

Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We obtain an analytical expression for the next-to-next-to-leading order of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) Pomeron eigenvalue in planar $\mathcal{N} = 4$ SYM using quantum spectral curve (QSC) integrability-based method. The result is verified with more than 60-digit precision using the numerical method developed by us in a previous paper [N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, arXiv:1504.06640]. As a by-product, we developed a general analytic method of solving the QSC perturbatively.

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Introduction.—QCD is notorious for being hard to explore analytically: perturbative calculations become impossibly complex after first few loop orders. However, there are regimes in which one can probe all orders of perturbation theory analytically. The Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation is applicable in processes like deep inelastic scattering or hadronic dijet production, which are characterized by a presence of at least two widely separated energy scales. The large logarithm of ratio of these energy scales Δy enters into perturbative expansion, so in order to make sense of the perturbation theory, one has to resume powers of Δy in every order of perturbation theory. Schematically, these large corrections exponentiate to an expression of the form

$$d\sigma(\Delta y, p_{\perp}) \propto \sum_{n=-\infty}^{+\infty} e^{in\bar{\phi}} \int_{-\infty}^{\infty} d\nu Q_{\nu}(p_{\perp}) e^{A g^2 \chi(\nu, n) \Delta y}, \quad (1)$$

where $g = \sqrt{\lambda}/4\pi$ with λ being the 't Hooft coupling constant. In particular, this can be done in the case of the high-energy hadron-hadron scattering [1]. In this case, Q absorbs the trivial dependence on the transverse momenta and the nontrivial part χ is the so-called LO BFKL eigenvalue [2,3]

$$\chi^{\text{LO}}(\nu, n) = 2\psi(1) - \psi\left(\frac{n+1+i\nu}{2}\right) - \psi\left(\frac{n+1-i\nu}{2}\right). \quad (2)$$

In this Letter, we focus on the case $n = 0$. Taking into account the next-to-leading, next-to-next-to-leading corrections we get a similar structure [4], where the BFKL eigenvalue χ in the exponent gets corrected. One usually introduces $j(i\nu)$, related to the BFKL eigenvalue as

$$\frac{j(i\nu) - 1}{4g^2} = \chi^{\text{LO}}(\nu, 0) + g^2 \chi^{\text{NLO}}(\nu, 0) + g^4 \chi^{\text{NNLO}}(\nu, 0) + \dots$$

The next-to-leading BFKL eigenvalue was obtained after nine years of laborious calculations in [5–8]; the result in

modern notation is presented below in the text [Eq. (3)]. The corrections turned out to be numerically rather large compared to the LO expression, which makes one question the validity of the whole BFKL resummation procedure and its applicability for phenomenology.

This and other indications make it clear that just the NLO expression may not be enough to match experimental predictions. It is important to understand the general structure of BFKL expansion terms, and this Letter is concerned with NNLO BFKL eigenvalue in $\mathcal{N} = 4$ SYM—a more symmetric analog of QCD. Notably, it was observed in [9] that the $\mathcal{N} = 4$ SYM reproduces correctly the part of the QCD result with maximal transcendentality. In particular, the LO expression coincides exactly in the two theories.

Another way of extracting the Pomeron eigenvalue, technically more convenient, is due to the observation of [2,7] who reformulated the problem in terms of a certain analytical continuation of anomalous dimensions of twist-2 operators. Fortunately, in planar $\mathcal{N} = 4$ SYM, the problem of computing the anomalous dimensions is solved for finite coupling and any operator by the quantum spectral curve (QSC) formalism [9,10].

