Blast Dynamics in a Dissipative Gas

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The blast caused by an intense explosion has been extensively studied in conservative fluids, where the Taylor–von Neumann–Sedov hydrodynamic solution is a prototypical example of self-similarity driven by conservation laws. In dissipative media, however, energy conservation is violated, yet a distinctive self-similar solution appears. It hinges on the decoupling of random and coherent motion permitted by a broad class of dissipative mechanisms. This enforces a peculiar layered structure in the shock, for which we derive the full hydrodynamic solution, validated by a microscopic approach based on molecular dynamics simulations. We predict and evidence a succession of temporal regimes, as well as a long-time corrugation instability, also self-similar, which disrupts the blast boundary. These generic results may apply from astrophysical systems to granular gases, and invite further cross-fertilization between microscopic and hydrodynamic approaches of shock waves.

A blast wave follows the rapid and localized release of a large amount of energy in a medium. The physics community got seasonably interested in the dynamics of such shocks in air in the early 1940s. Taylor [1], von Neumann [2] and Sedov [3] independently understood that, as a result of the global conservation of mass and energy, the extension R of the blast had to grow with time like a power law t^δ, with $\delta = 2/5$ [or $2/(d+2)$ in dimension d] [4]. From a few publicly available snapshots of the blast at different times, Taylor could estimate within 10% the strength of the Trinity detonation in 1945, at the time classified information [5].

Remarkably, the hydrodynamic description of the flow inside the blast, now known as the Taylor–von Neumann–Sedov solution (TvNS), is self-similar in time, depending only on the rescaled radial distance $r/R(t)$. This similarity is *of the first kind* [5], i.e., driven by global invariants, and all exponents can be derived by dimensional analysis. This solution found widespread relevance beyond its initial realm, notably in plasma physics to describe laser-induced shocks [6,7] and in astrophysics for the evolution of supernova remnants [8]. However, it proves essential for a wealth of applications to relax some of the conservation laws [5,9], especially allowing for energy production or dissipation; on the shock boundary (e.g., a chemical reaction front) or in the bulk (e.g., collisional or radiative losses). This is usually expected to entail self-similarity *of the second kind*, where scaling exponents are no longer globally fixed, but depend continuously on parameters of the dynamics [5].

The subject of our inquiry, a blast with bulk energy dissipation, runs contrary to that expectation.

Understanding its similarity properties requires connecting two levels of analysis, coarse-grained and microscopic, that have remained largely impervious to each other. The abundant literature following the TvNS model has focused on global scaling laws or hydrodynamic models. Meanwhile, the study of granular gases [10] provides prototypical dissipative media where experiments [11,12] and particle-based simulations [13] have been performed, but a continuum view is missing. It is our purpose here to bridge this gap [14]. We provide an analytical understanding for three numerical observations: the spatial profiles for hydrodynamic fields, the scaling regimes exhibited by $R(t)$, and a previously unreported corrugation instability that distorts the shock wave at late times. Unraveling the wave structure is key in explaining these properties; Fig. 1 summarizes its main features, with a dense shell divided in three regions. The corrugation instability, also self-similar, is unique to the dissipative blast and differs significantly from those described in various other blast waves [16,17]. We will argue that, under broad assumptions, our results are largely independent of the mechanism at play in energy dissipation.

Model and previous results.—Our model system is a granular gas of identical spherical grains with radius σ and unit mass, where inelastic binary collisions conserve momentum but dissipate kinetic energy. Core results will not depend on specific dissipation mechanisms; hence, we opt for simplicity: energy loss is quantified by a fixed restitution coefficient $0 \leq \alpha \leq 1$ [10]; encounters are dissipative when $\alpha < 1$ and energy conserving (elastic) for $\alpha = 1$. We set a break-shot initial condition, where all
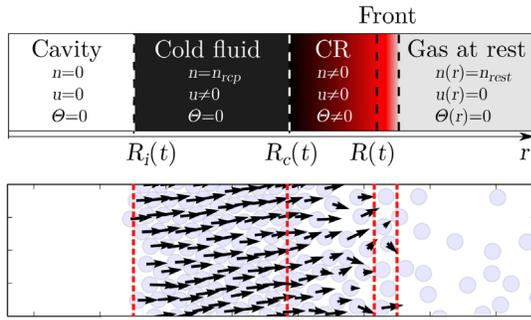
FIG. 1 (color online).   Section of the blast wave: cartoon (top) and hard sphere MD simulation (bottom). Dissipation causes particles to accrete into a dense, hollow shell. Equations (5)–(7) reveal its layered structure. First, the shock front where particles from the blast collide and mix with those at rest, creating incoherent motion. Then, a fixed-width cooling region (CR) where further compression occurs as temperature is dissipated, and finally a self-similar cold fluid region where velocities are well aligned, around a central cavity. This stands in stark contrast with the elastic case, where the bulk of the blast is comprised of a single, entirely self-similar region.

