Critical Exponents Can Be Different on the Two Sides of a Transition:
A Generic Mechanism

F. Léonard and B. Delamotte

Laboratoire de Physique Théorique de la Matière Condensée, CNRS UMR 7600,
UPMC–Sorbonne Universités, 75252 Paris Cedex 05, France

We present models where γ_+ and γ_-, the exponents of the susceptibility in the high- and low-temperature phases, are generically different. In these models, continuous symmetries are explicitly broken down by discrete anisotropies that are irrelevant in the renormalization-group sense. The $\mathbb{Z}_q$-invariant models are the simplest examples for two-component order parameters ($N = 2$) and the model with icosahedral symmetry for $N = 3$. We accurately compute $\gamma_+ - \gamma_-$ as well as the ratio ν/ν' of the exponents of the two correlation lengths present for $T < T_c$.

The question of the equality of the critical exponents on the two sides of a second-order phase transition has, apparently, not been raised for decades. The general renormalization group (RG) argument "showing" their equality goes as follows: Correlation functions are regular in the presence of an external field, which is sufficient to proceed continuously from one phase to the other. Moreover, if these functions satisfy the same RG equations above and below the critical temperature T_c, the same should hold true for the scaling behavior of quantities such as the susceptibility, the correlation length, or the specific heat. Since the renormalization properties of a theory are identical in its symmetric and spontaneously broken phases, it follows that the critical exponents are identical in both phases (see, for instance, [1–3]). This is indeed what happens generically.

To the best of our knowledge, Nelson [4] was the first to propose a counterexample based on the $O(2)$ model in dimension $d = 3$, to which is added either a cubic (CA) [4–12] or hexagonal (HA) anisotropy [4–6]. These anisotropies are taken into account in the Ginzburg-Landau Hamiltonian by terms of order 4 and 6, respectively, which are irrelevant in the RG sense at the transition. The corresponding fixed point is thus $O(2)$ symmetric. However, Nelson argued [4] that they are dangerously irrelevant [6,13] in the low-temperature phase and that they, therefore, induce a modification of the exponent γ_- of the susceptibility. A rather counterintuitive result is that the difference $\gamma_+ - \gamma_-$ is larger for HA than for CA, whereas HA is "more irrelevant" than CA. A detailed study of the literature shows that, up until now, this striking result has been completely ignored.

Because of its relationship with either deconfined quantum critical points [14] or pyrochlore [15] and the possible existence of two distinct phase transitions [16], the three-dimensional XY model with HA (and, more generally, the $\mathbb{Z}_q$-invariant models) has recently been studied again [17,18]. Although only one transition has been found,

the $\mathbb{Z}_q$ models were shown to exhibit two correlation lengths below T_c, ξ and ξ', that scale with two different critical exponents, ν and ν'. All authors agree that ν/ν' depends on the scaling dimension of the irrelevant HA term, but there are no less than three different scaling relations predicting this ratio, as well as several values obtained by Monte Carlo simulations [14,17,18].

In this Letter, we present a mechanism, valid not only for the XY case, to generate different critical exponents above and below T_c. The mechanism relies on the possibility of explicitly breaking a continuous symmetry down to a discrete one by terms that are irrelevant in the RG sense. In addition to the $\mathbb{Z}_q$-invariant models, we build an example for Heisenberg spins showing icosahedral symmetry. Using the nonperturbative renormalization group (NPRG), it is—contrary to perturbation theory—easy to show that the exponents γ_+ and γ_- are generically different for these models and easy to compute them, as well as ν/ν', accurately. Our approach allows us to completely clarify the physics of these models.

