# Tensor Network Renormalization Yields the Multiscale Entanglement Renormalization Ansatz

G. Evenbly[1,*] and G. Vidal[2,†]

[1]Department of Physics and Astronomy, University of California, Irvine, California 92697-4575, USA
[2]Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

(Received 25 August 2015; published 10 November 2015)

We show how to build a multiscale entanglement renormalization ansatz (MERA) representation of the ground state of a many-body Hamiltonian $H$ by applying the recently proposed tensor network renormalization [G. Evenbly and G. Vidal, Phys. Rev. Lett. 115, 180405 (2015)] to the Euclidean time evolution operator $e^{-\beta H}$ for infinite $\beta$. This approach bypasses the costly energy minimization of previous MERA algorithms and, when applied to finite inverse temperature $\beta$, produces a MERA representation of a thermal Gibbs state. Our construction endows tensor network renormalization with a renormalization group flow in the space of wave functions and Hamiltonians (and not merely in the more abstract space of tensors) and extends the MERA formalism to classical statistical systems.

DOI: 10.1103/PhysRevLett.115.200401 PACS numbers: 05.30.-d, 02.70.-c, 03.67.Mn, 75.10.Jm

Consider a strongly interacting quantum many-body system in $D$ spatial dimensions described by a microscopic Hamiltonian $H$. Understanding its collective, low energy behavior is a main goal in condensed matter and high energy physics, one that poses a formidable theoretical challenge. To tackle this problem, plenty of methods have been proposed based on the renormalization group (RG) [1–3], that is, on studying how the physics depends on the scale of observation. Weakly interacting systems can be addressed perturbatively using momentum space RG [2]. Instead, strongly interacting systems often require non-perturbative, *real space* RG methods, as pioneered by Kadanoff [1] and Wilson [2].

Improving on Kadanoff and Wilson's proposals, White's density matrix renormalization group for quantum spin chains [4] established how to systematically preserve the ground state wave function during real-space coarse graining, namely, by preserving the support of its reduced density matrix. Similarly, Levin and Nave's tensor renormalization group (TRG) [5] taught us how to coarse grain Euclidean path integrals of one-dimensional quantum systems (also partition functions of two-dimensional classical systems). Both the density matrix renormalization group and TRG are very successful, versatile approaches. However, they depart significantly from the spirit of the RG, in that they produce a coarse-grained, effective description of the system that still retains some irrelevant microscopic details. As a result, these methods (i) fail to define a proper RG flow, one with, e.g., the correct structure of fixed points, and (ii) struggle to deal with critical systems or systems in $D \geq 2$ dimensions, where the accumulation of irrelevant microscopic degrees of freedom is more significant and harmful.

These difficulties have been solved with two closely related proposals. First, *entanglement renormalization* was put forward to address the above two problems in the context of ground state wave functions [6,7]. By introducing *disentanglers*, which remove short-range entanglement, a proper RG flow is generated, as well as a RG transformation that is computationally sustainable even at criticality. In addition, entanglement renormalization leads to an efficient tensor network description of ground states for critical systems, the multiscale entanglement renormalization ansatz (MERA) [7], of interest not only as a many-body variational state [7–12] but also as a lattice realization of the holographic principle [13,14].

Tensor network renormalization (TNR), on the other hand, was more recently proposed to tackle the same problems in the context of Euclidean path integrals (and classical statistical systems) [15]. The Euclidean path integral $Z \equiv \mathrm{tr}(e^{-\beta H})$ is represented by a tensor network, consisting of copies of a single tensor $A$ [16], which extends both in space and Euclidean time directions. Through local manipulation of this tensor network, TNR produces a sequence of tensors,

$$A \to A' \to A'' \to \cdots \to A_{\mathrm{FP}}, \tag{1}$$

corresponding to increasing length scales, which flow towards some infrared fixed-point tensor, $A_{\mathrm{FP}}$. The later retains only the universal features of the phase or phase transition [15]. Once again, the key of the approach is the removal of short-range correlations by disentanglers.