grains are at rest except within a small region [13,15]. A cascade of collisions follows, with an ever growing number of particles in motion, which forms the blast as observed in Fig. 1. We define its radius $R(t)$ as the distance from the center to the innermost particle at rest, and the shock front as all moving particles that count grains at rest among their nearest neighbors. Since the external medium is motionless, strong shock (infinite Mach number) conditions are ensured for any initial energy release [18]. A priori, the dynamics are specified by two parameters: α and the volume fraction $\phi_{rest} = n_{rest}\mathcal{V}_d(\sigma)$, where n_{rest} is the number density of particles in the gas at rest, and $\mathcal{V}_d(\sigma)$ is the volume of a grain in dimension d. In the remainder, numerical results are obtained from molecular dynamics (MD) simulations [10,19].

The first property of interest is the scaling law(s) obeyed by the radius $R(t)$. As the TvNS scaling crucially hinges on energy conservation, relaxing the latter generically causes self-similarity to break down or cross over to the second kind—it is then sensitive to microscopic parameters and derivable through methods such as renormalization, but not dimensional analysis [5]. However, following an argument due to Oort [21] for astrophysical systems, a blast in a gas with *bulk* dissipation should tend to form a thin, hollow shell that slows down only by accreting more material. Its total radial momentum, of order $R^d dR/dt$, is thus constant, at odds with the energy-conserving case (as explained below). This implies $R \propto t^\delta$ with $\delta = 1/(d+1)$, or $1/4$ in three dimensions—smaller than its conservative counterpart $2/(d+2)$. This solution is known as the momentum conserving snowplow (MCS) [8] and is self-similar of the first kind. Recent numerical studies of granular gases have confirmed both this scaling law and the hollow structure of

the blast for any $\alpha < 1$, although the shell is thick due to the high densities considered [13].

Hydrodynamic description.—Previous works on the granular blast have stopped short of investigating its spatial structure beyond these simple arguments. We turn to a continuum description, which will shed light on the peculiar shell in Oort's argument, and reveal its long-term instability. To establish a closed set of hydrodynamics equations, we define the *granular temperature* (or energy of random motion) of the medium through the variance of local velocity fluctuations [10,22–24]. The coupling between this temperature field $\Theta(\mathbf{r})$ and the density and velocity fields $n(\mathbf{r})$ and $\mathbf{u}(\mathbf{r})$ is derived in the dense fluid transport framework for inelastic hard spheres [25], which generalizes earlier descriptions of dilute systems [22,26]

$$\partial_t n + \nabla(n\mathbf{u}) = 0, \tag{1a}$$

$$(\partial_t + \mathbf{u}.\nabla)\mathbf{u} + \frac{1}{n}\nabla.\mathbf{p} = 0, \tag{1b}$$

$$n(\partial_t + \mathbf{u}.\nabla)\Theta + \frac{2}{d}(\mathbf{p} \cdot \nabla) \cdot \mathbf{u} = -\Lambda. \tag{1c}$$

The energy sink term takes the form [25,27]

$$\Lambda = \omega(1 - \alpha^2)n\Theta \tag{2}$$

and $\omega = \omega_0 n\sigma^{d-1}\Theta^{1/2}$ is the local collision frequency, proportional to the average relative velocity and the inverse mean free path, with ω_0 a dimensionless constant. These equations are closed by specifying the pressure tensor $\mathbf{p}$ with a constitutive relation, discussed below, which is safely taken of zeroth order in gradients, neglecting heat conduction [28].