Let us discuss the general idea underlying the difference between γ_+ and γ_-. For concreteness, we consider a XY or Heisenberg model described by an $O(N)$-invariant Hamiltonian ($H_{O(N)}$), to which is added a discrete anisotropy term $\tau(x)$, $H = H_{O(N)} + \lambda_{\rm an} \int_x \tau(x)$. We assume that $\tau(x)$ is irrelevant in $d = 3$. The fixed point (FP) describing the phase transition is therefore $O(N)$ symmetric ($\lambda_{\rm an}^{\rm FP} = 0$). If this term were irrelevant in the ordinary sense—that is, could be neglected ($\lambda_{\rm an} = 0$)—the model would be identical to the $O(N)$ model. It is important to remember that, in this case, not only the transverse (χ_T) but also the longitudinal (χ_L) susceptibilities diverge for all $T < T_c$ because of the Goldstone modes [19–21]. However, since the symmetry is discrete when $\lambda_{\rm an} \neq 0$, there are no Goldstone modes and the susceptibilities cannot diverge for $T < T_c$. Thus, χ_T^{-1} vanishes only at T_c, and its scaling with $\Delta T = T - T_c$ obviously depends

on the way λ_{an} scales to zero close to the fixed point. Since this scaling is given by the scaling dimension of $\tau(x)$, the exponent γ_T defined by $\chi_T^{-1} \sim (T_c - T)^{\gamma_T}$ for $T < T_c$ cannot be equal to γ_+. This is why $\tau(x)$ is said to be dangerously irrelevant for $T < T_c$. The same holds true for χ_L^{-1}.

Let us now give two examples, with $N = 2$ and 3, of the kind of anisotropy that produces a difference between γ_+ and $\gamma_{T,L}$. We choose discrete subgroups G of either $O(2)$ or $O(3)$. These subgroups must satisfy two constraints. First, in order to have only one phase transition, there must exist only one invariant quadratic polynomial of G. Therefore, it must be $\varphi_i\varphi_i$, $i = 1, ..., N$, as in the $O(N)$ model. Second, the interaction term that explicitly breaks the $O(N)$ symmetry must be irrelevant. A term of order 4 can be irrelevant compared to the $O(N)$-invariant term $(\varphi_i\varphi_i)^2$. For $N = 2$ and $d = 3$, this is the case, for instance, of the term $\varphi_1^4 + \varphi_2^4$ of cubic anisotropy. For reasons that are explained below, this kind of term, being "weakly irrelevant," induces only small differences between γ_+ and $\gamma_{T,L}$ that, moreover, require very large systems to be observable. We therefore prefer to consider terms that are "strongly irrelevant," because they are of degrees higher than 4.

For $N = 2$, all the $\mathbb{Z}_q$-invariant models (q-state clock models) with $q > 4$ satisfy the two conditions above because the first invariant polynomial in φ_i that is not $O(2)$ symmetric is of degree q. For instance, for $\mathbb{Z}_6$, this invariant reads $\tau = 6\varphi_1^5\varphi_2 + 6\varphi_1\varphi_2^5 - 20\varphi_1^3\varphi_2^3$ [6,22].

For $N = 3$, the situation is more constrained because only the icosahedral group satisfies the two conditions above [23]. For all the other discrete subgroups of $O(3)$, the first invariant polynomials that are not $O(3)$ symmetric are of degree 4 and, therefore, are at best only weakly irrelevant. For the icosahedral symmetry, the first non-$O(3)$-symmetric invariant polynomial is of degree 6 and reads [23,24]

$$\tau = (4\Phi - 2)(\varphi_1^2 - \varphi_2^2)(\varphi_2^2 - \varphi_3^2)(\varphi_3^2 - \varphi_1^2)$$
$$+ 22(\varphi_1\varphi_2\varphi_3)^2 + (\varphi_1^4 + \varphi_2^4 + \varphi_3^4)(\varphi_1^2 + \varphi_2^2 + \varphi_3^2),$$

(1)

where Φ is the golden ratio [25].

