From the Physics to the Computational Complexity of Multiboson Correlation Interference

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We demonstrate how the physics of multiboson correlation interference leads to the computational complexity of linear optical interferometers based on correlation measurements in the degrees of freedom of the input bosons. In particular, we address the task of multiboson correlation sampling (MBCS) from the probability distribution associated with polarization- and time-resolved detections at the output of random linear optical networks. We show that the MBCS problem is fundamentally hard to solve classically even for nonidentical input photons, regardless of the color of the photons, making it also very appealing from an experimental point of view. These results fully manifest the quantum computational supremacy inherent to the fundamental nature of quantum interference.

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Motivation.—The interference of multiple bosons based on high-order correlation measurements [1–3] in a linear network is a phenomenon that is fundamental in atomic, molecular, and optical physics. The richness of its features gives rise to a wide variety of applications in quantum information processing [1,4,5], quantum metrology [6–8], and imaging [9]. Already correlated detections of two bosons after the interaction with a balanced beam splitter reveal an interference effect of truly quantum mechanical origin [10–13]: both particles always end up in the same output port due to the destructive interference of the two-boson quantum paths in which the bosons are either both reflected or both transmitted.

Going to higher-order correlation measurements in optical networks of large dimensions, multiboson interference becomes increasingly complex, promising a computational power that is not achievable classically [14,15]. Multiphoton correlation experiments with more than two photons have already been performed [16–26], providing an important milestone towards experiments of higher orders [27,28].

These experiments are usually based on joint measurements at the interferometer output ports "classically" averaging over the photons' degrees of freedom (e.g., time, polarization). In this context, Aaronson and Arkhipov argued the computational hardness of multiboson interference in linear optics for identical bosons by introducing the well-known boson sampling problem [14]. Does this computational hardness also occur for nonidentical photons? While the computational complexity for partially distinguishable photons is still not known [15], it is clear that boson sampling becomes computationally trivial for fully distinguishable photons when the information about the detection times and polarizations is completely ignored.

However, recent technological advances have enabled experimentalists to produce arbitrarily polarized single photons with near arbitrary spectral and temporal properties [29–31] which can be "read out" by time- and polarization-resolving measurements [1,32–35] with extremely fast detectors [36]. This makes it possible to encode entire "quantum alphabets" in the degrees of freedom of multiple photons [37,38] and to retrieve the encoded information by correlation measurements in those degrees of freedom, representing a valuable tool in quantum information processing [1,39–52].

All these remarkable technological achievements now allow experimentalists to fully address the following fundamental questions about the interplay between the physics and the complexity of multiboson interference: How do the spectral distributions of N nonidentical photons determine the occurrence of N-photon interference events in time- and polarization-resolving correlated measurements? How and to what degree is this occurrence connected with computational complexity? Does computational hardness really disappear for input bosons that are completely distinguishable in their spectra? This Letter aims to answer all these important questions, from both a fundamental and an experimental point of view, demonstrating the inherent computational complexity of the physics of multiboson correlation interference even for nonidentical photons.

Multiboson correlation sampling (MBCS).—We consider N single photons prepared at the N input ports of a linear interferometer (see Fig. 1) with $2M \gg 2N$ ports. The interferometer unitary transformation $\mathcal{U}$ is chosen randomly according to the Haar measure and is implemented by using a polynomial number (in M) of passive linear optical elements [53]. The state of N single photons injected in a set $\mathcal{S}$ of N input ports $s \in \mathcal{S}$ is given by

$$|\mathcal{S}\rangle \coloneqq \bigotimes_{s\in\mathcal{S}} |1[\boldsymbol{\xi}_s]\rangle_s \bigotimes_{s\notin\mathcal{S}} |0\rangle_s,$$

