Kagome Chiral Spin Liquid as a Gauged $U(1)$ Symmetry Protected Topological Phase

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While the existence of a chiral spin liquid (CSL) on a class of spin-$1/2$ kagome antiferromagnets is by now well established numerically, a controlled theoretical path from the lattice model leading to a low-energy topological field theory is still lacking. This we provide via an explicit construction starting from reformulating a microscopic model for a CSL as a lattice gauge theory and deriving the low-energy form of its continuum limit. A crucial ingredient is the realization that the bosonic spinons of the gauge theory exhibit a $U(1)$ symmetry protected topological (SPT) phase, which upon promoting its $U(1)$ global symmetry to a local gauge structure ("gauging"), yields the CSL. We suggest that such an explicit lattice-based construction involving gauging of a SPT phase can be applied more generally to understand topological spin liquids.

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Introduction.—Quantum spin liquids (QSLs) are long-range quantum entangled phases of interacting spins that support fractionalized excitations—generally referred to as spinons [1–4]. The task of predicting and controllably describing a QSL state relevant to a particular lattice spin model remains a challenge to date. This is all the more important at present in the context of understanding lattice Hamiltonians relevant for material [5,6]. Based on our current theoretical understanding of QSLs as deconfined phases of effective gauge theories that emerge as low-energy descriptions of spin models, answering the above questions requires us to address two key issues: (i) identify a faithful low-energy lattice gauge theory (LGT) for a given microscopic spin model and (ii) show that this lattice gauge model exhibits a QSL as a deconfined phase supporting fractionalized excitations.

An example of such a controlled LGT description of QSL is the quantum dimer model on the triangular lattice [7], which can be mapped to a Z_2 gauge theory [8,9], thereby potentially realizing a Z_2 QSL [7,10,11]. However, systematic implementations of such a construction to obtain other QSLs in two-dimensional magnets, such as chiral [12–14] or critical QSLs [3,4], have proven difficult. These QSLs can be obtained using LGT with fermionic spinons [3,4]. However, how to obtain such fermionic spinons in a controlled fashion is largely an open question. On the other hand, a faithful LGT with bosonic spinons and a compact $U(1)$ gauge field can be obtained for some microscopic models, e.g., the quantum dimer model on bipartite lattices [4,15,16] or spin models on the checkerboard lattice [17]. At low energies, these typically result in pure compact $U(1)$ gauge theories [4], for which, in two spatial dimensions, the LGT will be generically in the confining phase [18] leading to conventional ordering (e.g., valence bond solid), but in three spatial dimensions, it can host a stable QSL phase [19–21].

An interesting situation occurs in the low-energy limit of easy-axis spin-$1/2$ kagome antiferromagnets. In this case, the low-energy physics is described by a compact $U(1)$ LGT *coupled to dynamical bosonic spinons* [22,23] (in contrast to the above-mentioned dimer models, which are described by *pure* compact LGT). The dynamical bosonic spinons carry finite gauge charge, and their presence can have drastic influences on the system [22,24]. In addition, recent numerical simulations [25] show that such systems can stabilize QSLs, including the enigmatic kagome spin liquid [26–30] as well as a chiral spin liquid (CSL) [12,13] over large parameter regimes. The latter is a gapped QSL that breaks time-reversal symmetry (spontaneously or explicitly), exhibits topological ground-state degeneracy when put on a torus, and supports gapless chiral edge states with quantized (fractional) spin-Hall conductivity [12,13]. How can such kagome CSL be understood and described from the point of view of the above $U(1)$ LGT? This is the fundamental question that we will formulate an answer to.

In this Letter, taking clues from recent developments in symmetry protected topological phases (SPTs) [31–34], we explicitly construct a LGT description of the CSL phase and obtain its continuum limit a controlled way. Unlike QSLs, SPTs have no intrinsic topological order (and, therefore, no fractionalized bulk excitations) but support symmetry protected anomalous gapless [35] or topologically ordered edges [36]. We explore the idea of "gauging" a SPT to obtain a topologically ordered phase [35,37]. This means promoting the global symmetry that protects a given SPT to a local gauge structure will yield a topologically ordered phase. In particular, we derive a controlled description of the CSL as a gauged $U(1)$ SPT (bosonic integer quantum Hall state) [38–40]. We implement the idea for two microscopic easy-axis kagome spin models, which (or similar versions) were recently shown to host a

CSL by means of numerical simulations (density matrix renormalization group (DMRG) [41]) [25,42–44].