The NPRG is based on Wilson's idea of integrating fluctuations step by step [26]. In its modern version, it is implemented on the Gibbs free energy Γ [27–29]. A one-parameter family of models indexed by a scale k is thus defined such that only the rapid fluctuations, with wave numbers $|q| > k$, are summed over in the partition function $\mathcal{Z}_k$. The decoupling of the slow modes ($|q| < k$) in $\mathcal{Z}_k$ is performed by adding to the original Hamiltonian H a quadratic ("masslike") term which is nonvanishing only for these modes:

$$\mathcal{Z}_k[J] = \int D\varphi \exp(-H[\varphi] - \Delta H_k[\varphi] + J \cdot \varphi),$$

(2)

with $\Delta H_k[\varphi] = \frac{1}{2}\int_q R_k(q^2)\varphi_i(q)\varphi_i(-q)$—where, for instance, $R_k(q^2) = Z_k(k^2 - q^2)\theta(k^2 - q^2)$, with θ the step function and Z_k the field renormalization constant—and $J \cdot \varphi = \int_x J_i(x)\varphi_i(x)$. The coarse-grained Gibbs free energy $\Gamma_k[\phi]$ is defined as the (slightly modified) Legendre transform of $\log \mathcal{Z}_k[J_i]$,

$$\Gamma_k[\phi] + \log \mathcal{Z}_k[J] = J \cdot \phi - \frac{1}{2}\int_q R_k(q^2)\phi_i(q)\phi_i(-q),$$

(3)

where $\phi_i(x)$ is the thermal average of $\varphi_i(x)$. When k is of the order of the inverse lattice spacing Λ, all fluctuations in $\mathcal{Z}_k$ are frozen by the R_k term and the mean-field approximation becomes exact. With the definition (3), this implies that $\Gamma_{k=\Lambda}[\phi] = H[\phi]$ [27]. Since $R_{k=0}(q^2) \equiv 0$, $\Gamma_{k=0}[\phi] = \Gamma[\phi]$ and is, thus, the free energy that we want to compute.

The exact flow equation of Γ_k reads [27] (see Supplemental Material [30] for more details)

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\text{Tr}[\partial_t R_k(q^2)(\Gamma_k^{(2)}[q, -q; \phi] + R_k(q))^{-1}],$$

(4)

where $t = \log(k/\Lambda)$, Tr stands for an integral over q and a trace over group indices, and $\Gamma_k^{(2)}[q, -q; \phi]$ is the matrix of the Fourier transforms of the second functional derivatives of $\Gamma_k[\phi]$ with respect to $\phi_i(x)$ and $\phi_j(y)$.

Since it is impossible to solve Eq. (4) exactly, we must make use of approximations. To capture the critical physics, the simplest nonperturbative approximation is the derivative expansion [31,32]. We use the (improved) lowest order, the local potential approximation prime (LPA'), which consists of retaining only a potential term in $\Gamma_k[\phi]$ together with a field renormalization constant Z_k in front of the kinetic term [31,32],

$$\Gamma_k^{\text{LPA'}}[\phi] = \int_x \left\{\frac{1}{2}Z_k[\nabla\phi_i(x)]^2 + U_k(\phi_i(x))\right\}.$$

(5)

The running potential U_k is defined by $\Omega U_k(\phi_i) = \Gamma_k[\phi_i]$, where the fields ϕ_i are constant and Ω is the volume of the system. Its flow is obtained from Eq. (4), where $\Gamma_k^{(2)}$ is computed from (5) and is then evaluated in a constant field configuration ϕ_i.

On top of the LPA', $U_k(\phi_i)$ can be expanded around one of its running minima $\phi_{i,k}^{\min}$, which corresponds, at $k = 0$, to the stable state of the system when $J_i = 0$,

$$U_k = \frac{u_{20,k}}{2}(\rho - \kappa_k)^2 + u_{01,k}\tau + \frac{u_{30,k}}{3!}(\rho - \kappa_k)^3 + \cdots,$$