In order to obtain the BFKL eigenvalue in $\mathcal{N} = 4$ SYM from the anomalous dimension of twist operators, we consider the dimension $\Delta(S)$ of twist-two operator $\mathcal{O} = \text{Tr} Z D_{+}^S Z$. The inverse function $S(\Delta)$ is known to approach -1 perturbatively for Δ in the range $[-1, 1]$ and thus, the map to the BFKL regime is given by $\Delta = i\nu$ and $j = 2 + S(\Delta)$ (see, e.g., [11]). Then the goal is to compute $j(\Delta)$ as a series expansion in g^2 . Indeed, from the QSC formalism, it was shown in [12] that one reproduces correctly the LO result [Eq. (2)]. Here, we use some shortcuts to the direct approach of [12] to push the calculation to NNLO order, which already gives useful new information about the QCD result.

An essential for us observation was made in [13,14] where it was pointed out that both LO and NLO results can be represented as a simple linear combination of the nested

harmonic sums. Let us stress again that in our notation Δ is the full conformal dimension of the twist-two operator, related to the anomalous dimension γ as $\Delta = 2 + S + \gamma$. Then the expansion of $j(\Delta)$ can be written as

$$j(\Delta) = 1 + \sum_{\ell=1}^{\infty} g^{2\ell} \left[F_{\ell} \left(\frac{\Delta-1}{2} \right) + F_{\ell} \left(\frac{-\Delta-1}{2} \right) \right] \quad (3)$$

with the two first known orders given by [13]

$$F_1 = -4S_1, \\ F_2 = -\frac{3}{2}\zeta_3 + \pi^2 \ln 2 + \frac{\pi^2}{3}S_1 + 2S_3 + \pi^2 S_{-1} - 4S_{-2,1}, \quad (4)$$

where

$$S_{a_1, a_2, \dots, a_n}(x) = \sum_{y=1}^x \frac{[\text{sgn}(a_1)]^y}{y^{|a_1|}} S_{a_2, \dots, a_n}(y), \quad S(x) = 1.$$

We define harmonic sums for noninteger and negative arguments by the standard prescription, namely, analytic continuation from positive even integer values as in [15,16]. These analytically continued sums, which we denote as $S_{a_1, a_2, \dots}$, are denoted by $\tilde{S}_{a_1, a_2, \dots}^+$ in [15], see, e.g., Eq. (21) in that paper. A compatible but more general definition is given in [17]. This prescription is also implemented in the Mathematica files attached to the present Letter [18].

We assume the NNLO order can also be written in this form. After that, we only have to fix a finite number of coefficients which we do by expanding the QSC around some values of Δ where the result simplifies. Then we verify our result by comparing it with extremely high precision numerical evaluation proving this assumption to be correct.

Quantum Spectral Curve Generalities.—As it was already mentioned in the Introduction, there is a known relation between the anomalous dimensions of the twist-2 operators and the BFKL pomeron eigenvalue. Here, we describe the quantum spectral curve (QSC) solution of the spectral problem—a simple set of equations giving the full spectrum of the anomalous dimensions of the theory developed in [9,10]. Below we limit ourselves to the $sl(2)$ sector of the theory.

The simplest ingredient of the QSC is a set of four functions \mathbf{P}_a of the spectral parameter u which can be conveniently written as a convergent series expansion

$$\mathbf{P}_a(u) = \sum_{n=\tilde{M}_a}^{\infty} \frac{c_{a,n}}{x^n(u)}, \quad x(u) = \frac{u + \sqrt{u-2g}\sqrt{u+2g}}{2g}.$$

We see that \mathbf{P}_a has a branch cut and is powerlike at infinity. The constants \tilde{M}_a control the global charges of the state. For the case of the twist-2 operators, $\tilde{M}_a = \{2, 1, 0, -1\}$. The problem of solving the QSC consists in finding the coefficients $c_{a,n}$. They can be fixed in the following steps [19]:

First, find four linear independent analytic in the upper half plane solutions of the linear finite difference equation

$$\mathcal{Q}_{a|i} \left(u + \frac{i}{2} \right) - \mathcal{Q}_{a|i} \left(u - \frac{i}{2} \right) = -\mathbf{P}_a(u) \mathbf{P}^b(u) \mathcal{Q}_{b|i} \left(u + \frac{i}{2} \right), \quad (5)$$

where i labels the four solutions. Here and everywhere in this Letter indices are raised with a 4×4 matrix $\chi^{ab} = (-1)^a \delta_{a,5-b}$. The solutions can be always chosen to have “pure” asymptotics, which means that with exponential precision the large u (asymptotic) expansion of $\mathcal{Q}_{a|i}$ has the form

$$\mathcal{Q}_{a|i} \simeq u^{\tilde{M}_i - \tilde{M}_a} \sum_{n=0}^{\infty} \frac{A_{a,i,n}}{u^n}. \quad (6)$$

In the generic situation, it is always possible to choose the four solutions of that equation in this form. In what follows, we assume this to be done. We require in addition that \tilde{M}_i encode the conformal charges of the operators, i.e., $2\tilde{M}_i = \{\Delta - S + 2, \Delta + S, -\Delta - S + 2, S - \Delta\}$, where Δ is the dimension of the operator in question. This requirement fixes some of the leading coefficients $c_{a,0}$. There is an obvious rescaling freedom of $\mathcal{Q}_{a|i}$ which can be partially fixed by requiring that

$$\mathcal{Q}_{a|i} \mathcal{Q}^{aj} = -\delta^j_i. \quad (7)$$

Next, one finds four Q functions by dividing either side of Eq. (5) by $\mathbf{P}_a(u)$

$$\mathbf{Q}_i(u) = -\mathbf{P}^b(u) \mathcal{Q}_{b|i}(u + i/2). \quad (8)$$

They are now given implicitly in terms of the coefficients $c_{a,n}$. The main constraint comes from the condition that the analytic continuation, which we denote as $\tilde{\mathbf{Q}}_i$ is a linear combination with periodic coefficients of \mathbf{Q}_i themselves

$$\tilde{\mathbf{Q}}_i = \omega_{ij} \mathbf{Q}^j = -\tilde{\mathbf{P}}^b(u) \mathcal{Q}_{b|i}(u + i/2), \quad (9)$$

where $\tilde{\mathbf{P}}^b(u)$ is the same as $\mathbf{P}^b(u)$ with x replaced by $1/x$. In particular, from Eq. (9), we have

$$\tilde{\mathbf{Q}}_1(u) = \omega_{12} \mathbf{Q}_3(u) + \omega_{14} \mathbf{Q}_1(u) - \omega_{13} \mathbf{Q}_2(u), \\ \tilde{\mathbf{Q}}_3(u) = \omega_{34} \mathbf{Q}_1(u) - \omega_{14} \mathbf{Q}_3(u) + \omega_{13} \mathbf{Q}_4(u). \quad (10)$$

As we will see, ω_{ij} can be eliminated. To show this, we will need only to know that ω_{ij} is i -periodic, antisymmetric and should satisfy

$$\omega_{ij} \omega^{jk} = \delta_i^k, \quad \omega_{23} = \omega_{14}. \quad (11)$$

For all physical operators, ω_{ij} should go to a constant at large u ; however, it was emphasized, in particular in [19] and [20], that for noninteger S , one should allow for an exponential growth of ω_{24} as otherwise the system has no solution. Note that Eq. (11) implies that $\omega^{24} = \omega_{13}$ should decay exponentially at infinity. It is also known that ω_{14}

decays at infinity [19]. Condition (9) in fact imposes infinitely many constraints on the coefficients $c_{a,n}$ fixing them completely as well as the function $\Delta(S)$ or $S(\Delta)$.

Analytical data from QSC.—We describe now the details of our analytical method. We will focus on some particular points $\Delta_0 = 1, 3, 5, 7$. It can be seen already from the LO result [Eq. (2)] that the function $S(\Delta)$ is singular at these points; however, the coefficients of the expansion are relatively simple and are given by ζ functions. We will perform a double expansion first in g up to the order g^6 and then in $\delta = \Delta - \Delta_0$.