In this Letter, we establish a close connection between the two approaches. We show that, when applied to the Euclidean path integral restricted to the upper half plane, TNR generates a MERA for the ground state of Hamiltonian $H$. More generally, TNR also produces a MERA for the thermal Gibbs state $\rho_\beta \equiv e^{-\beta H}/Z$ at finite inverse temperature $\beta$, as well as for the low energy eigenstates of $H$ on a finite periodic chain. Our result provides an alternative route to the MERA, one that bypasses the costly energy minimization of previous algorithms [8,9] and has several other significant advantages, both conceptual and computational, that we also discuss. Among those, we emphasize that (i) we obtain the first correct MERA representation of the thermal state $\rho_\beta$ [17], together with an algorithm to find it, (ii) the connection implies that TNR produces a RG flow in the space of wave functions and Hamiltonians, and not just in the abstract space of tensors, Eq. (1), and (iii) the MERA formalism can be applied now also to classical statistical systems. For simplicity, we consider a translation invariant system in $D = 1$ dimensions, although the key results generalize to inhomogeneous systems in dimension $D \geq 1$.

*Tensor network for Euclidean time evolution.*—Given a translation invariant Hamiltonian $H$ in one dimension, we use a standard procedure (see Ref. [18] Sec. A) to produce a two-dimensional tensor network representation of the Euclidean time operator $e^{-\beta H}$ or Euclidean path integral $\mathrm{tr}(e^{-\beta H})$. This tensor network is made of copies of a single tensor $A$. If both the system size $L$ and the inverse temperature $\beta$ are infinite, then the network spans the entire $(x,\tau)$ plane, where $x$ and $\tau$ label space and Euclidean time, respectively. Here we will consider tensor networks for $e^{-\beta H}$ on three different geometries, obtained by introducing a horizontal cut at $\tau = 0$ and by choosing $L$ and $\beta$ to be either finite or infinite (see Ref. [18] Sec. B).

*TNR yields MERA.*—Let us start with the upper half plane, Fig. 1, which corresponds to the ground state $|\Psi\rangle$ of $H$ on an infinite lattice. The network has an open boundary at $\tau = 0$, with an infinite row of open indices, one for each site of the one-dimensional lattice on which $H$ acts. We first apply TNR everywhere on the upper half plane except near $\tau = 0$, where we keep the open indices of the tensor network untouched. TNR acts through an intricate sequence of local replacements [15]. Here we skip the technical details (reviewed in Ref. [18] Sec. C) and focus instead on describing the final result: a coarse-grained tensor network with effective tensor $A'$ for most of the upper half plane, in accordance with Eq. (1), together with a double row of special tensors, so-called disentanglers and isometries, which correspond to one layer of the MERA [25]. These tensors connect the microscopic degrees of freedom at length scale $s = 0$ (represented by the original open indices of the network) with the coarse-grained degrees of freedom at length scale $s = 1$ (represented by the lower indices of the lowest row of tensors $A'$). We can now repeat the process on tensors $A'$ to obtain coarse-grained tensors $A''$ and a second row of disentanglers and isometries, i.e., a second layer of the MERA, connecting scales $s = 1$ and 2. Iteration then produces a full MERA approximation for the ground state $|\Psi\rangle$ of $H$, encompassing all length scales $s = 0, 1, 2, \ldots,$.

*Thermal MERA.*—Let us now consider a horizontal strip of finite width $\beta$, Fig. 2(a), which is proportional to the thermal state, $\rho_\beta \equiv e^{-\beta H}/Z$. This time we have two boundaries, each with an infinite row of open indices: the incoming and outgoing indices of the Euclidean time evolution operator, $e^{-\beta H}$. As before, we use TNR to coarse grain the tensor network, except near its open boundaries, where we do not touch the open indices. Figure 2(b) shows the net result: a coarse-grained tensor network, with effective tensor $A'$, together with a double row of disentanglers and isometries both for the incoming and outgoing