(6)

with $\rho = \phi_i \phi_i / 2$ and $\kappa_k = \phi_{i,k}^{\min} \phi_{i,k}^{\min} / 2$. The flows of $u_{mn,k}$ and κ_k are obtained from that of U_k by acting with ∂_t on both sides of their definition,

$$u_{mn,k} = \left. \frac{\partial^{m+n} U_k}{\partial \rho^m \partial \tau^n} \right|_{\rho = \kappa_k, \tau = 0}, \qquad \left. \frac{\partial U_k}{\partial \rho} \right|_{\rho = \kappa_k, \tau = 0} = 0. \quad (7)$$

We have performed the calculations up to order 12 in the field-expansion equation (6). However, for the sake of simplicity, we present below the RG flow obtained for the XY model with HA within the simplest ansatz that includes only $u_{20,k}$ and $u_{01,k}$, which we call u and λ_6 (we omit the k index in the following to alleviate the notation). In this case, $\phi_{i,k}^{\min}$ corresponds to one of the six minima of U_k; we call the direction pointing towards $\phi_{i,k}^{\min}$ longitudinal and the perpendicular direction transverse. Once diagonalized, $\Gamma_{ij,k}^{(2)}$ splits as usual into the two inverse longitudinal and transverse propagators that each depend on a (running) "mass" $2u\kappa$ and $18\lambda_6 \kappa^2$, which we call m_L and m_T. Finding RG fixed points requires us to work with dimensionless and renormalized variables. We thus define $\tilde{\kappa} = K_d^{-1} Z_k k^{2-d} \kappa$, $\tilde{u} = K_d Z_k^{-2} k^{d-4} u$, and $\tilde{\lambda}_6 = K_d^2 Z_k^{-3} k^{2d-6} \lambda_6$, where $K_d^{-1} = d\Gamma(d/2) 2^{d-1} \pi^{(d/2)}$ has been included for convenience. The flow equations read (see Supplemental Material [30] for more details)

$$\partial_t \tilde{\kappa} = (2 - d - \eta_k) \tilde{\kappa} + \left(\frac{1}{2} + \frac{18 \tilde{\kappa} \tilde{\lambda}_6}{\tilde{u}} \right) I_2(\tilde{m}_T^2) + \frac{3}{2} I_2(\tilde{m}_L^2),$$
$$(8a)$$

$$\partial_t \tilde{u} = (d - 4 + 2\eta_k) \tilde{u} - 18 \tilde{\lambda}_6 I_2(\tilde{m}_T^2) + 9 \tilde{u}^2 I_3(\tilde{m}_L^2)$$
$$+ (\tilde{u} + 36 \tilde{\kappa} \tilde{\lambda}_6)^2 I_3(\tilde{m}_T^2), \quad (8b)$$

$$\partial_t \tilde{\lambda}_6 = (2d - 6 + 3\eta_k) \tilde{\lambda}_6 + 15 \tilde{\lambda}_6 (\tilde{u} + 6 \tilde{\kappa} \tilde{\lambda}_6) \frac{I_2(\tilde{m}_T^2) - I_2(\tilde{m}_L^2)}{\tilde{m}_L^2 - \tilde{m}_T^2},$$
$$(8c)$$

with $I_n(x) = 2(1+x)^{-n}[1 - \eta_k/(d+2)]$, $\tilde{m}_L^2 = 2\tilde{u} \tilde{\kappa}$, $\tilde{m}_T^2 = 18 \tilde{\lambda}_6 \tilde{\kappa}^2$. The running anomalous dimension is defined by $\eta_k = -\partial_t \log Z_k$ which tends, at criticality, to the anomalous dimension η for $k \to 0$ [31,32]. We show its flow in Fig. 1(a).