General iterative procedure for solving QSC We describe a procedure which for some given \mathbf{P}_a (or, equivalently, $c_{a,n}$) takes as an input some approximate solution of Eq. (5) $\mathcal{Q}_{a|i}^{(0)}$ valid up to the order ϵ^n (where ϵ is some small expansion parameter) and produces as an output new $\mathcal{Q}_{a|i}$ accurate to the order ϵ^{2n} . The method is very general and, in particular, is suitable for perturbative expansion around any background.

Let dS be the mismatch in the Eq. (5), *i.e.*,

$$\mathcal{Q}_{a|i}^{(0)}\left(u + \frac{i}{2}\right) - \mathcal{Q}_{a|i}^{(0)}\left(u - \frac{i}{2}\right) + \mathbf{P}_a \mathbf{P}^b \mathcal{Q}_{b|i}^{(0)}\left(u + \frac{i}{2}\right) = dS_{a|i}, \quad (12)$$

where $dS_{a|i}$ is small $\sim \epsilon^n$. We can always represent the exact solution in the form

$$\mathcal{Q}_{a|i}(u) = \mathcal{Q}_{a|i}^{(0)}(u) + b_i^j \left(u + \frac{i}{2}\right) \mathcal{Q}_{a|j}^{(0)}(u), \quad (13)$$

where the unknown functions b_i^j are also small. After plugging this ansatz into the Eq. (12), we get

$$-[b_i^j(u) - b_i^j(u+i)] \mathcal{Q}_{a|j}^{(0)} = dS_{a|i} + dS_{a|j} b_i^j. \quad (14)$$

Since b_i^j is small, it can be neglected in the r.h.s. where it multiplies another small quantity. Finally multiplying the equation by $\mathcal{Q}^{(0)a|k}$ and using Eq. (7) we arrive at

$$b_i^k(u+i) - b_i^k(u) = -dS_{a|i}(u) \mathcal{Q}^{(0)a|k} \left(u + \frac{i}{2}\right) + \mathcal{O}(\epsilon^{2n}).$$

We see that the rhs contains only the known functions dS and $\mathcal{Q}^{(0)}$ and does not contain b , which means that the original fourth order finite difference equation is reduced to a set of independent first order equations. In most interesting cases, the first order equation can be easily solved. After $\mathcal{Q}_{a|i}$ is found, one can use Eq. (8) to find \mathbf{Q}_i .

Iterations at weak coupling For our particular problem, we will take either $\epsilon = g$ or $\epsilon = \delta$. Applying this procedure a few times, we generate \mathbf{Q}_i for sufficiently high order both in g and in δ . Finally, by “gluing” \mathbf{Q}_i and \mathbf{Q}_i on the cut, we find $c_{a,n}$ and $S(\Delta)$ also as a double expansion.

For the above procedure, we need the leading order $\mathcal{Q}_{a,i}^{(0)}$. One can expect that to the leading order in g , the solution

should be very simple—indeed, the branch cuts collapse to a point making most of the functions polynomial or having very simple singular structure. Also, one can use that to the leading order in g functions \mathbf{P}_a are very simple and are already known from [12] for any Δ . By making a simple ansatz for \mathbf{Q}_i , we found for $\Delta_0 = 1$ to the leading order

$$\mathbf{Q}_1 \approx u, \quad \mathbf{Q}_2 \approx 1/u, \quad \mathbf{Q}_3 \approx 1, \quad \mathbf{Q}_4 \approx 1/u^2. \quad (15)$$

