Anomalous Energetics and Dynamics of Moving Vortices

Leo Radzihovsky
Department of Physics and Center for Theory of Quantum Matter, University of Colorado, Boulder, Colorado 80309, USA
(Received 14 August 2015; published 8 December 2015)

Motivated by the general problem of *moving* topological defects in an otherwise ordered state and specifically, by the anomalous dynamics observed in vortex-antivortex annihilation and coarsening experiments in freely suspended smectic-C films, I study the deformation, energetics, and dynamics of moving vortices in an overdamped XY model and show that their properties are significantly and qualitatively modified by the motion.

DOI: 10.1103/PhysRevLett.115.247801 PACS numbers: 61.30.Jf, 05.40.-a

Introduction.—Topological defects play a central role in phase transitions, relaxation of generalized strain (e.g., current dissipation in a superfluid, strain relaxation in a crystalline solid, etc.) [1], and coarsening dynamics after a quench into an ordered state [2]. They appear in a broad range of physical realizations from superfluids and liquid crystals [3] to early universe baryogenesis [4].

Many physical systems involve topological defects moving (stochastically or deterministically) through an otherwise ordered medium. Although it is usually tacitly assumed that defect's properties (texture structure, interaction, dynamics, etc.) are not modified by its motion, with the center $\mathbf{r}_0$ simply boosted $\mathbf{r}_0 \to \mathbf{r}_0(t)$ by the motion, there is no *a priori* reason for this to be the case. Instead, not unlike a relativistic charged particle, a moving defect is defined and governed by the dynamics of the associated vector field, requiring a nontrivial analysis that is the subject of this Letter.

Stimulated by this general question, and by the anomalous vortex-antivortex annihilation and coarsening dynamics [2] observed in freely suspended smectic-C films experiments [5–7], I explored the nature of moving vortices in an overdamped two-dimensional (2D) XY model, applicable to a broad range of soft matter systems. In this Letter, I report the results of these studies. With some modifications these may also extend to vortices in a nonzero-temperature superfluid and superconductor in the presence of a background supercurrent or dislocations in a strained crystal.

Results.—Before turning to the analysis, I summarize the results of this study. I find that a vortex imposed to move with a constant velocity v in an ordered medium of stiffness K and damping γ, beyond a length scale

$$\xi_v = \frac{K}{\gamma v} \equiv D/v \equiv k_v^{-1} \tag{1}$$

exhibits a nontrivial longitudinal distortion of its standard, purely transverse form [3], the latter retained on the length scale below ξ_v. In the *steady state* the resulting deformed vortex exhibits a parabolic cometlike tail, extending across the system to which most of the 2π phase winding is confined (see Figs. 1 and 2). While a motion-induced distortion is not surprising, its long-range nature and *qualitative* consequences (see below) indeed are. For a *transient state* at time t after a vortex begins to move, the steady-state distortion only extends out to a time-dependent anisotropic "horizon" $vt \times \sqrt{Kt/\gamma}$, beyond which the purely transverse vortex field is nearly undistorted by the motion. This is analogous to the Lienard-Wiechert potential of a moving point charge [8]. All other predictions follow from this result. Specifically, a $\pm 2\pi$ vortex steady-state mobility

$$\mu_v \approx \frac{1}{\pi\gamma \ln(2\xi_v/a)} \sim 1/|\ln v| \tag{2}$$

vanishes logarithmically with vanishing velocity ξ_v cutting off the $\ln L/a$ divergence of a stationary vortex drag coefficient, a result that was previously found via scaling and numerical analysis in earlier studies [1,2,9,10] (a is the vortex core size). Thus, a 2D vortex exhibits a breakdown of a linear response to an external force f, with truly nonlinear velocity-force characteristics $v(f) \sim f/|\ln f|$.

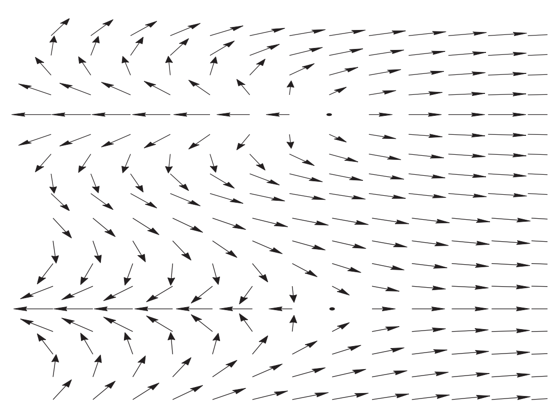

FIG. 1. A vector field corresponding to a phase $\theta(\mathbf{r})$ for a vortex-antivortex pair comoving to the right with velocity $\mathbf{v}$.