Lattice gauge theory.—We illustrate our main ideas for the chiral spin-1/2 easy-axis model on the kagome lattice described by the Hamiltonian

$$H_{\mathrm{chiral}} = J_z \sum_{\langle pq \rangle} S_p^z S_q^z + \lambda \sum_{p,q,r \in \nabla,\triangle} \vec{S}_r \cdot (\vec{S}_p \times \vec{S}_q), \quad (1)$$

with $J_z \gg \lambda > 0$, which is different from a previous work [44] that studied case $J_z = 0$. The three-spin term is the *scalar spin chirality*, which breaks time reversal and parity explicitly and has been used to engineer a CSL before [44,45]. After discussing the general structure of our construction, we apply it to the CSL phase of the generalized XXZ model on the kagome lattice [25] towards the end of the Letter.

In the classical Ising limit ($\lambda = 0$) of Eq. (1), the ground-state manifold has an extensive degeneracy and is given by all classical spin configurations that fulfill

$$\sum_{p \in \triangle,\nabla} S_p^z = \pm 1/2 \quad (2)$$

for each triangle of the kagome lattice. In this manifold, the three-spin terms act as a perturbation which lifts the classical degeneracy by forming a coherent superposition of classical configurations. We next formulate the resulting degenerate perturbation theory [22]. This is conveniently done in terms of degrees of freedom that live at the center of each triangle; these centers form a honeycomb lattice:

$$\sum_{p \in \triangle_i} S_p^z = a_i^\dagger a_i - \frac{1}{2}, \qquad \sum_{p \in \nabla_k} S_p^z = b_k^\dagger b_k - \frac{1}{2}. \quad (3)$$

Here, $a_i^\dagger$, $b_k^\dagger$ denote the creation operator for the hard-core boson on the A, B sublattice of the honeycomb lattice (Fig. 1).

We can now define a lattice electric field E_{ik} on the links of the honeycomb lattice

$$E_{ik} = -E_{ki} = (S_p^z + 1/2), \quad (4)$$

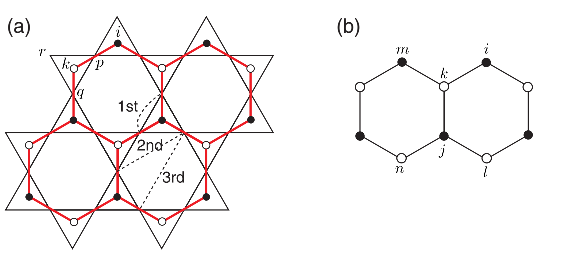

FIG. 1 (color online). (a) Kagome lattice and medial honeycomb lattice. (b) The lattice gauge theory is defined on a honeycomb lattice. The gauge field is only defined on the first neighbor link; hence, the gauge field on other links, i.e., the second neighbor, is $\mathcal{A}_{ij} = \mathcal{A}_{ik} + \mathcal{A}_{kj}$.

where $i(k) \in \triangle(\nabla)$, such that Gauss's law is fulfilled on each site of the honeycomb lattice:

$$\sum_{k;i \in \triangle} E_{ik} = n_i^a + 1, \qquad \sum_{i;k \in \nabla} E_{ki} = -(n_k^b + 1). \quad (5)$$

The summation runs over the three neighbors on the honeycomb lattice. Here, $n_i^a = a_i^\dagger a_i$, $n_k^b = b_k^\dagger b_k$, and ± 1 represent static background charges on the two sublattices. The spin flip operators are then given by

$$S_p^+ = \exp(i\mathcal{A}_{ik}) a_i^\dagger b_k^\dagger \quad \text{and} \quad S_p^- = \exp(i\mathcal{A}_{ki}) a_i b_k, \quad (6)$$

where $\mathcal{A} \in [0, 2\pi)$ is the vector potential conjugate to E_{ik} (here we have softened the hard-core constraint of E_{ik} as usually done in similar systems [4,16,20,21]). Conservation of S^z implies a conserved $U(1)$ charge, and the $a^\dagger$ and $b^\dagger$ that carry fractional $+1/2$ charges, hence, are spinons. We note that the ground state has vanishing magnetization $\sum S^z = 0$; thus, the spinons are effectively at half filling (per site). In addition, the two flavors of spinons carry opposite $U(1)$ gauge charges of $\mathcal{A}$, i.e., under gauge transformation: $a_i \to e^{i\theta_i} a_i$, $b_k \to e^{-i\theta_k} b_k$.