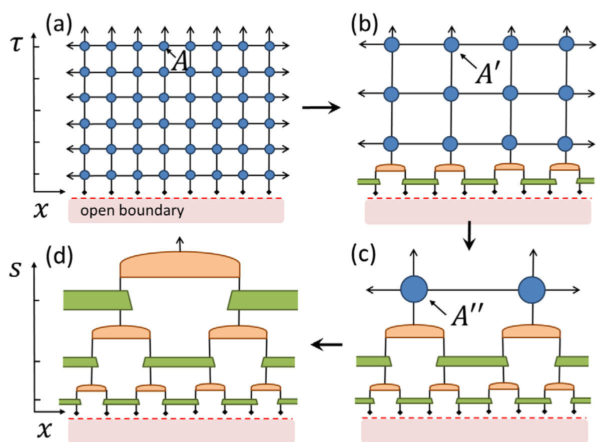


FIG. 1 (color online). (a) Tensor network, the ground state $|\Psi\rangle$ of $H$ on an infinite lattice. It is made of copies of tensor $A$ and restricted to the upper half plane $(x,\tau^+)$, with a row of open indices at $\tau = 0$. (b) By coarse graining the tensor network while leaving the open indices untouched, we obtain a new tensor network with tensors $A'$ together with one row of disentanglers and isometries. (c) Further coarse graining of the tensor network produces new coarse-grained tensors $A''$ and a second layer of disentanglers and isometries. (d) By iteration we obtain a full MERA approximation for state $|\Psi\rangle$.

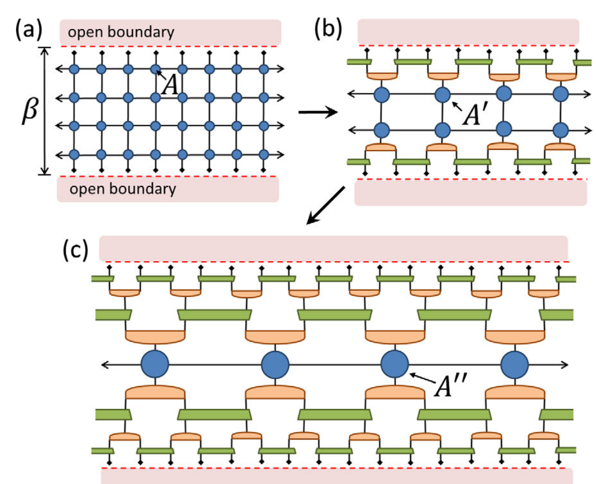


FIG. 2 (color online). (a) Tensor network on an infinite strip of finite width $\beta$, with two rows of open indices. It is proportional to the thermal state, $e^{-\beta H}/Z$. (b) By coarse graining the tensor network while leaving the open indices untouched, we obtain a new tensor network with tensors $A'$ together with an upper and lower row of disentanglers and isometries. (c) Further coarse graining produces a thermal MERA.

indices. After $O(\log_2(\beta))$ iterations of the coarse-graining procedure, we obtain a MERA representation of the thermal state, Fig. 2(c), made of $O(\log_2(\beta))$ double layers of disentanglers and isometries for both the incoming and outgoing indices, together with a central row of tensors. Disentanglers and isometries are isometric tensors and thus do not affect the spectrum of eigenvalues of the thermal MERA, which therefore depends exclusively on the central row of tensors [26].

The thermal MERA obtained from TNR resembles the form first suggested by Swingle in the context of holography [13], where a $1+1$ conformal field theory is dual to a gravity theory in three space-time dimensions. In this context, the thermal MERA is interpreted as describing a spacelike cross section of a black hole space-time geometry. A significant difference in our construction is the central row of tensors, which is absent in Swingle's proposal [13] and provides $\rho_\beta$ with the correct thermal spectrum of eigenvalues $\{e^{-\beta E_i}/Z\}$, where $\{E_i\}$ are the eigenvalues of $H$. This central row of tensors can be thought of as representing the Einstein-Rosen bridge connecting the two asymptotic anti–de Sitter regions [27], which seems to provide a manifestation of the ER = EPR conjecture [28].