The flow equations (8) are very simple, though they are nonperturbative. They show two crucial features. First, they automatically take into account the role of the masses $m_{L,T}$ and their decoupling: as long as $k \gtrsim m_{L,T}$, that is, $1 \gtrsim \tilde{m}_{L,T}$, the contributions coming from $I_n(\tilde{m}_L^2)$ [$I_n(\tilde{m}_T^2)$] are nonvanishing; they become negligible when $k \ll m_{L,T}$, and the longitudinal (transverse) mode is then said to decouple from the flow. Second, once generalized to arbitrary N, they reproduce the low-T expansion of the $O(N)$ nonlinear-sigma model at one loop in $d = 2 + \epsilon$ (when $\lambda_6 = 0$) and
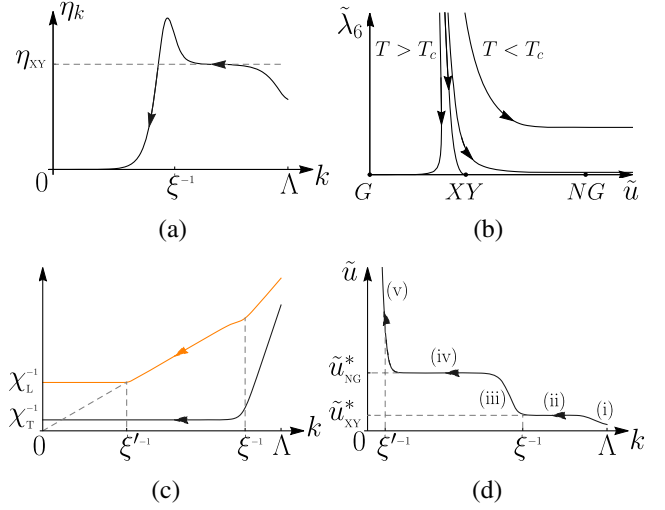

FIG. 1 (color online). XY model with HA in $d = 3$. (a) Flow for $T < T_c$ of the running anomalous dimension. (b) Projections of RG trajectories in the $\tilde{u} - \tilde{\lambda}_6$ plane. G, XY and NG are the Gaussian, critical $O(2)$, and low-temperature (Nambu-Goldstone) fixed points. (c) Flows for $T < T_c$ of the inverse transverse and longitudinal susceptibilities. The dashed line represents the flow of χ_L^{-1} in the pure $O(2)$ model. (d) Flow of $\tilde{u}$ for $T < T_c$ showing its five different regimes. The two plateaus correspond to the vicinity of the XY and NG FP.

are also one-loop exact in $d = 4 - \epsilon$ [32,33]. Gathering all these properties in a single set of flow equations is out of reach of the usual perturbative expansions.

Equations (8) admit three fixed points (with noninfinite couplings $\tilde{u}$ and $\tilde{\lambda}_6$) as shown in Fig. 1. The NG FP has coordinates $\tilde{u}_{NG}^* = 2 - d/2$, $\tilde{\lambda}_{6,NG}^* = 0$, $\tilde{\kappa}_{NG}^* = \infty$. By integrating Eqs. (8) in $d = 3$ with different initial conditions, we find the RG trajectories shown in Fig. 1(b). For $T < T_c$, λ_6 and κ reach fixed, nonvanishing values for $k \lesssim \xi^{-1}$. This is why the dimensionful inverse transverse susceptibility χ_T^{-1} stops running beyond this scale, see Fig. 1(c). The dimensionless analogues of these couplings, $\tilde{\lambda}_6$ and $\tilde{\kappa}$, keep running according to their canonical dimension. Thus, for $k \ll \xi^{-1}$,

$$\tilde{\kappa}(k) \sim \tilde{\kappa}(\xi^{-1})(k\xi)^{2-d} \sim \tilde{\kappa}_{XY}^*(k\xi)^{2-d},$$
$$\tilde{\lambda}_6(k) \sim \tilde{\lambda}_6(\xi^{-1})(k\xi)^{2d-6} \sim \tilde{\lambda}_6^{\text{in}} \xi^{-|y_6|}(k\xi)^{2d-6}, \quad (9)$$

where y_6 is the scaling exponent of $\tilde{\lambda}_6$ and $\tilde{\lambda}_6^{\text{in}}$ is the initial value of $\tilde{\lambda}_6$, that is, its value at the scale of the inverse lattice spacing. As for $\tilde{u}$, we find in $d = 3$, for T slightly below T_c, five different regimes represented on Fig. 1(d). In region (v), $\tilde{u}$ diverges and u reaches a finite value.