Note that the above mapping is similar to the one used in quantum dimer models [4,16] and quantum or classical spin ice [20,21] with the crucial difference [22] that the present model has dynamical bosonic spinons in addition to the compact $U(1)$ gauge field.

Using the above mapping, we can now represent the effective microscopic model as a lattice gauge theory in terms of the hard-core bosons (spinons) coupled to an emergent $U(1)$ gauge field (unimportant factors omitted):

$$H_{\mathrm{chiral}}^{\mathrm{LGT}} = \lambda \sum_{\langle\langle ij \rangle\rangle} [e^{i\mathcal{A}_{ij} + i\pi/2} (2n_k^b - 1) a_i^\dagger a_j + \mathrm{H.c.}] + \lambda \sum_{\langle\langle kl \rangle\rangle} [e^{i\mathcal{A}_{lk} + i\pi/2} (2n_j^a - 1) b_k^\dagger b_l + \mathrm{H.c.}], \quad (7)$$

where we have the correlated hopping: that bosons (spinons) are hopping within the second neighbors (e.g., $\langle\langle ij \rangle\rangle$) and are coupled to the boson density in the intermediate site (e.g., k) as shown in Fig. 1(b). In principle, there is also a Maxwell term for the gauge fields arising from ring exchange around hexagons. However, since this is third order in perturbation theory ($\sim \lambda^3/J_z^2 \ll \lambda$), we neglect it.

Gauge mean-field theory and the $U(1)$ SPT state.—We now adopt a gauge mean-field (GMF) treatment [46] to solve the above lattice gauge Hamiltonian Eq. (7) and find the SPT phase advertised above. The strategy is as follows: We start by treating the dynamical gauge fluctuation as a mean-field gauge flux $\mathcal{A} \sim \langle \mathcal{A} \rangle = \mathcal{A}^0$, and, consequently, the local $U(1)$ gauge structure is replaced by a $U(1)$ global symmetry. Then we solve the corresponding GMF Hamiltonian and find that the ground state is a gapped $U(1)$ SPT. Finally, we restore the local gauge invariance, and the gap protects the ground state against the gauge fluctuation. This gauged $U(1)$ SPT, as we shall show, is nothing but a CSL [37].

To completely define the GMF Hamiltonian, we need to determine the mean-field gauge flux $\sum_{\hexagon}\mathcal{A}^0$ for each hexagon. As we neglect the Maxwell term, the dynamics of the matter field alone determines the gauge flux. In other words, the flux pattern for the GMF Hamiltonian that has the lowest energy is chosen. Assuming that the solution preserves translation symmetry (as suggested by DMRG results), we consider two possibilities, $\sum_{\hexagon}\mathcal{A}^0=0$ or $\sum_{\hexagon}\mathcal{A}^0=\pi$. Using the infinite DMRG (iDMRG) method [47], we find that the ground state with the background flux $\sum_{\hexagon}\mathcal{A}^0=\pi$ (for each hexagon) has much lower energy $[(E_0-E_\pi)/E_\pi\sim 10\%]$.

With the background flux specified, the GMF Hamiltonian of Eq. (7) describes bosons with correlated hopping subject to a time-reversal symmetry breaking flux of $\pi/2$ (the sum of the gauge flux $\mathcal{A}^0$ and the original hopping flux $\pm\pi/2$). This is exactly the same model as studied in a previous work by us [40], where we found that the ground state is a gapped $U(1)$ SPT phase with Hall conductance $\sigma^c_{xy}=2$. Phenomenologically, the correlated hopping terms in Eq. (7) favor a mutual flux attachment [39] and, thus, stabilize a $U(1)$ SPT phase. In passing, we note that the correlated hopping term actually results from the projection of the scalar chirality to the effective lattice gauge model on the honeycomb lattice (see the Supplemental Material [48]).

Chiral spin liquid.—The above GMF theory provides insights into the correct continuum limit to describe the CSL in the original kagome spin model. This involves taking the low-energy continuum theory of the $U(1)$ SPT and restoring the local $U(1)$ gauge invariance.

