1}) \ldots P(a^M | x^M, \vec{\lambda}_{\mathcal{A}^M}), \end{aligned} \tag{1}$$

where each $P(a^j | x^j, \vec{\lambda}_{\mathcal{A}^j})$ is a valid probability distribution, which (without loss of generality) can be assumed to be deterministic. As we focus on binary measurements, it is convenient to consider correlators, i.e., the expectation values $\langle a^1_{x^1} a^2_{x^2} \ldots a^M_{x^M} \rangle$. In an $\mathcal{N}$-local model, these can be written as

$$\begin{aligned} \langle a^1_{x^1} \ldots a^M_{x^M} \rangle &= \int_{\Lambda_1} \mathrm{d}\lambda_1 \rho_1(\lambda_1) \ldots \int_{\Lambda_N} \mathrm{d}\lambda_N \rho_N(\lambda_N) \\ &\quad \times a^1_{x^1}(\vec{\lambda}_{\mathcal{A}^1}) \ldots a^M_{x^M}(\vec{\lambda}_{\mathcal{A}^M}), \end{aligned} \tag{2}$$

for some deterministic response functions $a^j_{x^j}(\vec{\lambda}_{\mathcal{A}^j}) = \pm 1$ of the party's input $x^j$ and of the random variables $\vec{\lambda}_{\mathcal{A}^j}$.

Characterizing the set of $\mathcal{N}$-local correlations is a challenging problem. The main technical difficulty, for cases beyond that of standard Bell locality, originates from the independence of the sources, which makes the set nonconvex. Here we will present a simple and efficient technique for generating Bell inequalities tailored for the problem of capturing $\mathcal{N}$-local correlations. Hence, a violation of such inequalities, which is usually possible considering quantum networks, certifies that no $\mathcal{N}$-local model can reproduce the given correlations. Below we state our main result, which is an iterative procedure for constructing Bell inequalities for $\mathcal{N}$-local correlations. We then illustrate the relevance of our method by applying it to simple networks, and discuss quantum violations.

*Main result.*—Consider a network $\mathcal{N}$, and a Bell inequality tailored for it. From $\mathcal{N}$, we construct a new network $\mathcal{N}'$ by adding one source, $\mathcal{S}_{N+1}$, linked to just one party of $\mathcal{N}$, say $\mathcal{A}_M$, and to one new party, $\mathcal{A}^{M+1}$ (see Fig. 1). The new party $\mathcal{A}^{M+1}$ gets an input $x^{M+1}$, which we choose to be binary ($x^{M+1} = 0, 1$), and gives a binary output $a^{M+1}_{x^{M+1}} = \pm 1$. Given a Bell inequality capturing $\mathcal{N}$-local correlations, we can now construct a Bell inequality tailored for $\mathcal{N}'$-local correlations using the following result:

*Theorem 1.*—Suppose that the correlators $\langle a^1_{x^1}, \ldots, a^M_{x^M} \rangle$ in any $\mathcal{N}$-local model satisfy a Bell inequality of the form

$$\sum_{x^1, \ldots, x^M} \beta_{x^1, \ldots, x^M} \langle a^1_{x^1}, \ldots, a^M_{x^M} \rangle \leq 1 \tag{3}$$

for some real coefficients $\beta_{x^1,\ldots,x^M}$. Then, $\mathcal{N}'$-local correlations (for the network $\mathcal{N}'$ obtained from $\mathcal{N}$ as described above) satisfy the following constraint: either there exists $q \in ]0,1[$ such that for any partition of the set of party $\mathcal{A}^M$'s inputs into two disjoint subsets $\mathcal{X}_+^M$ and $\mathcal{X}_-^M$, we have

$$\frac{1}{q}\Sigma_{\mathcal{X}_+} + \frac{1}{1-q}\Sigma_{\mathcal{X}_-} \le 1 \tag{4}$$

for

$$\Sigma_{\mathcal{X}_\pm} = \sum_{\substack{x^1,\ldots,x^{M-1},\\ x^M \in \mathcal{X}_\pm^M}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1}\ldots a^M_{x^M} \frac{a_0^{M+1} \pm a_1^{M+1}}{2} \right\rangle; \tag{5}$$

or, $\Sigma_{\mathcal{X}_-} = 0$ and $\Sigma_{\mathcal{X}_+} \le 1$ for all $k$ and all $\mathcal{X}_\pm^M$; or, $\Sigma_{\mathcal{X}_+} = 0$ and $\Sigma_{\mathcal{X}_-} \le 1$ for all $k$ and all $\mathcal{X}_\pm^M$.