The values of k at which the RG flow departs, respectively, from the XY and NG FP define two length scales called ξ and ξ', see Fig. 1. The first one, ξ, is the Josephson length (the correlation length of the amplitude mode) of the pure $O(2)$ model [34] [the anisotropy plays no role in part

(ii) of the flow, see Fig. 1(d), when ξ is large because $\tilde{\lambda}_6(k \sim \xi^{-1}) \ll 1$]. As long as the RG trajectory remains close to NG, the flows of $\tilde{u}$ and $\tilde{\kappa}$ remain similar to the flows of the $O(2)$ model in the low-T phase [region (iv) in Fig. 1(d)], and χ_L^{-1} decreases as it does in the pure $O(2)$ model where the fluctuations of the Goldstone modes make it vanish, see Fig. 1(c). However, because $\tilde{\lambda}_6 \neq 0$, the flow departs from the NG FP [14,17]; $u(k)$ stops flowing, as does χ_L^{-1}, see Fig. 1(c). This occurs when $\tilde{m}_T^2(k = \xi'^{-1}) \simeq 1$, which defines ξ'. This correlation length diverges at T_c according to $\xi' \sim \Delta T^{-\nu'}$, by definition of ν'. Using the definition of $\tilde{m}_T^2$ and Eqs. (9), it is now straightforward to show that

$$\nu' = \nu(1 + |y_6|/2). \quad (10)$$

This relation was already obtained in [18], although we do not fully agree with its derivation [35]. The derivation above shows that ν' is universal, contrary to what is suggested in [15].

The scaling relations among exponents are readily derived from the discussion above and the usual scaling behavior of the potential,

$$U = s^d U\big(s^{-1/\nu}\Delta T, \tilde{u}(s), \tilde{\lambda}_6(s), \ldots, \tilde{\phi}Z(s)^{1/2}\big), \quad (11)$$

where s is a rescaling factor, the dots stand for the infinite set of irrelevant couplings, and $Z(s)$ is the field renormalization factor. $Z(s) \sim s^{-d+2-\eta_{XY}}$ for $s \sim (\xi\Lambda)^{-1}$, where η_{XY} is the anomalous dimension at the XY FP, and $Z(s) \sim s^{-d+2}$ for $s \ll (\xi\Lambda)^{-1}$ since η_k vanishes away from it (in particular at the NG FP), see Fig. 1(a). In the low-T phase, by taking two derivatives of (11) with respect to $\tilde{\phi}$ in either the transverse or longitudinal directions, and then taking $\tilde{\phi}$ at the minimum of U, we obtain

$$\chi_T^{-1} \propto s^d Z(s)\tilde{\lambda}_6(s)\tilde{\kappa}^2(s), \quad (12)$$

$$\chi_L^{-1} \propto s^d Z(s)\tilde{u}(s)\tilde{\kappa}(s). \quad (13)$$

By taking $s \sim (\xi\Lambda)^{-1}$ in Eq. (12) we obtain $\gamma_T = \gamma_+ + \nu|y_6|$, and by taking $s \sim (\xi'\Lambda)^{-1}$ and using Eqs. (9), (10) we obtain $\gamma_L = \gamma_+ + (4-d)\nu|y_6|/2$ [36]. Notice that the scaling relations derived above for $\mathbb{Z}_6$ are generically valid.

We have computed y_q up to order 12 in the field-expansion equation (6) to obtain converged results, see Table I. We observe that our value of ν' for $\mathbb{Z}_6$ is very close to the one deduced from the scaling law [Eq. (10)] and Monte Carlo simulations in [18]. This validates our approach. We find, of course, that $|y_q|$ increases with q; we thus deduce the rather counterintuitive result that the more irrelevant the anisotropy term, the larger the difference between γ_+ and $\gamma_{T,L}$. However, since ξ' diverges extremely rapidly close to T_c^- for large $|y_q|$, it must be difficult to observe the scaling behavior of χ_L in a

TABLE I. Critical exponents in $d = 3$ for the XY and $\mathbb{Z}_q$ models. In both cases, the first row corresponds to our results. For the icosahedral symmetry ($N = 3$), we find, at order 8 in the field-expansion equation (6), $\nu' = 1.51$ and $\gamma_T - \gamma_+ = 1.54$.