For $\Delta_0 = 3, 5, \dots$, the solution involves also the η functions introduced in the QSC context in [21,22]

$$\eta_{s_1, \dots, s_k}(u) = \sum_{n_1 > n_2 > \dots > n_k \geq 0} \frac{1}{(u + in_1)^{s_1} \dots (u + in_k)^{s_k}}, \quad (16)$$

which are related in a simple way to the nested harmonic sums. For $\Delta = 3$, we found

$$\begin{aligned} \mathbf{Q}_1 &\approx u^2, & \mathbf{Q}_2 &\approx u^2 \eta_3 + i - \frac{1}{2u}, \\ \mathbf{Q}_3 &\approx u^2 \eta_2 + iu - \frac{1}{2}, & \mathbf{Q}_4 &\approx u^2 \eta_4 + \frac{i}{u} - \frac{1}{2u^2}, \end{aligned} \quad (17)$$

which reflects the general structure of the expansion of \mathbf{Q}_i around integer Δ 's which contain only η_2, η_3 , and η_4 with polynomial coefficients. As it was explained in [21,22], the η functions are closed under all essential for us operations: the product of any two η functions can be written as a sum of η functions, and most importantly, one can easily solve equations of the type

$$f(u+i) - f(u) = u^n \eta_{s_1, \dots, s_k} \quad (18)$$

for any integer n again in terms of a sum of powers of u multiplying η functions (which we call η polynomials). For example for $n = -1$ and $k = 1, s_1 = 1$ we get $f = -\eta_2 - \eta_{1,1}$, etc. Thus, for these starting points, we are guaranteed to get η polynomials on each step of the general procedure described above.

Proceeding in this way, we computed \mathbf{Q}_i up to the order g^6 and δ^{10} for $\Delta = 3, 5, 7$. After that, we fix the coefficients in the ansatz for \mathbf{P}_a from analyticity requirements described below.

Fixing remaining freedom Here, we will describe how to use \mathbf{Q}_i found before to finally extract relation between S and Δ and the constants $c_{a,n}$. This is done by using a relation between \mathbf{Q}_i and their analytical continuations \mathbf{Q}_i . On the one hand, we have the relation Eq. (10). On the other hand, we can use the $u \rightarrow -u$ symmetry [23] of the twist-2 operators to notice that $\mathbf{Q}_i(-u)$ should satisfy the same finite difference equation as $\mathbf{Q}_i(u)$ and thus, we should have $\mathbf{Q}_i(u) = \Omega_i^j(u) \mathbf{Q}_j(-u)$, where $\Omega_i^j(u)$ is a set of periodic coefficients. As $\mathbf{Q}_i(u)$ has a powerlike behavior at infinity, $\Omega_i^j(u)$ should not grow faster than a constant. Furthermore, since \mathbf{Q}_i has a definite asymptotic [Eq. (6)], only diagonal elements of $\Omega_i^j(u)$ can be nonzero at infinity. Combining these relations, we find

$$\tilde{\mathbf{Q}}_A(u) = \alpha_A^i \mathbf{Q}_i(-u), \quad A = 1, 3, \quad (19)$$

where $\alpha_A^i = \omega_{Ai} \chi^{ik} \Omega_k^j$ are i periodic (as a combination of i -periodic functions), analytic [as both $\tilde{\mathbf{Q}}_a(u)$ and $\mathbf{Q}_a(-u)$ should be analytic in the lower-half-plane], and growing not faster than a constant at infinity which implies that they are constants. Furthermore, most of them are zero because only ω_{12} , ω_{34} , and Ω_i^i are nonzero at infinity. Thus, we simply get

$$\tilde{\mathbf{Q}}_1(u) = \alpha_{13} \mathbf{Q}_3(-u), \quad \tilde{\mathbf{Q}}_3(u) = \alpha_{31} \mathbf{Q}_1(-u). \quad (20)$$