In the present Letter, we abuse the notation and write $q \in [0,1]$ to cover all cases; indeed, the particular cases where $\Sigma_{\mathcal{X}_\pm} = 0$ can easily be recovered in the limits $q \to 1$ or $q \to 0$. In part A of the Supplemental Material [18] we provide a more general statement of the above theorem—which allows one to consider several Bell inequalities at once and also allows for non-full-correlation terms in these inequalities—as well as a detailed proof. Interestingly, the technique used in our proof also provides an original way to derive the simplest Bell inequality of Clauser-Horne-Shimony-Holt (CHSH) [22], as discussed in part B of [18].

A remarkable feature of the "Bell inequality" (4) is that it involves the quantifier "$\exists q\ldots$". As a consequence, despite its appearance it actually defines a nonlinear constraint on $\mathcal{N}'$-local correlations. One could eliminate the quantifier by minimizing the left-hand side of Eq. (4) over $q$; this would indeed lead to explicitly nonlinear Bell inequalities (see below and parts C–F of the Supplemental Material [18]). However, it will be convenient in general to keep these quantifiers (in a practical test, they could be eliminated later, by optimizing the parameters $q$ directly for the specific values of the observed statistics). In fact, Theorem 1 also applies to an initial Bell inequality for $\mathcal{N}$-local correlations that features quantifiers itself. Our technique can therefore be used in an iterative manner, and allows one to construct Bell inequalities for a broad class of networks, as we shall see below.

*Bilocality.*—Let us first apply the above method to the simplest nontrivial network $\mathcal{N}$ consisting of $M = 2$ parties $\mathcal{A}^1$ and $\mathcal{A}^2$ connected to a single source $\mathcal{S}_1$, that is, the usual Bell scenario [1,2]. In that case, $\mathcal{N}$-local (i.e., here, simply "Bell local") correlations satisfy the well-known CHSH inequality [22]:

$$\left\langle \frac{a_0^1 + a_1^1}{2} a_0^2 \right\rangle + \left\langle \frac{a_0^1 - a_1^1}{2} a_1^2 \right\rangle \le 1. \tag{6}$$

The network $\mathcal{N}'$, obtained by adding an independent source $\mathcal{S}_2$ linked to party $\mathcal{A}^2$ and to a new party $\mathcal{A}^3$, corresponds here to the scenario of bilocality [8,9]; see Fig. 2(a). Applying Theorem 1 starting from the CHSH inequality and with $\mathcal{X}_+^2 = \{0\}$ and $\mathcal{X}_-^2 = \{1\}$, we find that $\mathcal{N}'$-local (i.e., bilocal) correlations satisfy the inequality

$$\begin{aligned} &\exists q \in [0,1] \quad \text{such that} \\ &\frac{1}{q}\left\langle \frac{a_0^1 + a_1^1}{2} a_0^2 \frac{a_0^3 + a_1^3}{2} \right\rangle \\ &\quad + \frac{1}{1-q}\left\langle \frac{a_0^1 - a_1^1}{2} a_1^2 \frac{a_0^3 - a_1^3}{2} \right\rangle \le 1. \end{aligned} \tag{7}$$

It is still fairly easy, in this first example, to eliminate the quantifier. As we show in part C of the Supplemental Material [18], this constraint (when combined with similar forms obtained from other versions of CHSH) is equivalent to the (nonlinear) "bilocal inequality" derived previously in Ref. [9].