Symmetry	$\mathbb{Z}_4$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	$\mathbb{Z}_{10}$
	0.71	1.06	1.44	3.84
ν'	0.72 [14]	1.05 [14]	1.6 [14]	2.8 [14]
			1.45 [18]	
$\gamma_T - \gamma_+$	0.029	0.74	1.49	6.29
	0.06 [12]		1.58 [4]	

finite-size system for "large" values of q. As for χ_T, its measurement should not be more difficult than in the pure $O(2)$ model. Reciprocally, if $|y_q|$ is too small, the transient regime before reaching the XY FP is very large [region (i) in Fig. 1(d)], and, thus, the corrections to scaling are also large; this spoils an accurate determination of the leading scaling behavior in finite-size systems. This is probably the case for CA in $d = 3$, where we find $|y_4| = 0.042$ [at six loops $|y_4| = 0.103(8)$ [12]].

We have presented a general mechanism leading to a large and measurable difference between critical exponents in the high- and low-T phases and a theoretical approach to compute them. For the XY case, we have resolved the existing discrepancies between the results obtained in $\mathbb{Z}_q$-invariant models [14,15,17,18]. Let us also emphasize that layered decagonal quasicrystals [37,38] showing tenfold anisotropies and XY spin systems with HA [4] exist, which would enable a direct measurement of $\gamma_{T,L} - \gamma_+$ and ν'. Another very interesting challenge is the possibility of measuring susceptibilities in Heisenberg systems with icosahedral anisotropy, possibly in quasicrystals. We recall that, for $N = 3$, there are probably many other anisotropies that are dangerously irrelevant; this likely would lead to differences between $\gamma_{T,L}$ and γ_+ that are smaller than in the icosadrehal case, but that are possibly also measurable. Finally, it would be extremely interesting to investigate the two-dimensional [39–48] case with the NPRG approach. At the price of avoiding any field truncation and working in at least the second order of the derivative expansion [49–51], this is reachable. We leave this for future work.

---

[1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 2nd ed. (Oxford University Press, New York, 1993), p. 610.

[2] M. Kardar, *Statistical Physics of Fields* (Cambridge University Press, Cambridge, England, 2007), p. 56.

[3] M. Le Bellac, *Quantum and Statistical Field Theory* (Oxford University Press, New York, 1991), p. 17.

[4] D. R. Nelson, Phys. Rev. B **13**, 2222 (1976).

[5] A. Houghton and F. J. Wegner, Phys. Rev. A **10**, 435 (1974).

[6] D. J. Amit and L. Peliti, Ann. Phys. (N.Y.) **140**, 207 (1982).

[7] A. Aharony, Phys. Rev. B **8**, 4270 (1973).

[8] I. J. Ketley and D. J. Wallace, J. Phys. A **6**, 1667 (1973).

[9] D. J. Wallace, J. Phys. C **6**, 1390 (1973).

[10] A. D. Bruce, J. Phys. C **7**, 2089 (1974).

[11] A. D. Bruce and A. Aharony, Phys. Rev. B **11**, 478 (1975).

[12] J. M. Carmona, A. Pelissetto, and E. Vicari, Phys. Rev. B **61**, 15136 (2000).

[13] M. E. Fisher, in Critical Phenomena, *Proceedings of the International School of Physics "Enrico Fermi," Course CI*, edited by M. S. Green (Academic, New York, 1971).

[14] J. Lou, A. W. Sandvik, and L. Balents, Phys. Rev. Lett. **99**, 207203 (2007).