with the single photon states

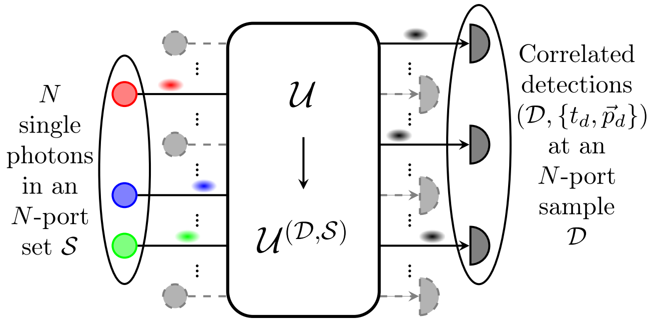

FIG. 1 (color online). General setup for multiboson correlation sampling. N single photons are injected into an N-port subset $\mathcal{S}$ of the $M \gg N$ input ports of a random linear interferometer. At the output of the interferometer, they are detected in one of the possible port samples $\mathcal{D}$ containing N of the M output ports at corresponding detection times and polarizations $\{t_d, \boldsymbol{p}_d\}_{d\in\mathcal{D}}$. For each output port sample $\mathcal{D}$ and given input configuration $\mathcal{S}$, the evolution through the interferometer is fully described by a $N \times N$ submatrix $\mathcal{U}^{(\mathcal{D},\mathcal{S})}$ of the $M \times M$ interferometer matrix $\mathcal{U}$.

$$|1[\boldsymbol{\xi}_s]\rangle_s \coloneqq \sum_{\lambda=1,2} \int_0^\infty d\omega [\boldsymbol{e}_\lambda \cdot \boldsymbol{\xi}_s(\omega)] \hat{a}^\dagger_{s,\lambda}(\omega)|0\rangle_s, \quad (1)$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is an arbitrary polarization basis and $\hat{a}^\dagger_{s,\lambda}(\omega)$ is the creation operator for the frequency mode ω and the polarization λ [54]. The complex spectral amplitude

$$\boldsymbol{\xi}_s(\omega) \coloneqq \boldsymbol{v}_s \xi_s(\omega - \omega_s) e^{i\omega t_{0s}} \quad (2)$$

is defined by the spectral shape $\xi_s(\omega - \omega_s) \in \mathbb{R}$ [centered around the central frequency (photon color) ω_s and with normalization $\int d\omega |\xi_s(\omega)|^2 = 1$], the polarization $\boldsymbol{v}_s$, and the time t_{0s} of emission of the photon injected in the port $s \in \mathcal{S}$. For simplicity, we consider input-photon spectra satisfying the narrow bandwidth approximation and a polarization-independent interferometric evolution with equal propagation time Δt for each possible path from an input source to a detector at the interferometer output.

Given such a multiboson interferometer and assuming identical photons, $\boldsymbol{\xi}_s = \boldsymbol{\xi}\ \forall s \in \mathcal{S}$, the boson sampling problem [14] was defined by Aaronson and Arkhipov as the task of sampling from the probability distribution over the output port samples $\mathcal{D}$, regardless of detection times and polarizations. We address here an interesting generalization of this famous problem by introducing the problem of multiboson correlation sampling (MBCS) [1,55]. The MBCS problem is defined as the task of sampling at the interferometer output from the probability distribution associated with time- and polarization-resolving correlation measurements. Each possible sample corresponds to an N-photon detection event at an N-port subset $\mathcal{D}$ of the M output ports at given times and polarizations $\{t_d, \boldsymbol{p_d}\}_{d\in\mathcal{D}}$, with $\boldsymbol{p}_d \in \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ [56]. The N-photon detection probability rate corresponding to a sample $(\mathcal{D}, \{t_d, \boldsymbol{p_d}\}_{d\in\mathcal{D}})$ depends [1] on both the $N \times N$ submatrix

$$\mathcal{U}^{(\mathcal{D},\mathcal{S})} \coloneqq [\mathcal{U}_{d,s}]_{\substack{d\in\mathcal{D}\\ s\in\mathcal{S}}}$$

of the $M \times M$ unitary matrix $\mathcal{U}$ describing the interferometer, and the Fourier transforms

$$\boldsymbol{\chi}_s(t) \coloneqq \mathcal{F}[\boldsymbol{\xi}_s](t - \Delta t) = \boldsymbol{v}_s \chi_s(t - t_{0s} - \Delta t) e^{i\omega_s(t - t_{0s} - \Delta t)} \quad (3)$$

of the single-photon spectra $\boldsymbol{\xi}_s(\omega)$ in Eq. (2) [with $\chi_s(t)$ being the Fourier transform of $\xi_s(\omega)$]. Defining the matrices

$$\mathcal{T}^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}} \coloneqq \{\mathcal{U}_{d,s}[\boldsymbol{p}_d \cdot \boldsymbol{\chi}_s(t_d)]\}_{\substack{d\in\mathcal{D}\\ s\in\mathcal{S}}}$$

and using the definition

$$\operatorname{perm} \mathcal{M} \coloneqq \sum_{\sigma\in\Sigma_N} \prod_{i=1}^{N} \mathcal{M}_{i,\sigma(i)}$$

of the permanent of a matrix $\mathcal{M}$, where the sum runs over all permutations σ in the symmetric group Σ_N, the probability rate of an N-fold detection event $(\mathcal{D}, \{t_d, \boldsymbol{p_d}\}_{d\in\mathcal{D}})$ is

$$G^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}} = \left|\operatorname{perm} \mathcal{T}^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}}\right|^2, \quad (4)$$

for ideal photodetectors.