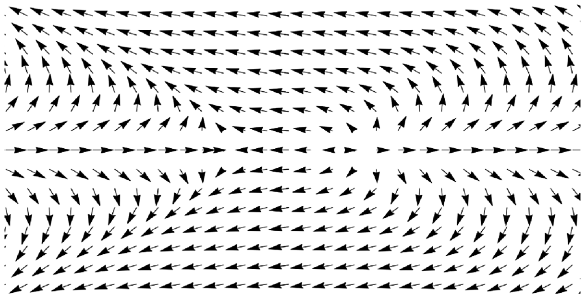

FIG. 2. A vortex-antivortex counterpropagating pair relevant to the annihilation problem. The motion-induced cometlike tails (that lead to a linearly diverging vortex elastic energy) and a suppression of the deformation between vortices (that leads to a weakened interaction) are clearly visible.

The "comet-tail" texture of a moving $\pm 2\pi$ vortex leads to an elastic energy that diverges linearly with system size

$$E_v \approx \pi K(L/\xi_v + \ln \xi_v/a), \tag{3}$$

with the usual logarithm cut off by the length $\xi_v \sim 1/v$ that diverges with a vanishing velocity. The interaction between two $\pm 2\pi$ moving vortices strongly depends on their velocities and orientation relative to the separation vector, $\mathbf{r}$. With the eye on the problems of a vortex-antivortex annihilation and nucleation by an imposed strain, I find the interaction potential $U^{\parallel}_{\mathbf{v},-\mathbf{v}}(r)$ for a vortex and antivortex moving toward each other $\mathbf{v}\|\mathbf{r}$ (Fig. 2)

$$\begin{aligned} U^{\parallel}_{\mathbf{v},-\mathbf{v}}(r) &= 2\pi K[c - \sinh^{-1}(\xi_v/r)], \\ &\approx 2\pi K \begin{cases} c - \xi_v/r, & a \ll \xi_v \ll r, \\ \ln r/a, & a \ll r \ll \xi_v, \end{cases} \end{aligned} \tag{4}$$

and a potential $U^{\perp}_{\mathbf{v},\mathbf{v}}(r)$ for a pair comoving with velocity $\mathbf{v}\perp\mathbf{r}$ (Fig. 1)

$$\begin{aligned} U^{\perp}_{\mathbf{v},\mathbf{v}}(r) &= 2\pi K\left[c - \sinh^{-1}(\xi_v/r) + \sqrt{r^2/\xi_v^2 + 1}\right], \\ &\approx 2\pi K \begin{cases} c + r/\xi_v, & a \ll \xi_v \ll r, \\ \ln r/a, & a \ll r \ll \xi_v, \end{cases} \end{aligned} \tag{5}$$

with $c = \ln \xi_v/a$. Thus, in the annihilation configuration (Fig. 2), vortex attraction for separation beyond ξ_v is suppressed by the motion. Conversely and even more dramatically, I predict that vortex-pair motion in the transverse configuration (Fig. 1) leads to a *linear confinement* on long length and time scales.

The above velocity-dependent vortex mobility and interaction qualitatively modify the equation of motion for the vortex-antivortex separation. This leads to a late-time slowed annihilation dynamics that may be an important ingredient in the anomalies observed in the experiments [6].

Analysis.—With the above motivation in mind, I now turn to the analysis of moving $\pm 2\pi$ vortices in a 2D overdamped XY model (in the isotropic approximation the generalization to higher vortex charge is straightforward)

$$\gamma\partial_t\theta = K\nabla^2\theta, \quad \text{with} \quad \nabla\times\nabla\theta = 2\pi\delta(\mathbf{r} - \mathbf{r}_v(t))\hat{\mathbf{z}}, \tag{6}$$

searching for a vortex solution $\theta(\mathbf{r},t)$ that for simplicity I take to be moving at constant velocity defined by $\mathbf{r}_v(t) = \mathbf{v}t$. Despite ignoring a number of ingredients [11], I expect it to be a core description of many systems where damping is dominant.