Next we discuss the quantum violation of the above Bell inequality, thus considering the entanglement swapping scenario. Assume that each source $\mathcal{S}_i$ ($i = 1, 2$) emits two particles in the 2-qubit Werner state $\varrho(v_i) = v_i|\Phi^+\rangle\langle\Phi^+| + (1-v_i)\mathbb{1}/4$, with $v_i \in [0,1]$, $|\Phi^+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$, and $\mathbb{1}/4$ the fully mixed state of two qubits. Moreover, the parties $\mathcal{A}^1$ and $\mathcal{A}^3$ perform single qubit projective measurements given by operators $\hat{a}_0^1 = \hat{a}_0^3 = [(\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{2}]$ (for $x^1, x^3 = 0$) or $\hat{a}_1^1 = \hat{a}_1^3 = [(\hat{\sigma}_z - \hat{\sigma}_x)/\sqrt{2}]$ (for $x^1, x^3 = 1$); here $\hat{\sigma}_z$ and $\hat{\sigma}_x$ are the Pauli matrices. Finally, the intermediate party $\mathcal{A}^2$ performs projective two-qubit measurements given by $\hat{a}_0^2 = \hat{\sigma}_z \otimes \hat{\sigma}_z$ (for $x^2 = 0$) or $\hat{a}_1^2 = \hat{\sigma}_x \otimes \hat{\sigma}_x$ (for $x^2 = 1$). Defining $V = v_1 v_2$, one finds

$$\langle a^1_{x^1} a^2_{x^2} a^3_{x^3} \rangle = (-1)^{x^1 x^2 + x^2 x^3} \frac{V}{2}, \tag{8}$$

so that

$$\left\langle \frac{a_0^1 + a_1^1}{2} a_0^2 \frac{a_0^3 + a_1^3}{2} \right\rangle = \left\langle \frac{a_0^1 - a_1^1}{2} a_1^2 \frac{a_0^3 - a_1^3}{2} \right\rangle = \frac{V}{2}. \tag{9}$$

Noting that $\min_{q\in[0,1]}\{(1/q)(V/2) + [1/(1-q)](V/2)\} = 2V$, we find that the quantum correlations thus obtained violate the Bell inequality (7)—and, hence, are nonbilocal—for any $V > \frac{1}{2}$, as already shown in Ref. [8]. Notably, these quantum correlations are Bell local for any $0 \le V \le 1$ [9] (see also part G of the Supplemental Material [18]), and thus can never violate any standard Bell inequality.

*Chain network.*—The above procedure can be iterated in order to characterize $\mathcal{N}$-local correlations on a one-dimensional chain network [see Fig. 2(b)]. First, starting from the previous bilocality network (with 2 sources and 3 parties), we add a new party $\mathcal{A}^4$, and a source $S_3$ connected to $\mathcal{A}^3$ and $\mathcal{A}^4$. Applying Theorem 1 to the Bell inequality (7), and choosing $\mathcal{X}_+^3 = \{0\}$ and $\mathcal{X}_-^3 = \{1\}$, we find that "trilocal" correlations satisfy the inequality

$$\exists q, r \in [0,1] \quad \text{such that}$$
$$\frac{1}{8}\left[\frac{1}{q}\frac{1}{r}\langle(a_0^1+a_1^1)a_0^2a_0^3(a_0^4+a_1^4)\rangle + \frac{1}{q}\frac{1}{1-r}\langle(a_0^1+a_1^1)a_0^2a_1^3(a_0^4-a_1^4)\rangle + \frac{1}{1-q}\frac{1}{r}\langle(a_0^1-a_1^1)a_1^2a_0^3(a_0^4+a_1^4)\rangle - \frac{1}{1-q}\frac{1}{1-r}\langle(a_0^1-a_1^1)a_1^2a_1^3(a_0^4-a_1^4)\rangle\right] \le 1. \quad (10)$$

Note that it is, in principle, possible to write the above constraint without quantifiers, and end up with a nonlinear Bell inequality (as in the case of bilocality above). We discuss this operation in part D of the Supplemental Material [18]. However, in this case the nonlinear form appears to be extremely cumbersome and of no practical use.

Next, we extend our analysis to chains of arbitrary lengths, focusing on linear Bell inequalities with quantifiers. By further iterating the argument, we obtain the following inequality for chains of $N$ independent sources and $M = N+1$ parties:

$$\exists q^2, \ldots, q^N \in [0,1] \quad \text{such that}$$
$$\frac{1}{2^N}\sum_{x^1,\ldots,x^{N+1}}\frac{1}{q_{x^2}^2}\cdots\frac{1}{q_{x^N}^N}(-1)^{x^1x^2+x^2x^3+\cdots+x^Nx^{N+1}} \times \langle a_{x^1}^1\ldots a_{x^{N+1}}^{N+1}\rangle \le 1, \quad (11)$$

with for each $j$, $q_0^j = q^j$ and $q_1^j = 1-q^j$.