By considering an integration time T_I short enough such that

$$\forall t_d\colon \chi_s(t - t_{0s} - \Delta t)\chi_{s'}(t - t_{0s'} - \Delta t) e^{i(\omega_s - \omega_{s'})t} \approx \text{const.}$$
$$\forall t \in [t_d - T_I, t_d + T_I], \quad \forall s, s' \in \mathcal{S}, \quad (5)$$

we obtain, for a detection sample $(\mathcal{D}, \{t_d, \boldsymbol{p}_d\}_{d\in\mathcal{D}})$, the probability

$$P^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}} = (2T_I)^N \left|\operatorname{perm} \mathcal{T}^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}}\right|^2 \quad (6)$$

of an N-fold detection in the time intervals $\{[t_d - T_I, t_d + T_I]\}_{d\in\mathcal{D}}$, where the detection time axes are discretized with step width $2T_I$.

We emphasize that, for each possible sample $(\mathcal{D}, \{t_d, \boldsymbol{p}_d\}_{d\in\mathcal{D}})$, the probability in Eq. (6) is at most exponentially small in N, as demonstrated in Theorem 1 in the Supplemental Material [57].

Exact MBCS.—Obviously, the complexity of sampling exactly from the probability distribution defined by Eq. (6) depends on the N-tuples $\{\boldsymbol{\xi}_s\}_{s\in\mathcal{S}}$ of single-photon input spectra in Eq. (2) [61].

With that in mind, in order to establish the complexity of exact MBCS, it is useful to define the N-photon interference matrix with elements

$$a(s,s') \coloneqq |\boldsymbol{v}_s \cdot \boldsymbol{v}_{s'}| \int_{-\infty}^{\infty} dt |\chi_s(t - t_{0s})||\chi_{s'}(t - t_{0s'})| \le 1, \quad (7)$$

with s, $s' \in \mathcal{S}$, depending on the pairwise overlaps of the absolute values of the temporal single-photon detection amplitudes [62] $\chi_s(t - t_{0s} - \Delta t)e^{i\omega_s(t - t_{0s} - \Delta t)}$ and of the polarizations $\boldsymbol{v}_s$ in Eq. (3). For nonvanishing elements

$$0 < a(s, s') \le 1 \quad \forall s, s' \in \mathcal{S}, \tag{8}$$

there exists a time interval T and at least a polarization $\boldsymbol{e}_{\bar{\lambda}} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, such that

$$\boldsymbol{e}_{\bar{\lambda}} \cdot \boldsymbol{\chi}_s(t_d) \neq 0 \quad \forall t_d \in T, \quad \forall s \in \mathcal{S}, \quad \forall d \in \mathcal{D}.$$

It is then ensured that for each detection sample $(\mathcal{D}, \{t_d, \boldsymbol{p}_d\}_{d\in\mathcal{D}})$, with $t_d \in T$, $\boldsymbol{p}_d = \boldsymbol{e}_{\bar{\lambda}}$ $\forall d \in \mathcal{D}$, the input photons are indistinguishable at the detectors: this leads to the interference of all possible $N!$ N-photon quantum paths manifested by the coherent superposition of all corresponding, nonvanishing $N!$ N-photon detection amplitudes in Eq. (4). Therefore, only the conditions Eqs. (5) and (8) for the nonidentical input spectra $\{\boldsymbol{\xi}_s\}_{s\in\mathcal{S}}$ in Eq. (2) are enough to ensure the occurrence of N-photon correlation interference events.