To this end, I take the solution to be $\theta(\mathbf{r},t) = \theta_v(\mathbf{r} - \mathbf{v}t) + \theta_s(\mathbf{r},t)$, where $\theta_v(\mathbf{r}) = \varphi = \arctan(y/x)$ is the azimuthal polar angle that is the standard purely transverse solution of the static problem ($\gamma = 0$) that enforces a moving unit of vorticity. The $\theta_s(\mathbf{r},t)$ part is a nonsingular, single-valued function (with a purely longitudinal, curl-free gradient) determined by the requirement that $\theta(\mathbf{r},t)$ satisfies the equation of motion (6). Thus, $\theta_s(\mathbf{r},t)$ describes the distortion of a moving vortex about the stationary form $\theta(\mathbf{r}) = \varphi$, with its spatial Fourier transform satisfying

$$\gamma\partial_t\theta_s(\mathbf{k}) + Kk^2\theta_s(\mathbf{k}) = \gamma\mathbf{v}\cdot\nabla\theta_v(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{v}t}. \tag{7}$$

The exact solution is easily found either directly for $\theta_s(\mathbf{r},t)$ or by first Galilean transforming to the moving vortex frame $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, $\partial_t \to \partial_t + \mathbf{v}\cdot\nabla_{\mathbf{r}'}$, where the distortion is $\theta_s{}'(\mathbf{r}',t) \equiv \theta_s(\mathbf{r}' + \mathbf{v}t, t)$.

For a vortex that has been moving forever, the Fourier transform of the steady-state distortion (vanishing for $v = 0$) is given by

$$\theta_s{}'(\mathbf{k}) = \frac{\gamma\mathbf{v}\cdot\nabla\theta_v(\mathbf{k})}{Kk^2 - i\gamma\mathbf{v}\cdot\mathbf{k}} = \frac{-2\pi i\mathbf{k}_v\cdot\hat{\mathbf{z}}\times\mathbf{k}}{k^2(k^2 - i\mathbf{k}_v\cdot\mathbf{k})}, \tag{8}$$

where $\mathbf{k}_v \equiv \gamma\mathbf{v}/K$. This leads to the "elastic" energy spectrum, $|\nabla\theta(\mathbf{k})|^2 = 4\pi^2(k^2 + k_v^2)/[(\mathbf{k}_v\cdot\mathbf{k})^2 + k^4]$, which, on length scales beyond ξ_v ($k \ll k_v$), is highly anisotropic, akin to that of a smectic liquid crystal. On shorter length scales it reduces to that of an isotropic stationary (undistorted) vortex [3].

In real space the steady-state distortion for a 2π vortex moving along the x axis in the vortex frame is given by

$$\begin{aligned} \theta_s'(\mathbf{r}) \approx &- \int_0^\infty \frac{dq}{q}\frac{e^{-q|\hat{x}|}\sin q\hat{y}}{q+1} \\ &- 2\Theta(-\hat{x})\int_0^\infty \frac{dq}{q}\frac{e^{-q|\hat{x}|}\sin q\hat{y}}{q^2 - 1}[1 - e^{-q(q-1)|\hat{x}|}], \end{aligned} \tag{9}$$

where $\hat{x} = x/\xi_v$, $\hat{y} = y/\xi_v$, and $\Theta(x)$ is the Heaviside step function. Evaluating above integrals numerically and adding the singular part of the vortex, $\theta_v(\mathbf{r}) = \varphi$, gives the real-space vector fields illustrated in Figs. 1 and 2.

A *transient-state* field of a vortex that has been moving for time t (particularly relevant for the annihilation problem) can also be computed exactly and is given by

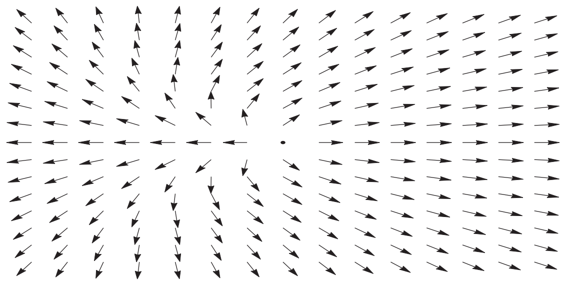

FIG. 3. A transient vector field of a vortex moving for time t, exhibiting a steady-state distortion out to an elliptical "horizon" $vt \times \sqrt{Kt/\gamma}$ and purely transverse vortex field beyond.

$$\theta'_s(\mathbf{r}, t) = \int_{\mathbf{k}} \mathbf{k}_v \cdot \nabla\theta_v(\mathbf{k}) \frac{1 - e^{-\frac{K}{\gamma}(k^2 - i\mathbf{k}_v\cdot\mathbf{k})t}}{k^2 - i\mathbf{k}_v \cdot \mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (10)$$