Let us discuss quantum violations. Consider that each source $\mathcal{S}_i$ sends two particles in the Werner state $\varrho(v_i)$; party $\mathcal{A}^1$ measures either $\hat{a}_0^1 = [(\hat{\sigma}_z+\hat{\sigma}_x)/\sqrt{2}]$ or $\hat{a}_1^1 = [(\hat{\sigma}_z-\hat{\sigma}_x)/\sqrt{2}]$; parties $j$, with $2 \le j \le N$ and $j$ even, measure either $\hat{a}_0^j = \hat{\sigma}_z \otimes \hat{\sigma}_z$ or $\hat{a}_1^j = \hat{\sigma}_x \otimes \hat{\sigma}_x$; parties $j$, with $3 \le j \le N$ and $j$ odd, measure either $\hat{a}_0^j = [(\hat{\sigma}_z+\hat{\sigma}_x)/\sqrt{2}]\otimes[(\hat{\sigma}_z+\hat{\sigma}_x)/\sqrt{2}]$ or $\hat{a}_1^j = [(\hat{\sigma}_z-\hat{\sigma}_x)/\sqrt{2}]\otimes[(\hat{\sigma}_z-\hat{\sigma}_x)/\sqrt{2}]$; for $N$ even, party $\mathcal{A}^{N+1}$ measures either $\hat{a}_0^{N+1} = \hat{a}_0^1 = [(\hat{\sigma}_z+\hat{\sigma}_x)/\sqrt{2}]$ or $\hat{a}_1^{N+1} = \hat{a}_1^1 = [(\hat{\sigma}_z-\hat{\sigma}_x)/\sqrt{2}]$; for $N$ odd, party $\mathcal{A}^{N+1}$ measures either $\hat{a}_0^{N+1} = \hat{\sigma}_z$ or $\hat{a}_1^{N+1} = \hat{\sigma}_x$. Defining $V = \prod_{i=1}^N v_i$, one finds

$$\langle a_{x^1}^1\ldots a_{x^{N+1}}^{N+1}\rangle = (-1)^{x^1x^2+x^2x^3+\cdots+x^Nx^{N+1}}\frac{V}{2^{N/2}}. \quad (12)$$

The left-hand side of inequality (11) is then given by

$$\frac{2^{N/2}V}{4^{N-1}}\sum_{x^2,\ldots,x^N}\frac{1}{q_{x^2}^2}\cdots\frac{1}{q_{x^N}^N}. \quad (13)$$

Noting that $\min_{q^2,\ldots,q^N}[\sum_{x^2,\ldots,x^N}(1/q_{x^2}^2)\ldots(1/q_{x^N}^N)] = 4^{N-1}$, we find that the quantum correlations thus obtained violate the Bell inequality (11)—and hence are non-$\mathcal{N}$-local—for $V > 2^{-N/2}$. This proves a conjecture made in Ref. [9].

Interestingly, note that although the global correlations become very weak for large $N$ and $V < 1$, their nonlocality can nevertheless be revealed using the Bell inequality (11). Notably, this would not be possible using standard Bell inequalities, as the correlations (12) admit a local model (assuming a single source) for $V = 1$ (for $N$ even) and for $V = 1/\sqrt{2}$ (for $N$ odd); see part G of the Supplemental Material [18]. This illustrates the advantage offered by $\mathcal{N}$ locality compared to the standard approach of Bell.

*Star network.*—To discuss star-shaped networks, we start from the bilocality network, i.e., a linear chain of 3 parties connected by 2 sources. For clarity, we relabel the parties by calling $\mathcal{A}^1$ and $\mathcal{A}^2$ the first and last parties in the chain, and $\mathcal{B}$ the middle one. The input and output of $\mathcal{B}$ are now denoted by $y$ and $b_y = \pm 1$, respectively. Clearly, $\mathcal{N}$-local correlations satisfy the Bell inequality (7), with $a_{x^2}^2$ replaced by $b_y$ and $a_{x^3}^3$ replaced by $a_{x^2}^2$.