In the particular elastic case ($\alpha = 1$, hence $\Lambda = 0$) with isotropic pressure $\mathbf{p} = p\mathbf{I}$ ($\mathbf{I}$ being the identity matrix), the Euler equations for a perfect fluid are recovered. Within the blast, the fields then assume a scaling form

$$n(\mathbf{r}, t) = n_{rest}M(\lambda), \qquad \mathbf{u}(\mathbf{r}, t) = \frac{\mathbf{r}}{t}V(\lambda),$$

$$\Theta(\mathbf{r}, t) = \frac{r^2}{t^2}T(\lambda), \qquad p(\mathbf{r}, t) = n_{rest}\frac{r^2}{t^2}P(\lambda), \tag{3}$$

where $\mathbf{r} = r\mathbf{e}_r$ denotes the position relative to the center of the blast with $r = |\mathbf{r}|$, and $\lambda = r/R(t)$ is the scaling variable. The profiles M, V, T, and P defined by Eq. (3) are dimensionless, isotropic and time-independent, and can be evidenced throughout the self-similar blast expansion by appropriately rescaling the density, velocity, temperature, and pressure fields. With the further assumption of an ideal gas constitutive relation $p = n\Theta$, Eqs. (1) and (3) together admit the classic TvNS solution [1–3,5]. It is noteworthy that there is a unique velocity scale in the elastic problem: $\mathbf{u}$ and $\sqrt{\Theta}$ exhibit the same scaling in $R(t)/t$, meaning that

coherent and incoherent motion remain coupled. This
solution exhibits a simple structure, with a boundary layer
of fixed size (the shock front) around an isotropic, self-
similar bulk. The front, where discontinuities in the hydro-
dynamic fields occur, can be defined microscopically as the
thin mixed region where mobile and immobile particles
coexist.

Results and discussion.—In the dissipative case, the
front is unchanged, but the core becomes more complex,
and cannot be described by fields with a simple scaling
form. Indeed, the dissipation term in Eq. (1c) depends
explicitly on an additional time scale, the collision time
ω^{-1}, and is consequently incompatible with the scaling
[Eq. (3)]. In the front, the spread in velocity is of order
$\dot{R}$, since particles advancing at that speed collide with
others at rest. Incoherent motion is continuously generated
by these collisions, so that at the boundary $\Theta \sim \dot{R}^2(t)$, as in
the elastic case. Inserting this ansatz in Eq. (1c), the
dissipation term grows in magnitude compared to transport
terms by a factor $\omega R(t)/\dot{R}(t) \sim R(t)$: asymptotically, tem-
perature is dissipated too fast to be advected on distances
comparable to $R(t)$. Dissipation instead acts over a distance
$\dot{R}/\omega$ that is here time independent; this creates a travelling
wavelike zone behind the front, the "cooling region." Then,
moving further toward the interior of the blast, we find a
distinct cold layer where temperature is negligible and
particle velocities are aligned; dissipation is eliminated, and
fields once again assume a self-similar form. This region
reaches a density close to random close packing, echoing
the clustering tendency of granular media [10]. Finally, as
all the inner material accretes into this dense layer, an
empty cavity opens at the center and the blast wave forms a
hollow shell, in line with Oort's argument [21]. The
validation of this multilayered structure in MD simulations
(see Fig. 1 and 2) is a crucial result that drives all further
analysis. We expect it to hold beyond our model system: the
decoupling of length or energy scales that shapes the layers
only requires ω to increase with n and Θ.

By contrast with past attempts at elucidating the blast's
structure [34], we argue that a consistent description
requires a dense fluid equation of state, to account for
compression in the shell. We find a solution to Eqs. (1) with
an anisotropic pressure tensor acting only along the radial
direction $\mathbf{e}_r$,

$$\mathbf{p} = n\Theta Z(n)\mathbf{e}_r \otimes \mathbf{e}_r, \qquad (4)$$

where $Z(n)$ is a diverging function of density, accounting
for steric effects [25,27,29]. Such strongly anisotropic
pressure is frequently observed in granular systems [35],
and the divergence in $Z(n)$ allows the cold region, despite
its vanishing temperature, to exhibit a finite pressure field.
This is essential for the existence of self-similar profiles in
analogy with Eqs. (3). Boundary conditions on these
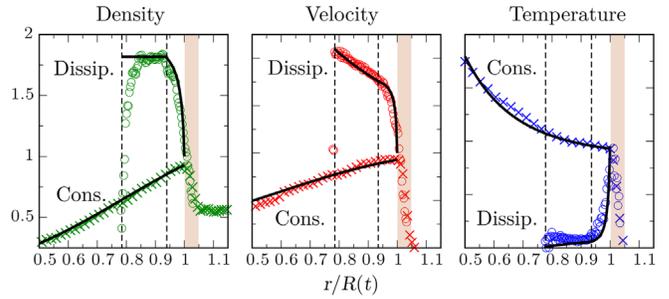profiles are set by a fixed-width layer, comprised of the