*Periodic chain of size L.*—In our third construction, we use TNR to coarse grain a tensor network for the ground state of Hamiltonian $H$ on a periodic chain of size $L$; see Fig. 3(a). The network consists of a semi-infinite, vertical

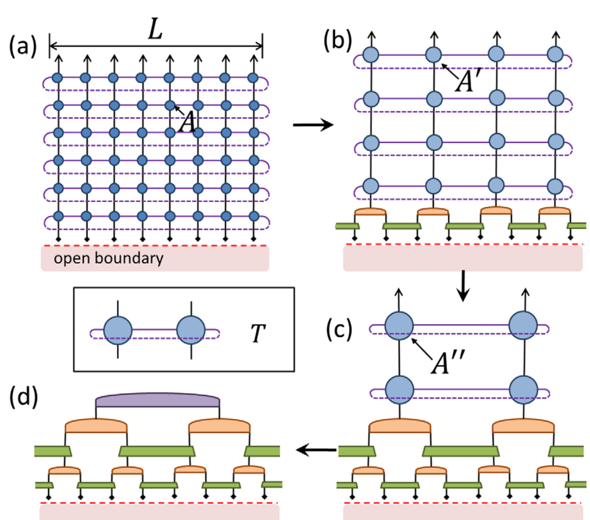


FIG. 3 (color online). (a) Tensor network on a semi-infinite vertical cylinder of finite width $L$ and with a row of open indices, proportional to the ground state of $H$ on a periodic chain made of $L$ sites. (b) Result of coarse graining the initial tensor network while not touching its open indices. (c) MERA connected to a semi-infinite vertical cylinder of $O(1)$ width. Inset: Transfer matrix $T$ of this cylinder. The eigenvectors of $T$ with the largest eigenvalues correspond to the low energy eigenstates of $H$. (d) MERA for the ground state or low energy excited states of $H$, where the top tensor is an eigenvector of the transfer matrix $T$.

cylinder of width $L$, with a row of open indices. After about $O(\log_2(L))$ coarse-graining steps, the size of the system has effectively become $O(1)$, see Fig. 3(c), and we have $O(\log_2(L))$ layers of the MERA connected to a semi-infinite cylinder of $O(1)$ width. This semi-infinite cylinder can be understood as the infinite product of a transfer matrix $T$. The dominant eigenvector of $T$ leads to the ground state of $H$, whereas subdominant eigenvectors describe low energy eigenstates.

To illustrate the computational possibilities offered by the new algorithm, we consider the one-dimensional quantum Ising model with the transverse magnetic field both at finite $\beta$ for an infinite chain, and at zero temperature for a finite periodic chain of length $L$ [29]. The calculation required less than 5 min on a 2.5 GHz dual core laptop with 4 Gbytes of memory (MERA bond dimension $\chi = 10$). First, for $L = \infty$, Fig. 4(a) shows the expectation value of the energy density $E_{\rm thermal} \equiv {\rm tr}(\rho_\beta H)/L$ as a function of the