The GMF Hamiltonian has two global $U(1)$ symmetries corresponding to (i) the overall particle or charge (n_a+n_b) conservation arising from the S^z conservation of the original spin model and (ii) the pseudospin (n_a-n_b) symmetry from the freezing of the gauge fluctuations, which should, hence, be promoted back to a local gauge structure. The $U(1)$ SPT is described by a condensation of composite bosons $\tilde{a}$, $\tilde{b}$, which is due to the mutual flux attachment: attaching flux of b to the density of a and vice versa [39]. Those composite bosons $\tilde{a}$, $\tilde{b}$ carry the same quantum numbers as bosons a and b; specifically, $\tilde{a}$ carries $+1/2$ S^z charge and $+1$ gauge charge, and $\tilde{b}$ carries $+1/2$ S^z charge and -1 gauge charge. The low-energy theory including the gauge fluctuations is then given by the Lagrangian [39]

$$\mathcal{L}=\mathcal{L}_a+\mathcal{L}_b+\mathcal{L}_{\rm int}+\mathcal{L}_{\rm CS}+\mathcal{L}_{\mathcal{A}}, \tag{8}$$

where

$$\mathcal{L}_a=i\tilde{a}^*\left[\partial_0-i\left(\frac{1}{2}A_0^{\rm ext}+\mathcal{A}_0\right)+i\alpha_0\right]\tilde{a} -\frac{1}{2m}\left|\nabla\tilde{a}-i\left(\frac{1}{2}\vec{A}^{\rm ext}+\vec{\mathcal{A}}\right)\tilde{a}+i\vec{\alpha}\,\tilde{a}\right|^2, \tag{9}$$

$$\mathcal{L}_b=i\tilde{b}^*\left[\partial_0-i\left(\frac{1}{2}A_0^{\rm ext}-\mathcal{A}_0\right)+i\beta_0\right]\tilde{b} -\frac{1}{2m}\left|\nabla\tilde{b}-i\left(\frac{1}{2}\vec{A}^{\rm ext}-\vec{\mathcal{A}}\right)\tilde{b}+i\vec{\beta}\,\tilde{b}\right|^2, \tag{10}$$

and

$$\mathcal{L}_{\rm CS}=\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}[\alpha_\mu\partial_\nu\beta_\lambda+\beta_\mu\partial_\nu\alpha_\lambda], \tag{11}$$

$$\mathcal{L}_{\mathcal{A}}=-\frac{1}{4e^2}(\partial\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu)^2. \tag{12}$$

Here, α, β are the statistical Chern-Simons (CS) fields that implement the mutual flux attachment. $\mathcal{L}_{\rm int}$ represent quadratic and quartic terms for the composite bosons that can be tuned to condense them. $\mathcal{A}$ represents the internal dynamic gauge field, and $A^{\rm ext}$ represents an external test field that couples to the S^z charge of the microscopic models.

When $\tilde{a}$, $\tilde{b}$ condense to stabilize the $U(1)$ SPT, the CS statistical gauge fields are locked as

$$\alpha=\frac{1}{2}A^{\rm ext}+\mathcal{A},\qquad \beta=\frac{1}{2}A^{\rm ext}-\mathcal{A}. \tag{13}$$

Thus, the CS term becomes the action of CSL,

$$\mathcal{L}_{\rm CS}=\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\left[\frac{1}{2}A_\mu^{\rm ext}\partial_\nu A_\lambda^{\rm ext}-2\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda\right]. \tag{14}$$

The last term on the right, namely, the CS term for the emergent gauge field, prohibits the creation of instantons (monopoles) in $2+1$ dimensions; hence, the system is deconfined [4], and along with the Maxwell term for $\mathcal{A}$ provides a mass $m\sim e^2$ for the photon and, hence, gaps it out, and the low-energy theory is given by the first term which gives a Hall conductivity (of S^z) $\sigma_{xy}=1/2$.

It is worth elaborating on two important points about the relation between the $U(1)$ SPT and CSL. First, the $U(1)$ SPT requires one global $U(1)$ symmetry, either the $U(1)$ charge or the $U(1)$ pseudospin, to protect it. The $U(1)$ pseudospin symmetry (n_a-n_b) corresponds to the local gauge structure in the context of the CSL phase, which can never be broken and, hence, protects the $U(1)$ SPT. On the contrary, breaking the $U(1)$ charge corresponds to breaking global S^z conservation of the CSL. As the CSL is topologically ordered, it is robust against such symmetry breaking. Second, we need to understand how to match edge modes, namely, the two counterpropagating edge modes of the $U(1)$ SPT and the single chiral edge mode of the CSL. The two edge modes of the $U(1)$ SPT are a left-mover carrying charge and a right-mover carrying pseudospin. Note that the pseudospin mode is coupled to the dynamical gauge field $\mathcal{A}$ and will, thus, be removed after integrating out $\mathcal{A}$ as it acquires a Higgs mass [49]. As a result, only the charge mode is left, which becomes the chiral mode of the CSL. This edge mode does not require the global S^z conservation to protect it.