Its key generic features are controlled by three length scales ξ_v, $\xi_\perp \equiv \sqrt{Kt/\gamma}$, and $\xi_\parallel = vt$. At time $t > t_* \equiv \xi_v/v$, such that $\xi_v \ll \xi_\perp \ll \xi_\parallel$, one can see from the solution, Eq. (10), that on scales shorter than an anisotropic domain $\xi_\parallel \times \xi_\perp$, the solution reduces to the comet-tail steady-state one, Eq. (9) (Figs. 1 and 2). On longer scales the vortex distortion reduces to $\theta'_s(\mathbf{r}, t) \approx \mathbf{v}t \cdot \nabla\theta_v(\mathbf{r}')$, which when combined with the singular part gives

$$\theta'(\mathbf{r}', t) \approx \theta_v(\mathbf{r}') + \mathbf{v}t \cdot \nabla\theta_v(\mathbf{r}') \approx \theta_v(\mathbf{r}' + \mathbf{v}_0 t) = \varphi. \quad (11)$$

Thus, on scales outside of the $vt \times \sqrt{Kt/\gamma}$ domain the vortex field reduces to that of an undistorted stationary vortex $\theta(\mathbf{r}, t) = \varphi$ at its initial, $t = 0$ position (see Fig. 3). This is a diffusive vortex analog of a "causal horizon" beyond which the distortion associated with a moving vortex had not had sufficient time to propagate out. Other results (e.g., a vanishing vortex mobility, vortex energy, and interaction between moving vortices) follow directly from the above moving vortex solution.

Vortex mobility.—In the steady state the power input by the external force F to drive the vortex at velocity v is balanced by the rotational power dissipated; $P_{\mathrm{rot}} = \int_{\mathbf{r}} (\partial_t\theta)(K\nabla^2\theta) = \int_{\mathbf{r}} \gamma(\partial_t\theta)^2 = \gamma v^2 \int_{\mathbf{r}} (\partial_x\theta)^2$ gives the vortex drag coefficient $\gamma_v \equiv \mu^{-1}$ (inverse mobility) [1,9,10]

$$\gamma_v = \gamma \int_0^{a^{-1}} dkk \int_0^{2\pi} d\theta \frac{\sin^2\theta}{k_v^2 \cos^2\theta + k^2},$$
$$= \pi\gamma \sinh^{-1}\left(\frac{1}{k_v a}\right) \approx \pi\gamma \ln(2\xi_v/a) \sim \gamma \ln v. \quad (12)$$

Thus, at finite velocity, a previously noted divergence with system size L or vortex separation r [2,5,7] is cut off by the velocity length $\xi_v \sim 1/v$, thereby displaying nonlinear velocity-force characteristics, i.e., an absence of linear response down to a vanishing force.

Vortex energy.—It is of interest to calculate the elastic energy $E_v = (K/2) \int d^2r |\nabla\theta|^2$ stored in a moving vortex. In steady state, using Eq. (8) I find

$$E_v = \pi K \left[\sqrt{(L/\xi_v)^2 + 1} + \ln(\xi_v/a) - \sinh^{-1}(\xi_v/L)\right],$$

which, for vanishing velocity $L \ll \xi_v$ reduces to $\ln L/a$ of a stationary vortex, but for a rapidly moving vortex $L \gg \xi_v$ gives the energy, Eq. (3), that diverges *linearly* with L and with the standard logarithm cut off by the velocity length, ξ_v. This later result is due to the confinement of the elastic distortion (that in a stationary vortex is uniformly azimuthally distributed) to a comet-tail wake of a moving vortex.

Vortex interaction.—To further characterize the nature of moving vortices I study vortex-antivortex interaction, which strongly depends on their velocities and orientation relative to the initial separation vector, $\mathbf{r}_\parallel = \mathbf{r}_+ - \mathbf{r}_-$.