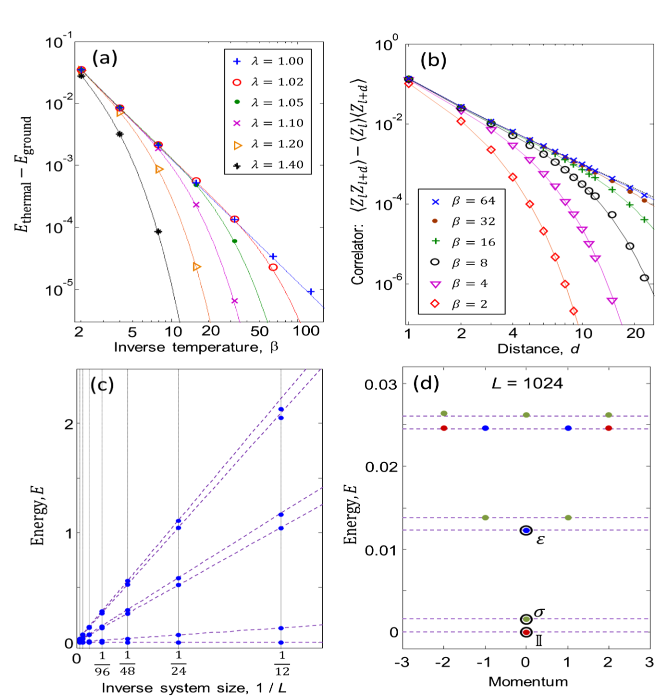


FIG. 4 (color online). (a) Thermal energy per site (above the ground state energy) as a function of the inverse temperature $\beta$, for the quantum Ising model $H = \sum_i X_i X_{i+1} + \lambda \sum_i Z_i$ in an infinite chain, for different values of magnetic field $\lambda$. Continuous lines correspond to the exact solution. (b) Connected two-point correlators at the critical magnetic field $\lambda = 1$, as a function of the distance $d$, for several values of $\beta$. Continuous lines correspond again to the exact solution. (c) Low energy eigenvalues of $H$ for critical $\lambda = 1$ as a function of $1/L$. (d) Low energy spectrum of $H$ for critical $\lambda = 1$ and corresponding momentum (in units of $2\pi/L$) for $L = 1024$ sites, which appear organized according to the conformal towers of the identity $\mathbb{I}$ (red), spin $\sigma$ (green), and energy density $\epsilon$ (blue) primary fields of the Ising conformal field theory, [30]. Discontinuous lines in (c) and (d) correspond to the finite-size conformal field theory prediction, which ignores corrections of order $L^{-2}$.

inverse temperature $\beta$, while Fig. 4(b) displays, at the critical magnetic field, the crossover between the polynomial decay of correlations at short distances (due to quantum fluctuations at criticality) and their exponential decay at longer distances (due to finite temperature statistical fluctuations). Then, for $\beta = \infty$, Figs. 4(c) and 4(d) show, respectively, the low energy spectra of $H$ as a function of the inverse system size $1/L$ and the energy and momentum of low energy states for $L = 1024$. In all cases, an accurate approximation to the exact result is obtained.

*Discussion.*—Since its proposal a decade ago [7], the MERA has been regarded as a variational class of states. Accordingly, one attempts to approximate the ground state $|\Psi\rangle$ of $H$ by optimizing the variational parameters contained in the disentanglers and isometries of the MERA, for instance, by iteratively minimizing the expectation value of $H$ [8,9]. Such energy optimization is costly (it may require thousands of sweeps over scale) and prone to becoming trapped in local minima. Moreover, there is no guarantee that the end result is an approximation to the ground state $|\Psi\rangle$—one just has a wave function with, hopefully, reasonably low energy. Here we have argued that, instead, an approximate MERA representation of the ground state $|\Psi\rangle$ can be obtained with TNR by coarse graining an initial, quasiexact tensor network representation of $|\Psi\rangle$ [31]. This only requires one sweep over scale, and it is therefore computationally much more efficient (see Ref. [18] Sec. D). In addition, at each coarse-graining step TNR introduces a truncation error [15] that can be explicitly computed. If this truncation error is sufficiently small, then one can certify that the resulting MERA approximates the true ground state $|\Psi\rangle$ within that small error [32].

Importantly, TNR acts *locally*: the coarse graining of the tensor network at point $(x, \tau)$ only depends on the tensors in an immediate neighborhood [15]. Let us mention three consequences for the resulting MERA. (i) Since TNR is not aware of the system size $L$ or inverse temperature $\beta$, it produces the same tensors $A, A', A'', \ldots$, and disentanglers and isometries for the ground state $|\Psi\rangle$ in the thermodynamic limit ($L = \beta = \infty$) as it does for the states $\rho_\beta$ and $|\Psi^{(L)}\rangle$ at finite $\beta$ or $L$. Thus, a single TNR calculation produces an accurate MERA approximation for all these states. (ii) In the absence of translation invariance, where a different tensor $A_i$ may be required for each site $i$ of the lattice, the coarse graining of different parts of the system can be conducted in parallel, leading to a massive reduction in computational time. (iii) At a conceptual level, locality of TNR implies the validity of the theory of minimal updates [34]. This theory, which asserts that only certain parts of the MERA need to be changed in order to account for a localized change in the Hamiltonian $H$ [34], is particularly useful in the study of systems with boundaries, defects, or interfaces [9,12].