Even more interestingly, the same simple conditions lead to the computational hardness of the exact MBCS problem, establishing a connection between the occurrence of multiphoton correlation interference and complexity. Indeed, for approximately equal detection times $t_d \approx t \in T$ and equal polarizations $\boldsymbol{p}_d = \boldsymbol{e}_{\bar{\lambda}}$, $\forall d \in \mathcal{D}$, the multiphoton detection probabilities in Eq. (6) become

$$P^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}} = |\mathrm{perm}\,\mathcal{U}^{(\mathcal{D},\mathcal{S})}|^2 (2T_I)^N \prod_{s\in\mathcal{S}} |\boldsymbol{e}_{\bar{\lambda}} \cdot \boldsymbol{\chi}_s(t)|^2. \tag{9}$$

The interference of all N-photon quantum paths in Eq. (9) depends, apart from an overall factor, only on the permanent of a submatrix $\mathcal{U}^{(\mathcal{D},\mathcal{S})}$ of the interferometer random unitary matrix $\mathcal{U}$. For $N \ll M$, these matrices have elements given by approximately independent and identically distributed (i.i.d.) Gaussian random variables and the approximation of their respective permanents is a #P-hard task [14]. We emphasize that the presence of only an arbitrarily small fraction of samples with probabilities as in Eq. (9) would be enough to ensure the hardness of the exact MBCS. This can be shown analogously to the hardness proof of the original problem of exact boson sampling in Ref. [14]. Indeed, the ability to perform exact MBCS with a polynomial number of resources would imply that the task of approximating any given, fixed permanent associated with the probability distribution, Eq. (6), is in the complexity class $\mathrm{BPP}^{\mathrm{NP}}$. Since this would also include the task of approximating the #P-hard permanents emerging in Eq. (9), the polynomial hierarchy would collapse to the third level, which is strongly believed to be highly unlikely. We refer to Sec. II of the Supplemental Material for more details [57].

Interestingly, differently from the original boson sampling problem [14], the classical intractability of exact MBCS is not conditioned on input photons with approximately identical spectra $\boldsymbol{\xi}_s$ in Eq. (2). Only the simple conditions Eqs. (5) and (8) on the spectra are enough to guarantee its computational hardness.

Approximate MBCS.—Is approximate MBCS also not tractable with a classical computer? Such a question is obviously of fundamental importance from an experimental point of view, since it takes into account the inevitable experimental errors in an MBCS quantum interferometer which make only approximate sampling possible [61]. We consider, for simplicity, the case of an N-photon interference matrix in Eq. (7) with unit elements

$$a(s, s') \cong 1 \quad \forall s, s' \in \mathcal{S}. \tag{10}$$

This corresponds to two possible scenarios. Either all the input photons are completely identical or they differ only by their color, i.e., central frequency. In these cases the input photons have equal polarizations and are always indistinguishable at the detectors independently of the detection times and polarizations.

To simplify the expressions, we consider here polarization-insensitive detectors.

(i) Identical input photons: For approximately identical frequency spectra

$$\boldsymbol{\xi}_s(\omega) \cong \boldsymbol{\xi}(\omega) \quad \forall s \in \mathcal{S},$$

by using Eq. (6), the polarization-insensitive detection probability reads

$$\begin{aligned} P^{(\mathcal{D},\mathcal{S})}_{\{t_d\}} &\coloneqq \sum_{\{\boldsymbol{p}_d\}\in\{\boldsymbol{e}_1,\boldsymbol{e}_2\}^N} P^{(\mathcal{D},\mathcal{S})}_{\{t_d,\boldsymbol{p}_d\}} \\ &= |\mathrm{perm}\,\mathcal{U}^{(\mathcal{D},\mathcal{S})}|^2 (2T_I)^N \prod_{d\in\mathcal{D}} |\boldsymbol{\chi}(t_d)|^2, \end{aligned}$$

where we used the property $\sum_{\boldsymbol{p}_d=\boldsymbol{e}_1,\boldsymbol{e}_2} |\boldsymbol{p}_d \cdot \boldsymbol{v}| = |\boldsymbol{v}|^2 = 1$. Of course the only possible events occur within a detection-time interval where the function $|\boldsymbol{\chi}(t_d)| = |\mathcal{F}[\boldsymbol{\xi}](t_d - \Delta t)|$ is not negligible. Here, independently of the detection times $\{t_d\}_{d\in\mathcal{D}}$, all the probability rates associated with each possible sample $(\mathcal{D}, \{t_d\}_{d\in\mathcal{D}})$ are given, apart from a prefactor, by the permanents of $N \times N$ submatrices $\mathcal{U}^{(\mathcal{D},\mathcal{S})}$ of the interferometer transformation $\mathcal{U}$.

When the observer ignores the information about the detection times the approximate MBCS problem reduces to the well-known standard formulation of the approximate boson sampling problem, which Aaronson and Arkhipov argued to be intractable with a classical computer [14]. Therefore, the approximate MBCS problem is at least as complex as the original approximate boson sampling problem.