Similarly to our previous constructions, let us add a source $S_3$, connected now to party $\mathcal{B}$ and to a new party $\mathcal{A}^3$. The network $\mathcal{N}'$ thus obtained has a 3-branch star shape [see Fig. 2(c)]. Applying Theorem 1 to the Bell inequality of Eq. (7) (and with the two subsets of party $\mathcal{B}$'s inputs $\mathcal{Y}_+ = \{0\}$ and $\mathcal{Y}_- = \{1\}$), we find that $\mathcal{N}'$-local correlations satisfy the inequality

$$\exists q, r \in [0,1] \quad \text{such that}$$
$$\frac{1}{q}\frac{1}{r}\left\langle\frac{a_0^1+a_1^1}{2}\frac{a_0^2+a_1^2}{2}\frac{a_0^3+a_1^3}{2}b_0\right\rangle + \frac{1}{1-q}\frac{1}{1-r}\left\langle\frac{a_0^1-a_1^1}{2}\frac{a_0^2-a_1^2}{2}\frac{a_0^3-a_1^3}{2}b_1\right\rangle \le 1. \quad (14)$$

Iterating the above procedure, we obtain a star-shaped network $\mathcal{N}$ consisting of $N$ independent sources $\mathcal{S}_i$, each connected to one out of $N$ parties $\mathcal{A}^i$ and to a single central party $\mathcal{B}$, as depicted in Fig. 2(d). For such a network, we find that $\mathcal{N}$-local correlations satisfy the inequality

$$\exists q_1, \ldots, q_{N-1} \in [0,1] \quad \text{such that}$$
$$\frac{1}{q_1}\cdots\frac{1}{q_{N-1}}\left\langle\frac{a_0^1+a_1^1}{2}\cdots\frac{a_0^N+a_1^N}{2}b_0\right\rangle + \frac{1}{1-q_1}\cdots\frac{1}{1-q_{N-1}}\left\langle\frac{a_0^1-a_1^1}{2}\cdots\frac{a_0^N-a_1^N}{2}b_1\right\rangle \le 1. \quad (15)$$

As shown in part E of the Supplemental Material [18], by eliminating the quantifiers one can recover here the nonlinear Bell inequalities derived in Ref. [10], which generalize the bilocal inequalities of Ref. [9] to the star-shaped network considered here. For violations of these inequalities in quantum theory, we refer the reader to Ref. [10].

*Other topologies.*—To illustrate the versatility of our framework, we now discuss a network which is neither a linear chain nor star shaped. Specifically, we start from a network $\mathcal{N}$ consisting of a single source $\mathcal{S}_1$ connected to 3 parties $\mathcal{A}^1$, $\mathcal{A}^2$, and $\mathcal{A}^3$. Here, $\mathcal{N}$-local (i.e., Bell-local) correlations satisfy the Mermin inequality [23]:

$$\left\langle\frac{a_0^1a_1^2+a_1^1a_0^2}{2}a_0^3\right\rangle + \left\langle\frac{a_0^1a_0^2-a_1^1a_1^2}{2}a_1^3\right\rangle \le 1. \quad (16)$$

Adding a source $\mathcal{S}_2$, linked to party $\mathcal{A}^3$ and to a new party $\mathcal{A}^4$, we obtain a network $\mathcal{N}'$ sketched in Fig. 2(e). Using Theorem 1 and choosing $\mathcal{X}^3_+ = \{0\}$ and $\mathcal{X}^3_- = \{1\}$ we find that $\mathcal{N}'$-local correlations have to obey the following Bell inequality:

$$\exists q \in [0,1] \quad \text{such that}$$
$$\frac{1}{q}\left\langle \frac{a_0^1 a_1^2 + a_1^1 a_0^2}{2} a_0^3 \frac{a_0^4 + a_1^4}{2} \right\rangle + \frac{1}{1-q}\left\langle \frac{a_0^1 a_0^2 - a_1^1 a_1^2}{2} a_1^3 \frac{a_0^4 - a_1^4}{2} \right\rangle \leq 1. \qquad (17)$$

We examine the quantum violations of the inequality (17) in part G of the Supplemental Material [18], where the $\mathcal{N}'$-locality independence condition exhibits again a significant advantage compared to Bell standard locality.

*Discussion.*—We presented a simple and efficient method for generating Bell inequalities tailored for networks with independent sources. The relevance of our method was illustrated with various examples, featuring strong quantum violations, which cannot be detected using standard Bell inequalities.