FIG. 2 (color online). Hydrodynamic profiles from MD sim-
ulations for dense conservative (crosses) and dissipative (circles)
fluids with $\phi_{\text{rest}} = 0.3$, and analytical solutions (solid lines).
Profiles are rescaled by the Rankine-Hugoniot boundary value
(see Ref. [29]), from left to right: $n(\lambda)/n_{\text{RH}}$, $u(\lambda)/u_{\text{RH}}$, and
$\Theta(\lambda)/\Theta_{\text{RH}}$ with $\lambda = r/R(t)$. The conservative solution is parti-
tioned into the gas at rest, the shock front (shaded area), and the
self-similar bulk. This is the first validation of a TvNS-like
solution in a dense fluid. For any dissipation, i.e., $\alpha < 1$ (here
$\alpha = 0.8$), the bulk structure becomes threefold, as shown by
dashed vertical lines: the cooling region between the front and
$R_c \approx 0.93R$; the maximally dense cold fluid down to $R_i \approx 0.78R$;
and the central cavity.

shock front where temperature is created, and the cooling
region where it is dissipated. We describe this layer in a
flux-difference form [27], which generalizes Rankine-
Hugoniot conditions [32] to the dissipative case and over
a finite distance [29]: in this thin layer the system is
assumed to be quasi-1D, allowing us to simplify Eqs. (1)
and integrate them between $R(t)$ and $r = R(t) - x$ giving

$$n(x) = n_{\text{rest}}M(x), \qquad u = \dot{R}\left[1 - \frac{1}{M}\right], \qquad p = n_{\text{rest}}\dot{R}u \qquad (5)$$

where all three fields are parameterized by the compression
$M(x)$ obeying the following equation [which may be
integrated numerically for any choice of $Z(n)$]:

$$(M - 1)\left[\frac{d}{2}Z^{-1} + 1\right]$$
$$= \frac{M^2}{2} - \omega_0(1 - \alpha^2)\int_{0^-}^x dx'\left(\frac{d(M-1)}{2Z}\right)^{3/2}. \qquad (6)$$

Higher-order transport terms neglected in Eqs. (1) may in
fact intervene in this layer, which has no growing typical
length scale; however, this simplified analysis proves
accurate enough.

In the cold region, we can assume scaling forms similar
to Eqs. (3), although temperature is vanishingly small. An
analytical solution is possible, as the fluid density is fairly
approximated by its random close packing value $n_{\text{rcp}} =
n_{\text{rest}}M_{\text{rcp}}$ corresponding to a volume fraction $\phi_{\text{rcp}} \approx 0.84$ for
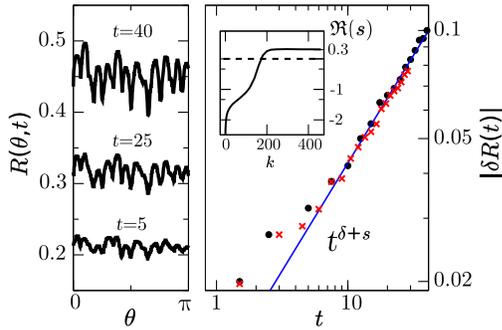$d = 2$ or 0.64 for $d = 3$; hence,

FIG. 3 (color online). Left: Shock radius by angular sector $R(\theta, t)$ at successive times (see Ref. [29] for its definition). The line thickness represents the mean free path in the shock front, or about half its average width $w \approx 0.01$. Right: Corrugation width $\delta R \sim t^{\delta+s}$ with theoretical exponent $s \simeq 0.3$ (solid line) and numerical validation for $\alpha = 0.3$ (circles) and $\alpha = 0.8$ (crosses). Inset: Real part of the dispersion relation $s(k)$ for the unstable mode, crossing the marginal stability line $\Re(s) = 0$ (dashed line), with a plateau at $\Re(s) \approx 0.3$.

$$V(\lambda) = \delta(1 - M_{\text{rcp}}^{-1})\lambda^{-d},$$
$$P(\lambda) = \delta^2\lambda^2(1 - M_{\text{rcp}}^{-1})[M_{\text{rcp}}(\lambda^d - 1) + 1]. \qquad (7)$$

The profiles thus obtained are seen in Fig. 2 to fare remarkably against MD simulations, despite the piecewise nature of the model. No fitted parameter is necessary. This is a crisp validation of the hydrodynamic view developed here as, to the best of our knowledge, no other model for blasts (dissipative or conservative) has been successfully applied to a dense fluid or supported by microscopic simulations. As anticipated, the flow velocity is maximal on the inner boundary of the cold region: the shell is pushed outward by its innermost particles, while the outermost slow down with dissipation and accrete onto the incoming "snowplow." At odds with the conservative case, coherent flow thus decouples from thermal agitation; we now see that this is the cornerstone of the similarity solution.