(ii) Photons of different colors: We now address the case of input photons in Eq. (1) with spectral distributions

$$\boldsymbol{\xi}_s(\omega) = \boldsymbol{v}\xi(\omega - \omega_s) e^{i\omega t_0},$$

with equal emission times $t_{0s} = t_0$ and equal polarizations $\boldsymbol{v}_s = \boldsymbol{v}$ but different colors ω_s. For simplicity, we consider spectral shapes

$$\xi(\omega) = \frac{1}{\sqrt{\pi\Delta\omega}} \mathrm{sinc}\left(\frac{\omega}{\Delta\omega}\right)$$

with equal bandwidths $\Delta\omega_s = \Delta\omega \lesssim |\omega_s - \omega_{s'}| \ \forall s, s'$, where $\mathrm{sinc}\, x \coloneqq \sin x / x$. The N-photon interference at the detectors is therefore characterized by the Fourier transforms

$$\boldsymbol{\chi}_s(t) = \boldsymbol{\nu}\sqrt{\frac{\Delta\omega}{2}} \mathrm{rect}\left(\frac{\Delta\omega(t - t_0 - \Delta t)}{2}\right) e^{i\omega_s(t - t_0 - \Delta t)},$$

with the rectangular function

$$\mathrm{rect}\, x \coloneqq \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2}. \\ 0 & \text{else} \end{cases}$$

Therefore, the condition Eq. (10) is satisfied, and the probability rates in Eq. (4) are nonvanishing only for detection times $t_d \in T \coloneqq [t_0 + \Delta t - 1/\Delta\omega, t_0 + \Delta t + 1/\Delta\omega] \ \forall d \in \mathcal{D}$. Moreover, Eq. (5) is fulfilled for integration times

$$T_I \ll |\omega_s - \omega_{s'}|^{-1} \quad \forall s, s' \in \mathcal{S}, \tag{11}$$

where $(T_I\Delta\omega)^{-1}$ defines the number of discrete steps of length $2T_I$ along the time interval T. As is known, detectors with such high time resolution cannot distinguish photons of different colors ω_s and multiphoton interference can be observed. Indeed, from Eq. (6), the polarization-independent detection probabilities are

$$P^{(\mathcal{D},\mathcal{S})}_{\{t_d\}} = (\Delta\omega T_I)^N |\mathrm{perm}\,([\mathcal{U}^{(\mathcal{D},\mathcal{S})}_{d,s} e^{i\omega_s t_d}]_{\substack{d\in\mathcal{D}\\ s\in\mathcal{S}}})|^2 \tag{12}$$

for all possible detection time intervals $[t_d - T_I, t_d + T_I] \subset T$. Such probabilities are proportional to permanents of matrices whose elements are the elements of $\mathcal{U}^{(\mathcal{D},\mathcal{S})}$ multiplied by the complex phases $\exp(i\omega_s t_d)$.

Since the elements $\mathcal{U}^{(\mathcal{D},\mathcal{S})}_{d,s}$ of the submatrices $\mathcal{U}^{(\mathcal{D},\mathcal{S})}$ are i.i.d. Gaussian random variables and the phase factors $e^{i\omega_s t_d}$ only rotate such elements in the complex plane, the entries of the matrices $[\mathcal{U}^{(\mathcal{D},\mathcal{S})}_{d,s} e^{i\omega_s t_d}]_{\substack{d\in\mathcal{D}\\ s\in\mathcal{S}}}$ are also i.i.d. Gaussian random variables as shown in Appendix C of the Supplemental Material [57]. Therefore, the probability distribution of the interferometer output interestingly depends, for all possible samples, on permanents whose approximation to within a multiplicative factor is a #P-hard problem [14]. Consequently, even for input photons of different colors, it is possible to show in analogy with Ref. [14] that approximate MBCS is of at least the same complexity as the standard boson sampling with identical photons [63]. As a "bonus," the number of possible samples $(\mathcal{D}, \{t_d\}_{d\in\mathcal{D}})$ is exponentially larger [by a factor $(T_I\Delta\omega)^{-N}$ with $T_I\Delta\omega \ll 1$ according to Eq. (11)] with respect to the standard boson sampling problem.