Motivated by the vortex-pair *annihilation* dynamics, I first compute the energy $E_{\mathbf{v},-\mathbf{v}}(r_\parallel) = (K/2) \int d^2r |\nabla\theta_{\mathbf{v},-\mathbf{v}}|^2$ of a vortex-antivortex pair moving toward each other with velocity $\pm\mathbf{v} = \pm v\hat{\mathbf{r}}$ along the separation vector, $\mathbf{r}_\parallel$. In steady state the solution is given by $\theta_{\mathbf{v},-\mathbf{v}}(\mathbf{r}, t) = \theta_s^+(\mathbf{r} - \mathbf{r}_+ - \mathbf{v}t) + \theta_v^+(\mathbf{r} - \mathbf{r}_+ - \mathbf{v}t) + \theta_s^-(\mathbf{r} - \mathbf{r}_- + \mathbf{v}t) + \theta_v^-(\mathbf{r} - \mathbf{r}_- + \mathbf{v}t)$, with singular (v) and smooth (s) components for vortex (at $\mathbf{r}_+$) and antivortex (at $\mathbf{r}_-$), respectively. The corresponding elastic energy $E_{\mathbf{v},-\mathbf{v}}(r_\parallel) = K\int \frac{d^2k}{k^2}[1 - e^{i\mathbf{k}\cdot\mathbf{r}_\parallel(t)} + (k^2k_v^2 - (\mathbf{k}_v\cdot\mathbf{k})^2)(\frac{1}{k^4 + (\mathbf{k}_v\cdot\mathbf{k})^2} + \frac{e^{i\mathbf{k}\cdot\mathbf{r}_\parallel(t)}}{(k^2 - i\mathbf{k}_v\cdot\mathbf{k})^2})]$, is given by

$$E_{\mathbf{v},-\mathbf{v}}(r_\parallel) \approx 2\pi K\left[\frac{L}{\xi_v} + \ln\frac{\xi_v}{a} - \sinh^{-1}\frac{\xi_v}{r_\parallel}\right]$$
$$\approx 2\pi K \begin{cases} L/\xi_v + \ln(\xi_v/a) - \xi_v/r_\parallel, & a \ll \xi_v \ll r_\parallel, \\ L/\xi_v + \ln(r_\parallel/a), & a \ll r_\parallel \ll \xi_v, \end{cases} \quad (13)$$

where $\mathbf{r}_\parallel(t) = \mathbf{r}_+ - \mathbf{r}_- - 2\mathbf{v}t$ and above I evaluated the asymptotic $r_\parallel$ dependence using an approximate hard cutoff $\xi_v/r_\parallel$ on low $\mathbf{k}$. Even for coinciding vortex-antivortex positions a linear in system size contribution L/ξ_v remains due to elastic energy associated with the comet tail of each moving vortex (see Fig. 2). Subtracting this constant self-energy piece I obtain the vortex-antivortex interaction $U^\parallel_{\mathbf{v},-\mathbf{v}}(r_\parallel)$ advertised in Eq. (4), which is qualitatively weaker and of a shorter range, falling off as $1/r_\parallel$ at large separations, $r_\parallel \gg \xi_v$.

Before moving on, I stress that a full vortex annihilation problem is far richer, requiring the analysis of a full transient dynamics as vortices accelerate from rest, with their velocity length $\xi_v(t)$ evolving nontrivially and tails limited by the causal horizon, growing with t from below to beyond their separation, $r_\parallel(t)$. Consequently, the nature of the interaction $U^\parallel_{\mathbf{v},-\mathbf{v}}(r_\parallel, \xi_v)$ is nontrivially velocity dependent. I analyze the associated dynamics of $\mathbf{r}(t)$ below.

Another geometry of interest is a comoving vortex-antivortex pair (see Fig. 1), with the velocity $\mathbf{v}$ perpendicular to the separation vector, $\mathbf{r}_\perp = \mathbf{r}_+ - \mathbf{r}_-$.

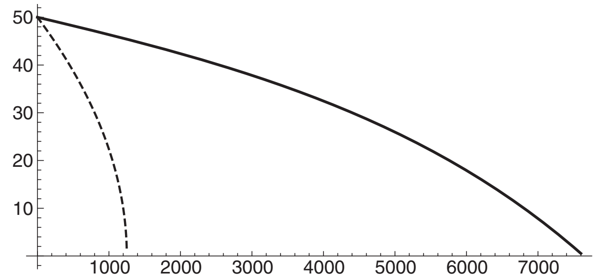

FIG. 4. Vortex-antivortex separation $r(t)$ as a function of time, a solution of Eq. (15) (solid curve) is significantly slowed down compared to the "naive" dynamics $\gamma_v dr/dt = -K/r$ (dashed curve).