From the tensorial form of pressure, Eq. (4), it can be shown by integrating Eq. (1b) that radial momentum $\Pi = \int nur^{d-1}dr$ within a given solid angle ($d = 3$) or angular sector ($d = 2$), is conserved. This formally demonstrates the invariant suggested by previous authors [13,21] which yields the growth exponent $\delta = 1/(d+1)$, as sketched above. The reason for the conservation of Π is twofold: first, the central pressure at $r = 0$ vanishes in the cavity, and second, there are no orthoradial exchanges of momentum, due to the decoupling of coherent (radial) and incoherent (partly orthoradial) motion. In the elastic case ($\alpha = 1$), however, the opposite statements hold, and the expansion is dominated by the central pressure, which causes Π to increase with time, leading to $\delta > 1/(d+1)$. On shorter time scales, the evolving interplay of these forces gives rise to a succession of intermediate regimes,

such as the pressure-driven snowplow [8], which we confirm in MD simulations [29].

Finally, we evidence an instability arising at late times, which disrupts the MCS solution analyzed above and manifests as a growing corrugation of the shell (see Fig. 3). Numerics show that this growth follows a power law, suggesting that it stems from the self-similar cold region. While other instabilities can be found in TvNS shock waves under specific conditions [36], we stress the generic character of the instability discussed here. Necessary conditions are the conservation of radial momentum and a vanishing pressure on the inner boundary of the shell, which explain the absence of this phenomenon in conservative gases, and distinguish it from other instabilities of granular systems [37]. We perform a linear stability analysis, focusing here on the bidimensional case which lends itself to experimental confirmation. For each field ψ (among density, velocity and pressure) in the cold region, we look for a solution of the form

$$\psi(\mathbf{r}, t) = \psi_0(\lambda)[1 + \delta\psi(\lambda)\cos(k\theta)t^s], \qquad (8)$$

with $\psi_0(\lambda)$ the unperturbed self-similar profile, $\delta\psi(\lambda)$ the relative perturbation, and $s(k)$ the exponent of relative growth for a given angular frequency k. This analysis is complicated by the fact that the underlying solution is neither uniform nor stationary. We have to resort to a method previously used in conservative blasts [17]: the exponent s is used as a free parameter, selected for each value of k to minimize the difference between numerically integrated profiles and theoretical values on the boundaries [29]. We thereby sample the dispersion relation $s(k)$: as seen in Fig. 3, we predict the fastest growing perturbation with $s \approx 0.3$. The value of the exponent and its independence on parameters α and ϕ_{rest} are both confirmed by simulations.

Conclusion.—Most extensions to the Taylor-von Neumann-Sedov blast exhibit either a breakdown of similarity, or self-similarity of the second kind, i.e., continuous dependence on dynamical parameters [5]. The dissipative blast studied here is exceptional in that its asymptotic expansion remains self-similar of the first kind, driven by inertial motion rather than fine-tuned by the dissipative processes themselves. This property is generic to a large class of fluids: regardless of its mechanism, bulk energy dissipation only comes into play to enforce the layered structure of the shock. Under weak conditions, the energy scales for coherent and incoherent motion decouple and each comes to dominate a different region: temperature resides in a thin layer, pushed by a self-similarly growing cold region. We have given a full hydrodynamic description of this structure, in excellent agreement with particle-level simulations, and thus brought to the fore the mechanisms underlying the similarity solution. Its cornerstone, the conservation of radial momentum per angular sector, stems

from this decoupling of energy scales, reflected by the anisotropic pressure within the shell. Using the hydrodynamic profiles, we could perform a stability analysis, and successfully predict the existence and exponent of a corrugation instability rooted in the cold region. These results invite further contact between kinetic and continuum approaches, and between fields, from plasmas to granular systems, to deepen our understanding of dissipative fluids.

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