Unbounded Violation of Quantum Steering Inequalities

M. Marciniak,[1] A. Rutkowski,[1,2,*] Z. Yin,[3,1] M. Horodecki,[1,2] and R. Horodecki[1,2]

[1]Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland

[2]National Quantum Information Centre of Gdańsk, 81-824 Sopot, Poland

[3]School of Mathematics and Statistics, Wuhan University, Wuhan, China

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We construct steering inequalities that exhibit unbounded violation. The concept was to exploit the relationship between steering violation and the uncertainty relation. To this end, we apply mutually unbiased bases and anticommuting observables, known to exhibit the strongest uncertainty. In both cases, we are able to procure unbounded violations. Our approach is much more constructive and transparent than the operator space theory approach employed to obtain large violation of Bell inequalities. Importantly, using anticommuting observables we are able to obtain a dichotomic steering inequality with unbounded violation. Thus far, there is no analogous result for Bell inequalities. Interestingly, both the dichotomic inequality and one of our inequalities cannot be directly obtained from existing uncertainty relations, which strongly suggest the existence of an unknown kind of uncertainty relation.

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Introduction.—Quantum theory is the primary mainstay of our understanding and formal description of nature. Moreover, it constitutes a perfect empirically confirmed formal construction. Despite many years of continuous attempts, a commonly accepted interpretation of the mathematical formalism of quantum mechanics has not been found. The phenomenon of quantum correlations, especially entanglement, is believed to be extremely amazing and eludes the schemes of classical thinking. Multiannual conceptual efforts to grapple the "spooky actions for spatially separated systems" began with the fundamental work of Einsten, Podolsky, and Rosen (EPR) [1] and continue until this day. Nowadays, we possess the knowledge that quantum correlations—which still remain a great mystery—allow experimental realization. Additionally, they can be controlled and implemented in nontrivial tasks. Secure quantum communication as well as quantum calculations are among them. Such promising perspectives to practically use quantum correlations as a resource clearly demonstrate the importance of the undertaken efforts to improve our deep understanding of this phenomenon.

The concept of quantum steering was first introduced by Schrödinger in 1935 [2] as a generalization of the EPR paradox [1] for bipartite systems in arbitrary pure entangled states and arbitrary measurements by one party. Consider two separated observers sharing entanglement. The first observer, by measurement on his system, can steer the state of the system held by the second observer. Like the debate of the EPR paradox, the notion of quantum steering had been ignored for a long time until it was recovered by Wiseman, Jones, and Doherty [3], where they introduced quantum steering as an information task. Like in the Bell scenario, the nonclassicality revealed by the steering phenomenon is expressed by means of violation of the so-called steering inequalities. It should be noted that not all entangled states lead to steering, and there are states that violate steering inequalities but do not violate any Bell inequality [3,4].

Recently, unbounded violations of Bell inequalities were intensively analyzed, mostly by means of advanced tools of mathematical physics [5–7] as well as communication complexity methods [8,9]. The existing results either are mostly random constructions having their origin in existing knowledge from the field of operator spaces or are derivatives of quite complicated communication complexity protocols.

In this Letter we analyze the unbounded violation of steering inequalities. We exploit an intrinsic relationship of steering phenomenon with the uncertainty principle (see, e.g., Ref. [10]) and apply the measurements that offer strong uncertainty, such as mutually unbiased bases (MUBs) and Clifford observables [11]. Until now it was not known whether quantum-correlation-type steering is equivalent to Bell-type correlations in the regime of large violation. Here, we provide two results that address this issue: (i) using a mutually unbiased basis, we obtain a larger violation than the largest quantum violation of Bell inequalities, and (ii) by means of Clifford observables, we provide unbounded violation of steering inequality with binary outputs—a feature that is still unknown for Bell inequalities with binary outputs for one of the parties.

Our inequalities are extremely simple in comparison to the existing Bell inequalities exhibiting large violation [5,6,12], as well as to random constructions of steering inequalities based on the operator space approach, provided in the companion paper [13]. While one of our violations is a consequence of the existing fine-grained uncertainty principle [14] obtained in Ref. [15], our other results—the unbounded violation for binary observables and a variant of large violation with MUBs—cannot be derived from any existing uncertainty principles.