Numerical verification.—Using numerical techniques, we can show that the original kagome system Eq. (1) has indeed a CSL ground state—in keeping with our gauge theory analysis. Similar studies have been discussed in detail of other systems with CSL elsewhere [25,42–44], and, thus, here we will only briefly review our numerical results. In particular, we use the iDMRG method [47] to obtain the twofold topological degenerate ground states [50], calculate fractional Hall conductance [51], obtain the braiding statistics (via modular matrices [13,52,53]), and probe the chiral gapless edge mode from entanglement spectra [54] (see the Supplemental Material [48]). All those calculations validate the characteristic topological properties of a CSL, which are robust against finite size effects, do not involve any data extrapolation, and, hence, are quite reliable.

Interestingly, based on the lattice gauge theory, we can identify the Wilson loop operator $\mathcal{W}_y$ in the microscopic spin model. The Wilson loop operator is a global operator that distinguishes different topological degenerate ground states and can be obtained as follows: first create a pair of spinons, then wind them around the torus or infinite cylinder, and finally annihilate them. Then the expectation value of the Wilson loop operator for the two topologically degenerate ground states ($\psi_{1(s)}$) is $\mathcal{W}_y \sim \pm 1$ [Fig. 2(a)]. From our microscopic lattice gauge theory, the Wilson loop corresponds to a gauge flux around the noncontractible loop along the y direction [Fig. 2(b)]:

$$\mathcal{W}_y = \exp\left(i\sum_{\mathcal{NC}_y}\mathcal{A}_{ij}\right) = \mathcal{P}\left(\prod_{\mathcal{NC}_y} S_i^+ S_j^-\right)\mathcal{P} = \mathcal{P}\mathcal{T}_y\mathcal{P}. \quad (15)$$

$\mathcal{P}$ represents a projection into the classically degenerate manifold. Technically, the projection can be treated as a renormalization and approximated by $\langle\psi|\mathcal{W}_y|\psi\rangle \approx \langle\psi|\mathcal{T}_y|\psi\rangle/\langle\psi|\mathcal{T}_y^\dagger\mathcal{T}_y|\psi\rangle$. Numerically, we find that $\langle\psi|\mathcal{T}_y|\psi\rangle/\langle\psi|\mathcal{T}_y^\dagger\mathcal{T}_y|\psi\rangle \sim \pm 0.8$, which is close to the above theoretical expectation.

Anisotropic kagome XXZ model.—We finally discuss the relevance of our findings for the extended kagome XXZ model [25]

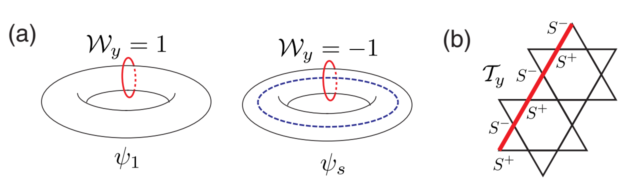

FIG. 2 (color online). (a) Topological degenerate ground states distinguished by the Wilson loop operator. (b) Approximation of the Wilson loop operator of the microscopic kagome model.

$$H_{XXZ} = J_z\sum_{\langle pq\rangle} S_p^z S_q^z + \frac{J_{xy}}{2}\sum_{\langle pq\rangle}(S_p^+ S_q^- + \text{H.c.}) + \frac{J'_{xy}}{2}\sum_{\langle\langle pq\rangle\rangle}(S_p^+ S_q^- + \text{H.c.}) + \frac{J'_{xy}}{2}\sum_{\langle\langle\langle pq\rangle\rangle\rangle}(S_p^+ S_q^- + \text{H.c.}), \quad (16)$$

with first-neighbor $\langle\rangle$ XXZ interactions, second- $\langle\langle\rangle\rangle$, and third- $\langle\langle\langle\rangle\rangle\rangle$ neighbor XY interactions (see Fig. 1).