Next, we note that if we analytically continue this relation and change $u \rightarrow -u$, we should get an inverse transformation which implies $\alpha_{13} = 1/\alpha_{31} \equiv \alpha$. The coefficient α depends on relative normalization of \mathbf{Q}_1 and \mathbf{Q}_3 . Let us see how to use the identity Eq. (20) to constrain the constants $c_{a,i}$. We observed that all the constants are fixed from the requirement of regularity at the origin of the combinations $\mathbf{Q}_1 + \tilde{\mathbf{Q}}_1$ and $(\mathbf{Q}_1 - \tilde{\mathbf{Q}}_1)/\sqrt{u^2 - 4g^2}$, which now can be written as

$$\begin{aligned} \mathbf{Q}_1(u) + \alpha \mathbf{Q}_3(-u) &= \text{regular}, \\ \frac{\mathbf{Q}_1(u) - \alpha \mathbf{Q}_3(-u)}{\sqrt{u^2 - 4g^2}} &= \text{regular}. \end{aligned}$$

This relation is used in the following way: one first expands in g the lhs and then in u around the origin. Then requiring the absence of the negative powers will fix α , all the coefficients $c_{a,n}$, and the function $\Delta(S)$. So we can completely ignore ω_{ij} , \mathbf{Q}_2 , and \mathbf{Q}_4 in this calculation. This observation can be used in more general situations and allows avoiding construction of ω_{ij} , and, in particular, can simplify the numerical algorithm of [19] considerably.

Constraints from poles We use the procedure described above to compute the expansion of $S(\Delta)$ around

$\Delta_0 = 3, 5, 7$. In particular, for $\Delta = 5 + \epsilon$, we computed the first eight terms

$$\begin{aligned} \chi^{\text{NNLO}} &= -\frac{1024}{\epsilon^5} + \frac{64(4\pi^2 - 33)}{3\epsilon^3} + \frac{16(-36\zeta_3 + 2\pi^2 + 31)}{\epsilon^2} \\ &+ \frac{-288\zeta_3 + \frac{232\pi^4}{45} - 16\pi^2 - 296}{\epsilon} \\ &- \frac{2}{15} [20(4\pi^2 - 75)\zeta_3 + 6300\zeta_5 \\ &+ \pi^4 - 215\pi^2 + 285] + \dots \end{aligned} \quad (21)$$

The terms with ϵ , ϵ^2 , and ϵ^3 , which we also evaluated explicitly, are omitted for the sake of brevity. We also reproduced expansions extracted from [24] for $\Delta = 1 + \epsilon$. In our calculations, we used several Mathematica packages for manipulating harmonic sums and multiple zeta values [25].

The result.—By inspecting Eq. (4) for LO and NLO, we notice that the transcendentality of these expressions is uniform if one assigns to S_{a_1, \dots, a_k} transcendentality equal to $\sum_{j=1}^k |a_j|$. The principal assumption of our calculation states that $F_3(x)$ can also be written as a linear combination of nested harmonic sums with coefficients made out of several transcendental constants π^2 , $\log(2)$, ζ_3 , ζ_5 , $\text{Li}_4(1/2)$, $\text{Li}_5(1/2)$ of uniform transcendentality 5. The final basis obtained after taking into account the constants contains 288 elements.

Hence, we build the linear combination of these basis elements with free coefficients and constrained them by imposing the expansion at $\Delta = 1, 3, 5, 7$ to match the results of the analytic expansion of QSC [in particular, requiring Eq. (21)]. This gave an overdefined system of linear equations for the unknown coefficients which happen to have a unique solution presented below:

$$\begin{aligned} \frac{F_3(x)}{256} &= -\frac{5S_{-5}}{8} - \frac{S_{-4,1}}{2} + \frac{S_1 S_{-3,1}}{2} + \frac{S_{-3,2}}{2} - \frac{5S_2 S_{-2,1}}{4} + \frac{S_{-4} S_1}{4} + \frac{S_{-3} S_2}{8} + \frac{3S_{3,-2}}{4} - \frac{3S_{-3,1,1}}{2} - S_1 S_{-2,1,1} \\ &+ S_{2,-2,1} + 3S_{-2,1,1,1} - \frac{3S_{-2} S_3}{4} - \frac{S_5}{8} + \frac{S_{-2} S_1 S_2}{4} + \pi^2 \left[\frac{S_{-2,1}}{8} - \frac{7S_{-3}}{48} - \frac{S_{-2} S_1}{12} + \frac{S_1 S_2}{48} \right] - \pi^4 \left[\frac{2S_{-1}}{45} - \frac{S_1}{96} \right] \\ &+ \zeta_3 \left[-\frac{7S_{-1,1}}{4} + \frac{7S_{-2}}{8} + \frac{7S_{-1} S_1}{4} - \frac{S_2}{16} \right] + \left[2\text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^2 \log^2 2}{12} + \frac{\log^4 2}{12} \right] (S_{-1} - S_1) \\ &+ \frac{\log^5 2}{60} - \frac{\pi^2 \log^3 2}{36} - \frac{2\pi^4 \log 2}{45} - \frac{\pi^2 \zeta_3}{24} + \frac{49\zeta_5}{32} - 2\text{Li}_5\left(\frac{1}{2}\right). \end{aligned} \quad (22)$$

The simplicity of the final result is quite astonishing: only 37 coefficients out of 288 turned out to be nonzero. Furthermore, they are significantly simpler than the coefficients appearing in the series expansion around the poles [Eq. (21)]. These are all clear and expected indications of the correct result similar to what was observed in the usual perturbation theory [26]. In addition, we also performed the numerical test described below.

Numerical tests.—Using the method of [19], we evaluated 40 values of spin S for various values of the coupling g in the range (0.01, 0.025) with exceptionally high 80-digit precision and then fit this data to get the following prediction (see Table I) for the N^{th} LO BFKL coefficients at the fixed value of $\Delta = 0.45$:

We found that our result Eq. (22) reproduces perfectly the first line in the table within the numerical error 10^{-61} , which leaves no room for doubt in the validity of our result.

TABLE I. Numerical predictions for the N^{th} LO BFKL coefficients at $\Delta = 0.45$.

	Value	Error
$N^2\text{LO}$	10774.635818847176637957593127192456995929170948057653783424533229	10^{-61}
$N^3\text{LO}$	-366393.2052053917038937903507478544549935531959333919163403836	10^{-56}
$N^4\text{LO}$	$1.332736355681126915694044310369828561521940588979476878854 \times 10^7$	10^{-51}
$N^5\text{LO}$	$-4.921740136657916500913955552075070060721450958436559876 \times 10^8$	10^{-47}

Summary.—In this Letter, we have applied the quantum spectral curve method [9,10] to the calculation of the NNLO correction to the BFKL eigenvalue. We check our result numerically with high precision using the algorithm developed in [19] and gave numerical predictions for a few next orders. We also developed a general efficient analytic method suitable for systematic perturbative solution of QSC.

There are numerous packages such as [25] available for the evaluation of the nested harmonic sums. Yet to simplify future applications of our results, we attached a small Mathematica notebook [18,27], which allows one to evaluate our result numerically and also generate expansions about singularities and at infinity.

We hope that our findings could shed some light on the QCD counterpart of our result and resolve some mysteries shrouding the BFKL physics.