Steering inequality.—Herein, we will consider the following steering scenario [13,16], which is equivalent to the one in Ref. [3]. Suppose there are two observers (Alice and Bob). Alice can choose among n different measurement settings labeled by $x = 1, \ldots, n.$, each of which can result in one of m outcomes, labeled by $a = 1, \ldots, m$. Suppose the local Hilbert space dimension for Bob is d. The available data are the steered states, they are positive operators on $\mathcal{H}_B$: $\sigma_x^a \geq 0$, and by the no-signaling principle [17], we have that $\mathrm{Tr}(\sum_a \sigma_x^a) = 1$ and it is independent of x. We will denote the set of those operators as $\sigma = \{\sigma_x^a : x = 1, \ldots, n, a = 1, \ldots, m\}$ and call it $(n, m, \dim(\mathcal{H}_B))$ assemblage or simply assemblage. The set of all assemblages will be denoted by $\mathcal{Q}$. It is well known [18,19] that any assemblage σ has a quantum realization; i.e., it can be generated remotely by performing measurements on a subsystem of bipartite quantum states. More precisely, for any assemblage σ, there exists a Hilbert space $\mathcal{H}_A$ such that

$$\sigma_x^a = \mathrm{Tr}_A((E_x^a \otimes \mathbb{1}_B)\rho), \tag{1}$$

for every x and a, where $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a density matrix and $\{E_x^a\}_{a=1}^m \subset B(\mathcal{H}_A)$ [by $B(\mathcal{H})$ we mean the algebra of all bounded linear operators on $\mathcal{H}$] is a positive operator valued measurement on Alice for every x; i.e., $E_x^a \geq 0$ for every x, a, and $\sum_a E_x^a = \mathbb{1}$, for every x.

If the shared state is separable, by measuring its subsystems, one can only generate assemblages that possess a local hidden state model, defined as follows. The assemblage has a local hidden state (LHS) model if there is a finite set of indices Λ, non-negative coefficients q_λ such that $\sum_\lambda q_\lambda = 1$, density matrices σ_λ in $B(\mathcal{H}_B)$ for $\lambda \in \Lambda$, and probability distributions $\{p_\lambda(a|x)\}_a$ for every x and λ [i.e., $p_\lambda(a|x) \geq 0$ and $\sum_a p_\lambda(a|x) = 1$ for every x, λ], such that

$$\sigma_x^a = \sum_{\lambda \in \Lambda} q_\lambda p_\lambda(a|x)\sigma_\lambda, \tag{2}$$

for every x, a. We denote the set of LHS assemblages by $\mathcal{L}$.

As a Bell functional (inequality) can be used to show the incompatibilities between the local hidden variable model and the quantum theory, we can use the steering inequalities [20] to study the difference between the two sets $\mathcal{L}$ and $\mathcal{Q}$. First, let us define a steering inequality in the spirit of Ref. [16]. Let F be some function from $\mathcal{Q}$ assemblages to the real numbers. If $S_{\mathrm{LHS}}(F)$ is the maximum of S over all assemblages that admit LHS models, then $S \leq S_{\mathrm{LHS}}(F)$ is called a steering inequality. Let $S_{\mathcal{Q}}(F)$ be the maximum of F over all assemblages (recall that all assemblages have a quantum realization [18,19]). If $S_{\mathcal{Q}}(F) > S_{\mathrm{LHS}}(F)$, then the steering inequality is called nontrivial; i.e., it can be violated using entangled states. We will consider only the linear functional from the space of assemblages to the real numbers. In other words, we can define the steering functional in the following way: for given natural numbers n, m, and d, we define a steering functional F as a set $\{F_x^a : x = 1, \ldots, n, a = 1, \ldots, m\}$ of $d \times d$ real matrices. For a given assemblage σ, we get a real number

$$\langle F, \sigma \rangle = \mathrm{Tr}\left(\sum_{x=1}^n \sum_{a=1}^m F_x^a \sigma_x^a\right). \tag{3}$$

Additionally, let us define two quantities: for a given steering functional F, we define the LHS bound of F as the number,

$$S_{\mathrm{LHS}}(F) = \sup\{|\langle F, \sigma \rangle| : \sigma \in \mathcal{L}\}, \tag{4}$$

and the quantum bound of F as

$$S_{\mathcal{Q}}(F) = \sup\{|\langle F, \sigma \rangle| : \sigma \in \mathcal{Q}\}. \tag{5}$$

Now we are ready to define the quantum violation of F as the number

$$V(F) = \frac{S_{\mathcal{Q}}(F)}{S_{\mathrm{LHS}}(F)}. \tag{6}$$

A steering functional with large violation will tell us that the sets $\mathcal{L}$ and $\mathcal{Q}$ are prominently different. Apart from the above theoretical aspect, there will be many benefits when we apply it to practical experiments [21,22] and applications [23]. However, for a given Bell or steering functional, it is difficult to calculate its violation. Operator space theory was shown to be a powerful tool to overcome this difficulty. See Refs. [5,6] in the Bell scenario and Ref. [13] in steering. For example, in scenario (d, d, d), the following random steering functional was considered in the companion article [13]:

$$F_x^a = \frac{1}{d} \sum_{k=1}^d \epsilon_{x,a}^k |1\rangle\langle k|, \qquad x, a = 1, \ldots, d, \tag{7}$$

where $\epsilon_{x,a}^k, x, a, k = 1, \ldots, d$ are independent Bernoulli variables. The violation of this inequality is $O[\sqrt{(d/\log d)}]$.