In summary, in this Letter we have shown that, when applied to the Euclidean path integral restricted to several geometries, TNR [15] produces a MERA for the ground state and thermal states of a quantum Hamiltonian, and have explored a number of consequences of this result. We conclude by briefly mentioning two more implications. First, thanks to this connection, TNR inherits from the MERA its ability to define a RG flow in the space of wave functions and Hamiltonians [6–8]. Notice that extracting the emergent physics from these RG flows should be easier (both conceptually and computationally) than extracting it from the RG flow in the more abstract space of tensors, Eq. (1). Second, although we have focused on quantum systems, TNR can also be applied to statistical partition functions [15]. Therefore the present construction extends the MERA formalism (including strategies to extract universal critical properties [10–12]) to classical statistical systems.

The authors thank Rob Myers for insightful comments. G. V. acknowledges support by the John Templeton Foundation and thanks the Australian Research Council Centre of Excellence for Engineered Quantum Systems. The authors also acknowledge support by the Simons Foundation (Many Electron Collaboration). Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

*gevenbly@uci.edu
†gvidal@perimeterinstitute.ca

[1] L. P. Kadanoff, Physics (Long Island City, N.Y.) **2**, 263 (1966).
[2] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); **4**, 3184 (1971); Rev. Mod. Phys. **47**, 773 (1975).
[3] M. E. Fisher, Rev. Mod. Phys. **70**, 653 (1998).
[4] S. R. White, Phys. Rev. Lett. **69**, 2863 (1992); Phys. Rev. B **48**, 10345 (1993).
[5] M. Levin and C. P. Nave, Phys. Rev. Lett. **99**, 120601 (2007).
[6] G. Vidal, Phys. Rev. Lett. **99**, 220405 (2007).
[7] G. Vidal, Phys. Rev. Lett. **101**, 110501 (2008).
[8] G. Evenbly and G. Vidal, Phys. Rev. B **79**, 144108 (2009).
[9] G. Evenbly and G. Vidal, J. Stat. Phys. **157**, 931 (2014).
[10] V. Giovannetti, S. Montangero, and R. Fazio, Phys. Rev. Lett. **101**, 180503 (2008); R. N. C. Pfeifer, G. Evenbly, and G. Vidal, Phys. Rev. A **79**, 040301(R) (2009); J. C. Bridgeman, A. O'Brien, S. D. Bartlett, and A. C. Doherty, Phys. Rev. B **91**, 165129 (2015).
[11] G. Evenbly and G. Vidal, in *Strongly Correlated Systems: Numerical Methods*, edited by A. Avella and F. Mancini, Springer Series in Solid-State Sciences, Vol. 176 (Springer, New York, 2013), Chap. 4.
[12] G. Evenbly, R. N. C. Pfeifer, V. Pico, S. Iblisdir, L. Tagliacozzo, I. P. McCulloch, and G. Vidal, Phys. Rev. B **82**, 161107(R) (2010); P. Silvi, V. Giovannetti, P. Calabrese, G. E. Santoro, and R. Fazio, J. Stat. Mech. (2010) L03001; Y.-L. Lo, Y.-D. Hsieh, C.-Y. Hou, P. Chen, and Y.-J. Kao, Phys. Rev. B **90**, 235124 (2014).

[13] B. Swingle, Phys. Rev. D **86**, 065007 (2012); H. Matsueda, M. Ishihara, and Y. Hashizume, Phys. Rev. D **87**, 066002 (2013); T. Hartman and J. Maldacena, J. High Energy Phys. 05 (2013) 014; J. Molina-Vilaplana, and J. Prior Gen. Relativ. Gravit., **46**, 1823 (2014); A. Mollabashi, M. Nozaki, S. Ryu, and T. Takayanagi, J. High Energy Phys. 03 (2014) 098.