[15] M. E. Zhitomirsky, P. C. W. Holdsworth, and R. Moessner, Phys. Rev. B **89**, 140403 (2014).

[16] S. Miyashita, J. Phys. Soc. Jpn. **66**, 3411 (1997).

[17] M. Oshikawa, Phys. Rev. B **61**, 3430 (2000).

[18] T. Okubo, K. Oshikawa, H. Watanabe, and N. Kawashima, Phys. Rev. B **91**, 174417 (2015).

[19] A. Patashinskii and V. Pokrovskii, Zh. Eksp. Teor. Fiz. **64**, 1445 (1973) [Sov. Phys. JETP **37**, 733 (1973)].

[20] E. Brézin and D. J. Wallace, Phys. Rev. B **7**, 1967 (1973).

[21] N. Dupuis, Phys. Rev. E **83**, 031120 (2011).

[22] We notice an error in Eq. (4.1) of [4] regarding the Hamiltonian $\mathcal{H}_w$ corresponding to HA.

[23] M. V. Jaric, L. Michel, and R. T. Sharp, J. Phys. (Paris) **45**, 1 (1984).

[24] C. Cummins and J. Patera, J. Math. Phys. (N.Y.) **29**, 1736 (1988).

[25] Two other independent invariant polynomials, of degrees 10 and 15, exist for this group.

[26] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971).

[27] C. Wetterich, Phys. Lett. B **301**, 90 (1993).

[28] U. Ellwanger, Z. Phys. C **58**, 619 (1993).

[29] T. R. Morris, Int. J. Mod. Phys. A **09**, 2411 (1994).

[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.115.200601 for the detailed derivation of Eqs. (4) and (8).

[31] N. Tetradis and C. Wetterich, Nucl. Phys. **B422**, 541 (1994).

[32] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).

[33] B. Delamotte, D. Mouhanna, and M. Tissier, Phys. Rev. B **69**, 134413 (2004).

[34] B. Josephson, Phys. Lett. **21**, 608 (1966).

[35] Contrary to what is shown in Fig. 1 of [18], we find only three fixed points with $\tilde{\lambda}_6 < \infty$. Moreover, in $d = 3$, $\tilde{\lambda}_6$ cannot be of order 1 for $T \to T_c$ and $k \to 0$.

[36] Notice that we disagree with Eq. (4.3) of [4], though we agree with Eq. (4.12) of the same paper.

[37] V. S. Kraposhin, D. A. Bazhanov, and P. V. Bocharov, EPJ Web Conf. **15**, 03012 (2011).

[38] D. Rau, J. Gavilano, S. Mushkolaj, C. Beeli, and H. Ott, J. Magn. Magn. Mater. **272–276**, Part 2, 1330 (2004).

[39] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[40] E. Domany and E. K. Riedel, Phys. Rev. Lett. **40**, 561 (1978).

[41] J. L. Cardy, J. Phys. A **13**, 1507 (1980).

[42] J. Tobochnik, Phys. Rev. B **26**, 6201 (1982).

[43] D. Landau, J. Magn. Magn. Mater. **31–34**, 1115 (1983).

[44] M. S. S. Challa and D. P. Landau, Phys. Rev. B **33**, 437 (1986).

[45] A. Yamagata and I. Ono, J. Phys. A **24**, 265 (1991).

[46] S. V. Isakov and R. Moessner, Phys. Rev. B **68**, 104409 (2003).

[47] S.-Z. Lin, Y. Kamiya, G.-W. Chern, and C. D. Batista, Phys. Rev. Lett. **112**, 155702 (2014).

[48] T. Coletta, S. E. Korshunov, and F. Mila, Phys. Rev. B **90**, 205109 (2014).

[49] H. Ballhausen, J. Berges, and C. Wetterich, Phys. Lett. B **582**, 144 (2004).

[50] T. R. Morris, Phys. Lett. B **345**, 139 (1995).

[51] P. Jakubczyk, N. Dupuis, and B. Delamotte, Phys. Rev. E **90**, 062105 (2014).