While we focused here on the case of binary inputs and outputs for each observer, our technique can also be used for deriving Bell inequalities with more inputs and outputs. In fact, the only requirements that we explicitly made use of is that party $\mathcal{A}^M$ has binary outputs and the added observer $\mathcal{A}^{M+1}$ has binary inputs and outputs. In part F of the Supplemental Material [18], we illustrate for instance a case with ternary inputs for parties $\mathcal{A}^1$ and $\mathcal{A}^2$ in the bilocality scenario, which also includes non-full-correlation terms. In principle, our technique could also allow for any numbers of outputs for parties $\mathcal{A}^1, \ldots \mathcal{A}^{M-1}$; it would just become quite cumbersome to write without resorting to correlators. Extending our method to the case where the party $\mathcal{A}^M$ has more outputs, and party $\mathcal{A}^{M+1}$ has an arbitrary number of inputs and outputs, is left for future work.

Finally, it would be interesting to derive Bell inequalities tailored for networks featuring loops. In the present work we could only discuss acyclic networks, as our method allows us to "add a leaf" to a graph, but not to create a cycle. Note, however, that given a Bell inequality tailored for a network with a loop, our method can readily be applied in order to add a leaf; however, we are not aware of any nontrivial Bell inequality for networks containing a loop, despite intense research efforts in this direction [9,12].

We thank Ben Garfinkel for useful comments on the manuscript. C. B. acknowledges financial support from the "Retour Post-Doctorants" program (ANR-13-PDOC-0026) of the French National Research Agency and from a Marie Curie International Incoming Fellowship (PIIF-GA-2013-623456) of the European Commission. N. B. from the Swiss National Science Foundation (Grant No. PP00P2_138917 and Starting Grant DIAQ), SEFRI (COST action MP1006) and the EU SIQS.

*Note added.*—While writing up this Letter, we became aware of related and complementary work. First, Lee and Spekkens [24] presented a method for characterizing DAGs of any shape, but restricted to two binary variables. Second, Chaves [25] discussed polynomial Bell inequalities applicable to multivalued variables, but for a restricted class of DAGs. This opens up the interesting possibilities of combining these different methods in order to obtain Bell inequalities in situations where none could be obtained by any of the methods individually.

[1] J. S. Bell, Physics **1**, 195 (1964).
[2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[3] A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. **98**, 230501 (2007).
[4] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. **82**, 665 (2010).
[5] D. Rosset, J.-D. Bancal, and N. Gisin, J. Phys. A **47**, 424022 (2014).
[6] H. J. Kimble, Nature (London) **453**, 1023 (2008).
[7] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Rev. Mod. Phys. **83**, 33 (2011).
[8] C. Branciard, N. Gisin, and S. Pironio, Phys. Rev. Lett. **104**, 170401 (2010).
[9] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Phys. Rev. A **85**, 032119 (2012).
[10] A. Tavakoli, P. Skrzypczyk, D. Cavalcanti, and A. Acin, Phys. Rev. A **90**, 062109 (2014).
[11] C. Branciard, N. Brunner, H. Buhrman, R. Cleve, N. Gisin, S. Portmann, D. Rosset, and M. Szegedy, Phys. Rev. Lett. **109**, 100401 (2012).
[12] T. Fritz, New J. Phys. **14**, 103001 (2012); T. Fritz, arXiv:1404.4812 [Commun. Math. Phys. (to be published)].
[13] R. Chaves and T. Fritz, Phys. Rev. A **85**, 032113 (2012).
[14] R. Chaves, L. Luft, and D. Gross, New J. Phys. **16**, 043001 (2014); R. Chaves, C. Majenz, and D. Gross, Nat. Commun. **6**, 5766 (2015).
[15] C. J. Wood and R. W. Spekkens, New J. Phys. **17**, 033002 (2015).
[16] J. Henson, R. Lal, and M. F. Pusey, New J. Phys. **16**, 113043 (2014).
[17] R. Chaves, R. Kueng, J. B. Brask, and D. Gross, Phys. Rev. Lett. **114**, 140403 (2015).
[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.010403,which includes Refs. [19–21], for detailed proofs of the results presented in the main text.
[19] D. Collins and N. Gisin, J. Phys. A **37**, 1775 (2004).
[20] B. Sturmfels, *Solving Systems of Polynomial Equations* (Am. Mathematical Soc., Providence, RI, 2002).
[21] H. Derksen and G. Kemper, *Computational Invariant Theory* (Springer, New York, 2002).
[22] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[23] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[24] C. M. Lee and R. Spekkens, arXiv:1506.03880.
[25] R. Chaves, preceding Letter, Phys. Rev. Lett. **116**, 010402 (2016).