The general phase diagram Fig. 3(a) [25] shows an extended chiral spin liquid phase that spontaneously breaks time-reversal symmetry for sufficiently strong second- and third-neighbor XY interactions $J_{xy} \sim J_{xy}'$. Analogous to the derivation for the chiral model Eq. (1) above, we find the effective lattice gauge Hamiltonian for the limit J_{xy}, $J'_{xy} \ll J_z$, which reads (with unimportant factors omitted)

$$H_{XXZ}^{\text{LGT}} = J_{xy}\left[\sum_{\langle\langle ij\rangle\rangle} e^{i\mathcal{A}_{ij}} a_i^\dagger a_j + \sum_{\langle\langle kl\rangle\rangle} e^{i\mathcal{A}_{lk}} b_k^\dagger b_l + \text{H.c.}\right] + J'_{xy}\sum_{\langle ik\rangle,\langle jl\rangle\in\bigcirc}\left[(e^{i\mathcal{A}_{ik}} a_i^\dagger b_k^\dagger)(e^{i\mathcal{A}_{lj}} b_l a_j) + \text{H.c.}\right]. \quad (17)$$

Again we use the GMF approach to solve this Hamiltonian. An interesting observation is that its GMF Hamiltonian can be written as the product of correlated hopping terms (see the Supplemental Material [48]):

$$\tilde{H}_{XXZ} = -J_{xy}\left[\sum_{ijm,k}\chi^a_{im,k}\chi^a_{mj,k} + \sum_{kln,j}\chi^b_{kn,j}\chi^b_{nl,j} + \text{H.c.}\right] - J'_{xy}\sum_{\langle ik\rangle,\langle jl\rangle\in\bigcirc}\chi^a_{ij,k}\chi^b_{kl,j}. \quad (18)$$

We use a generalized form of correlated hopping,

$$\chi^a_{ij,l} = i(2n_l^b - 1)e^{i\mathcal{A}^0_{ij}} a_i^\dagger a_j, \chi^a_{ji,l} = (\chi^a_{ji,l})^\dagger, \quad (19)$$

and similarly for χ^b. Here we do not require site l to be the nearest neighbor of site i and j. Thus, it is reasonable to expect that a finite J_{xy}' could induce a spontaneous

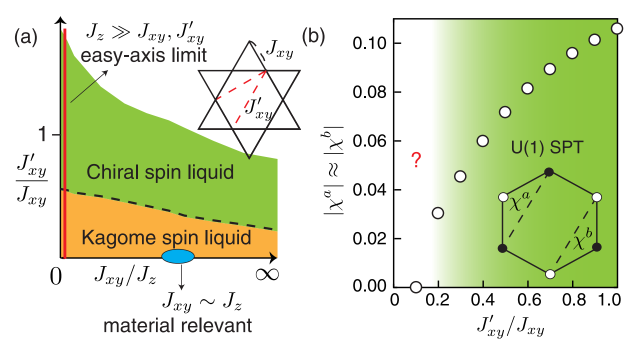

FIG. 3 (color online). (a) Phase diagram of the XXZ kagome model in Eq. (16). (b) Order parameter for spontaneous time-reversal symmetry breaking in the GMF Hamiltonian; here, the width of the cylinder is $W_y = 8$ sites.

time-reversal symmetry breaking with $\chi^a, \chi^b \neq 0$ and result in a $U(1)$ SPT phase (in the GMF Hamiltonian) as in the correlated hopping model [40]. With the help of iDMRG simulations, we find that this is indeed true for systems with sufficiently large J'_{xy} (see the Supplemental Material [48]). For example, the order parameter χ^a, χ^b for the time-reversal symmetry breaking is shown in Fig. 3(b). Therefore, the CSL in the kagome XXZ model can also be explained as a gauged $U(1)$ SPT following our construction.

Conclusion.—We have achieved a theoretical understanding of recent numerically discovered chiral spin liquids in kagome antiferromagnets, which turns out to be a gauged $U(1)$ symmetry protected topological phase. It is a rare example that starting from a microscopic model, a (non-Z_2) spin liquid phase can be described in a controlled way in two spatial dimensions, via a faithful gauge theoretic model accounting for the QSL as its deconfined phase. This framework might be applicable to numerous interesting problems, such as solving the precise nature of the kagome spin liquid as well as realizing exotic topological phases through the gauging procedure in related kagome systems or doped quantum dimer models.

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