Does approximate MBCS retain its complexity even for photons which are completely pairwise distinguishable in their colors ω_s (i.e., $\omega_s - \omega_{s'} \gg \Delta\omega \ \forall s \neq s'$)? We first emphasize that, since these photons are characterized by a pairwise overlap

$$\int_0^\infty d\omega\, \boldsymbol{\xi}_s(\omega) \cdot \boldsymbol{\xi}_{s'}(\omega) \cong 0 \quad \forall s \neq s', \tag{13}$$

the approximate boson sampling problem is trivial [14]. Indeed, in this case, by averaging the rate in Eq. (4) over all possible detection times and polarizations, one finds that the boson sampling probability [1]

$$P^{(\mathcal{D},\mathcal{S})} = \mathrm{perm}\,[|\mathcal{U}^{(\mathcal{D},\mathcal{S})}_{d,s}|^2]_{\substack{d\in\mathcal{D}\\ s\in\mathcal{S}}},$$

for an output port sample $\mathcal{D}$, is given by the permanent of a non-negative matrix that can be approximated with a polynomial number of resources [64]. Consequently, one might guess that also the approximate MBCS is computationally trivial. Nonetheless, the complexity emerging from the result in Eq. (12) is independent of the colors ω_s of the input photons, demonstrating that also in this case approximate MBCS is classically intractable.

Two essential physical aspects are behind the demonstrated complexity of approximate MBCS: *all* possible detection-time events can be an outcome of the sampling experiment (none of the events is disregarded) and *all* these time samples arise from the interference of $N!$ multiphoton quantum paths. In conclusion, the physics of sampling among all possible N-photon interference events behind our proposal is at the heart of the complexity of approximate MBCS.

Discussion.—In this Letter, we demonstrated how and to what degree the occurrence of multiphoton interference in time- and polarization-resolving correlation measurements leads to computational hardness in linear optical interferometers.

The definition of an N-photon interference matrix $a(s, s')$ in Eq. (7) allowed us to formulate the simple sufficient condition Eq. (8) on the spectra of the input photons for the occurrence of N-photon interference, provided sufficiently small integration times [see Eq. (5)].

Remarkably, these two simple conditions are also sufficient to guarantee the complexity of exact MBCS. In contrast, the complexity of the original exact boson sampling problem has only been proven for identical input photons.

For approximate MBCS on the other hand, not only the existence of samples exhibiting full N-photon interference [guaranteed by Eq. (8)] is important but also their fraction with respect to the total number of samples. Interestingly, this is encoded in the magnitude of the entries $a(s, s')$ of the N-photon interference matrix in Eq. (7).

It was thus natural to consider the simple case of full overlap of the modulus of the single-photon detection amplitudes $[a(s, s') = 1]$ where all possible detection events correspond to N-photon interference samples [65]. In this case, corresponding to identical input photons or photons with arbitrary colors, approximate MBCS is at least of the same complexity as boson sampling with identical photons.

This is particularly interesting if the differences in the central frequencies are much larger than the width of the single photons' spectral shapes, corresponding to fully distinguishable photons in the sense of Eq. (13). While approximate boson sampling becomes trivial in this case [14], approximate MBCS is at least as complex as when perfectly identical input photons are used.

Since detectors with high temporal resolution (<1 ns) and single photons with large coherence times ($>1\mu$s) are readily available today experimentally [36], the requirement of time-resolved measurements in the implementation of the MBCS problem can be readily fulfilled. Moreover, an implementation of MBCS has the advantage to ease the difficulties faced in the production of identical photons. Indeed, photons of approximately equal colors ($\Delta\omega \gg |\omega_s - \omega_{s'}|\ \forall s \neq s'$) are not needed anymore, unlike in the original approximate boson sampling problem. This furthermore paves the way towards the use of photons of arbitrarily small bandwidth $\Delta\omega$, where the indistinguishability in the emission times ($1/\Delta\omega \gg |t_{0s} - t_{0s'}|\ \forall s, s'$) can be easily achieved.

In conclusion, all these results represent an important stepping-stone towards a full fundamental understanding of the complexity of multiphoton interference of photons of arbitrary spectra in linear optical networks, when the information about detection times and polarizations is not ignored. This may lead to "real world" applications in quantum information processing [52] and in quantum optics overcoming the experimental challenge in the production of identical bosons.

Finally, our results can be extended to bosonic interferometric networks with atoms [12,13,66], plasmons [67], or mesoscopic many-body systems [68] and are also relevant to the study of the complexity of multiboson correlation interference for different input states [2,69,70] and different correlation measurements [71].

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