[14] G. Evenbly and G. Vidal, J. Stat. Phys. **145**, 891 (2011); B. Swingle, arXiv:1209.3304; J. Haegeman, T. J. Osborne, H. Verschelde, and F. Verstraete, Phys. Rev. Lett. **110**, 100402 (2013); F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, J. High Energy Phys. 06 (2015) 149; M. Miyaji and T. Takayanagi, Prog. Theor. Exp. Phys. **2015**, 073B03 (2015); X.-L. Qi, arXiv:1309.6282; N. Bao, C. Cao, S. M. Carroll, A. Chatwin-Davies, N. Hunter-Jones, J. Pollack, and G. N. Remmen, Phys. Rev. D **91**, 125036 (2015).

[15] G. Evenbly and G. Vidal, Phys. Rev. Lett. **115**, 180405 (2015).

[16] We assume translation invariance. For a system that is not invariant under translations, a different tensor $A_i$ may be needed for each site $i$ of the lattice.

[17] Previous proposals for a thermal MERA discussed in Ref. [13] in the context of holography have a flat spectrum of eigenvalues and therefore do not reproduce the correct, finite temperature spectrum of $\rho_\beta$.

[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.115.200401[url], which includes Refs. [19–24], for further details on (a) constructing the tensor network for the Euclidean path integral, (b) four geometries for a Euclidean evolution, (c) tensor network renormalization, (d) TNR versus energy minimization, and (e) TRG yields tree tensor networks.

[19] Suzuki, Phys. Lett. A **146**, 319 (1990); J. Math. Phys. (N.Y.) **32**, 400 (1991).

[20] A. T. Sornborger and E. D. Stewart, Phys. Rev. A **60**, 1956 (1999).

[21] G. Evenbly and G. Vidal, arXiv:1510.00689.

[22] Y. Y. Shi, L. Duan, and G. Vidal, Phys. Rev. A **74**, 022320 (2006); L. Tagliacozzo, G. Evenbly, and G. Vidal, Phys. Rev. B **80**, 235127 (2009).

[23] Z.-Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, Phys. Rev. B **86**, 045139 (2012).

[24] Z.-Y. Xie, H.-C. Jiang, Q.-N. Chen, Z.-Y. Weng, and T. Xiang, Phys. Rev. Lett. **103**, 160601 (2009).

[25] On the plane, the double row of disentanglers and isometries will annihilate with a double row of conjugate disentanglers and isometries corresponding to $\tau < 0$. Instead, on the half upper plane, these tensors remain.

[26] The normalization factor $Z$ corresponds to the trace of the thermal MERA. Since the conjugate layers of disentanglers and isometries do not change this trace, $Z$ is also the trace of the central row of tensors $A^{\max}$, or an infinite power of some (transfer) matrix $M$ obtained from tensor $A^{\max}$ by contracting its vertical indices. A normalized thermal MERA is then obtained by replacing $A^{\max}$ with $A^{\max}/\lambda$, where $\lambda$ is the dominant eigenvalue of $M$.

[27] J. M. Maldacena, J. High Energy Phys. 04 (2003) 021.

[28] J. M. Maldacena and L. Susskind, Fortschr. Phys. **61**, 781 (2013).

[29] The numerical results for finite or infinite values of $\beta$ and $L$ were all actually extracted from a single calculation, where TNR was applied to an infinite system at zero temperature ($\beta = L = \infty$).

[30] See, e.g., Chap. 10 of M. Henkel, *Conformal Invariance and Critical Phenomena* (Springer-Verlag, Berlin, 1999).

[31] The initial tensor network representation of $|\Psi\rangle$ is quasiexact, in that the Suzuki-Trotter decomposition used to produce it (Secs. A and B of Supplemental Material [18]) introduces only small errors, which can be systematically reduced.

[32] We emphasize, however, that TNR is not committed to exclusively producing a MERA. For instance, if at some scale $s$ the ground state approximately decomposes as the product of two ground states, then the corresponding tensor $A^{(s)}$ will approximately factorize as $A_1^{(s)} \otimes A_2^{(s)}$, and TNR will give rise instead to a branching MERA [33], which was proposed to describe, e.g., highly entangled phases of matter in $D \geq 2$ dimensions.

[33] G. Evenbly and G. Vidal, Phys. Rev. Lett. **112**, 220502 (2014); **112**, 240502 (2014).

[34] G. Evenbly and G. Vidal, Phys. Rev. B **91**, 205119 (2015).