In the steady state, the solution $\theta_{\mathbf{v},\mathbf{v}}(\mathbf{r},t) = \theta_s^+(\mathbf{r}-\mathbf{r}_+-\mathbf{v}t) + \theta_v^+(\mathbf{r}-\mathbf{r}_+-\mathbf{v}t) + \theta_s^-(\mathbf{r}-\mathbf{r}_--\mathbf{v}t) + \theta_v^-(\mathbf{r}-\mathbf{r}_--\mathbf{v}t)$ leads to the elastic energy $E_{\mathbf{v},\mathbf{v}}(r_\perp) = \frac{K}{2}\int d^2r|\nabla\theta_{\mathbf{v},\mathbf{v}}|^2 = K\int d^2k \frac{k^2+k_v^2}{k^4+(\mathbf{k}_v\cdot\mathbf{k})^2}(1-e^{i\mathbf{k}\cdot\mathbf{r}_\perp})$ given by

$$E_{\mathbf{v},\mathbf{v}}(r_\perp) \approx 2\pi K\left[\ln(\xi_v/a) - \sinh^{-1}(\xi_v/r_\perp) + \sqrt{r_\perp^2/\xi_v^2+1}\right] \approx 2\pi K\begin{cases} r_\perp/\xi_v + \ln(\xi_v/a), & a \ll \xi_v \ll r_\perp, \\ \ln(r_\perp/a), & a \ll r_\perp \ll \xi_v, \end{cases} \tag{14}$$

evaluated in the same hard cutoff approximation as in Eq. (13), and giving $U^\perp_{\mathbf{v},\mathbf{v}}(r_\perp)$ advertised in Eq. (5). This is a striking result as it predicts for $r_\perp > \xi_v$ a *linear confinement* of a moving vortex-antivortex pair, replacing logarithmic potential for a stationary pair. As is clear from Fig. 1 this elastic energy is associated with the $r_\perp$ length of the nonoverlapping parts of the "comet" tails, and the rest, beyond the $r_\perp$ parts, canceling between the comoving vortex and antivortex.

Vortex-antivortex annihilation dynamics, approximately described [neglecting [11] transients in Eq. (10)] by $\gamma_v dr/dt = -\partial U_{-\mathbf{v},\mathbf{v}}(r,v)/\partial r = -(2\pi K/r)(1/\sqrt{r^2/\xi_v^2+1})$,

$$\dot{\hat{r}}\ln(|\dot{\hat{r}}|/2) = -\frac{1}{\hat{r}}\frac{1}{\sqrt{\hat{r}^2\dot{\hat{r}}^2+1}} \tag{15}$$

is significantly enriched [11] by the velocity-dependent mobility, Eq. (2) and interaction, Eq. (4), as compared to the naive dynamics $\gamma dr/dt = -K/r$ that predicts a vortex separation $r(t) = \sqrt{r_0^2 - (2K/\gamma)t}$, initially separated by r_0, annihilating in time $t_0 = r_0^2\gamma/(2K)$ [6]. Above $\hat{r}$ and $\hat{t}$ are, respectively, measured in the microscopic units of a and $t_a = a^2\gamma/(2K)$. Equation (15) predicts in units of $v_a = a/t_0$ that $\hat{r}\,\hat{v} = (1/\sqrt{2})[\sqrt{1+4/\ln^2(\hat{v}/2)} - 1]^{1/2}$ (rather than $\hat{r}\,\hat{v} =$ const of the naive dynamics) and can be solved numerically, with the result illustrated in Fig. 4. It shows a significant modification and slowing of the dynamics by the effects studied here.

Beyond the transient time ξ_v/v, the enriched dynamics is expected only in the regime of large separation and high velocity $rv \gg av_a = K/\gamma$, corresponding to $r \gg \xi_v$. Using $K/\gamma \approx 10^{-5}$ cm^2/ sec and $v = 1$ μm/ sec, I estimate $\xi_v \approx 1$ mm and $v_a \approx 1$ mm/ sec for $a \approx 1$ μm, a limited regime of the current experiment's [6] applicability. Also, above prediction for the product rv decreasing with r is inconsistent with measurements [6]. Thus, I conclude that in current vortex annihilation experiments, the high velocity effects studied here are not sufficient to account for the observed anomalies [6] and other effects [11] may need to be considered. Further systematic experiments on moving vortices would be highly desirable to sort out various contributions.

I also leave the extension of the present London limit analysis to a superfluid, beyond a linearized *XY* model treatment [13], incorporating the full Galilean invariance [14] for a future study.

I thank Noel Clark for sharing his vortex annihilation data prior to publication, and acknowledge him and Sandy Fetter, Bert Halperin, Dan Arovas, and John Toner for stimulating discussions. This research was supported by the NSF through DMR-1001240, MRSEC DMR-1420736, and by the Simons Investigator grant from the Simons Foundation.

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