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- [1] L. N. Lipatov, *Yad. Fiz.* **23**, 642 (1976) [*Sov. J. Nucl. Phys.* **23**, 338 (1976)]; E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Zh. Eksp. Teor. Fiz.* **72**, 377 (1977) [*Sov. Phys. JETP* **45**, 199 (1977)]; I. I. Balitsky and L. N. Lipatov, *Yad. Fiz.* **28**, 1597 (1978) [*Sov. J. Nucl. Phys.* **28**, 822 (1978)].
- [2] T. Jaroszewicz, *Phys. Lett. B* **116**, 291 (1982).
- [3] L. N. Lipatov, *Zh. Eksp. Teor. Fiz.* **90**, 1536 (1986) [*Sov. Phys. JETP* **63**, 904 (1986)].
- [4] At least in the maximally supersymmetric $\mathcal{N} = 4$ SYM theory.
- [5] V. S. Fadin and L. N. Lipatov, *Phys. Lett. B* **429**, 127 (1998).
- [6] M. Ciafaloni and G. Camici, *Phys. Lett. B* **430**, 349 (1998).
- [7] A. V. Kotikov and L. N. Lipatov, *Nucl. Phys.* **B582**, 19 (2000).
- [8] A. V. Kotikov and L. N. Lipatov, *Nucl. Phys.* **B661**, 19 (2003); **B685**, 405(E) (2004).
- [9] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Phys. Rev. Lett.* **112**, 011602 (2014).
- [10] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *J. High Energy Phys.* **09** (2015) 187.
- [11] R. C. Brower, J. Polchinski, M. J. Strassler, and C. I. Tan, *J. High Energy Phys.* **12** (2007) 005.
- [12] M. Alfimov, N. Gromov, and V. Kazakov, *J. High Energy Phys.* **07** (2015) 164.
- [13] M. S. Costa, V. Goncalves, and J. Penedones, *J. High Energy Phys.* **12** (2012) 091.
- [14] We are grateful to S. Caron-Huot for bringing our attention to this paper.
- [15] A. V. Kotikov and V. N. Velizhanin, [arXiv:hep-ph/0501274](https://arxiv.org/abs/hep-ph/0501274).
- [16] D. I. Kazakov and A. V. Kotikov, *Nucl. Phys.* **B307**, 721 (1988); **B345**, 299 (1990); C. Lopez and F. J. Yndurain, *Nucl. Phys.* **B183**, 157 (1981); A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, and V. N. Velizhanin, *Phys. Lett. B* **595**, 521 (2004); **632**, 754 (2006).
- [17] J. Blumlein, *Comput. Phys. Commun.* **180**, 2218 (2009).
- [18] Ancillary Mathematica files attached to the present Letter.
- [19] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, [arXiv:1504.06640](https://arxiv.org/abs/1504.06640).
- [20] N. Gromov, F. Levkovich-Maslyuk, G. Sizov, and S. Valatka, *J. High Energy Phys.* **07** (2014) 156.
- [21] C. Marboe and D. Volin, *Nucl. Phys.* **B899**, 810 (2015).
- [22] S. Leurent and D. Volin, *Nucl. Phys.* **B875**, 757 (2013).
- [23] More generally, one can also use complex conjugation symmetry.
- [24] C. Marboe, V. Velizhanin, and D. Volin, *J. High Energy Phys.* **07** (2015) 084.
- [25] D. Maitre, *Comput. Phys. Commun.* **174**, 222 (2006); D. Maitre, *Comput. Phys. Commun.* **183**, 846 (2012); J. Ablinger, Johannes Kepler University, Diploma Thesis, 2009, [arXiv:1011.1176](https://arxiv.org/abs/1011.1176); J. Ablinger, Johannes Kepler University, Ph.D. Thesis, 2012; J. Ablinger, J. Blümlein, and C. Schneider, *J. Math. Phys. (N.Y.)* **54**, 082301 (2013); J. Ablinger, J. Blümlein, and C. Schneider, *J. Math. Phys. (N.Y.)* **52**, 102301 (2011); J. Blümlein, *Comput. Phys. Commun.* **180**, 2218 (2009); E. Remiddi and J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **15**, 725 (2000); J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **14**, 2037 (1999).
- [26] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher, and V. N. Velizhanin, *J. Stat. Mech.* (2007) P10003.
- [27] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.115.251601> for several Mathematica notebooks which should be helpful for working with our NNLO result.
- [28] B. Basso, in *The Scattering Amplitudes and the Multi-Regge Limit Conference*, Madrid, Spain, 2014.
- [29] B. Basso, S. Caron-Huot, and A. Sever, *J. High Energy Phys.* **01** (2015) 027.
- [30] L. Lipatov, in *IGST-2014 Conference*, DESY, Hamburg, 2014.
- [31] A. Sever, in *IGST-2014 Conference*, DESY, Hamburg, 2014.