In this Letter, we are able to derive steering functionals by using MUBs and Clifford algebra, More precisely, for the scenario $(d+1, d, d)$, when the dimension of Hilbert space d is equal power of prime number, then we know there exit exactly $d+1$ MUBs. By using MUBs, we can construct a steering functional with unbounded violation of order $O(\sqrt{d})$. It can be seen that we can obtain larger violation compared to the random one. On the other hand, for the scenario $(n, 2, 2^n)$, we are able to find a dichotomic steering functional with unbounded violation of order $O\sqrt{(n/2)} \simeq O(\sqrt{\log d})$ by using Clifford observables. Therefore, this unbounded violation reveals an interesting and particular property of quantum steering.

Unbounded violation: Mutually unbiased bases.—Now we are going to study a steering functional constructed by means of mutually unbiased bases [24]. Let $M_1 = \{|\phi_1^a\rangle : a = 1, \ldots, d\}$ and $M_2 = \{|\phi_2^a\rangle : a = 1, \ldots, d\}$ be orthonormal bases in the d-dimensional Hilbert space. Then they are said to be mutually unbiased if $|\langle\phi_1^a|\phi_2^b\rangle| = 1/\sqrt{d}$ for all $a, b = 1, \ldots, d$. A set $M = \{M_x : x = 1, \ldots n\}$ of orthonormal bases of $\mathbb{C}^d$ is said to be a set of MUBs if M_x and M_y are mutually unbiased for every $x \neq y$.

Given MUBs M, we define the steering functional $F = \{F_x^a\}$, where

$$F_x^a = |\phi_x^a\rangle\langle\phi_x^a|, \qquad x = 1, \ldots, n, a = 1, \ldots, d. \tag{8}$$

Our aim is to calculate $V(F)$. First, we would like to estimate the quantity $S_Q(F)$. We propose the following. ***Lemma 1.***—Let F be the steering inequality defined in Eq. (8). Then

$$S_Q(F) = n, \tag{9}$$

and the maximal value is attained on the maximally entangled state.

Proof of this lemma is in Appendix A in Supplemental Material [25]. To estimate the bound of S_{LHS}, we use the operator norm estimation of some operator (see Appendix A in Supplemental Material [25]). Comparing this result with Lemma 1, we are ready to formulate one of the main results.

Theorem 1.—If F is a steering functional determined by MUBs as in Eq. (8), then we have

$$V(F) \geq \frac{n\sqrt{d}}{n + 1 + \sqrt{d}}. \tag{10}$$

Proof of this theorem is in Appendix A in Supplemental Material [25]. If the dimension d is an integer power of a prime number, then we can always find $d + 1$ MUBs [24]. In this case $n = d + 1$; hence, we can find a steering functional F, with violation $O(\sqrt{d})$. It is better than the random one in the sense that it has a higher order of violation.

Our result is connected to the results obtained in Ref. [14]. In that paper the authors revealed that "nonlocality of quantum mechanics and Heisenberg's uncertainty principle are inextricably and quantitatively linked." They introduced a notion named "fine-grained uncertainty relations" to characterize the "amount of uncertainty" in a particular physical theory. For the given set of measurements A_x, with $x = 1, \ldots, n$ and the set of outputs $\vec{a} = \{a(x) : x = 1, \ldots, n\}$, consider the following quantity introduced in Ref. [14];

$$\xi_{\vec{a}} = \max_{\rho}\left\{\sum_{x=1}^{n} p_x p(a(x)|x)_{\rho}\right\}, \tag{11}$$

where $\{p_x\}$ is a probability distribution given *a priori* and $p(a(x))|x)_\rho$ is the probability of $a(x)$ when we measure x. This quantity forms a fine-grained uncertainty relation for this set of measurement settings. For the noncommuting observables, this quantity is bounded by 1. In Ref. [15], the author considered a special fine-grained uncertainty relation of MUBs by letting $p_x = 1/n$, for every x. An upper bound of $\xi_{\vec{a}}$ was obtained for all possible strings $\vec{a}$. Namely, we have the following. *Proposition 1* (Ref. 15).—Let $\mathcal{M} = \{M_x : x = 1, \ldots, n\}$ be a set of MUBs in a d-dimensional Hilbert space. For an arbitrary density matrix ρ, we have

$$\frac{1}{n}\sum_{x=1}^{n}\mathrm{Tr}(|\phi_x^a\rangle\langle\phi_x^a|\rho) \leq \frac{1}{d}\left(1 + \frac{d-1}{\sqrt{n}}\right), \quad \forall\ a = 1, \ldots, d. \tag{12}$$

Therefore, $\xi_{\vec{a}} \leq \frac{1}{d}\{1 + [(d-1)/\sqrt{n}]\}$, where we have chosen $p_x = 1/n$.

Using the above proposition, we can obtain an alternative violation of the steering inequality defined by the steering functional F [see Eq. (8)]. To end with, let us consider the LHS bound first. Assume that $\sigma \in \mathcal{L}$, then

$$\begin{aligned}\langle F, \sigma\rangle &= \sum_{x=1}^{n}\sum_{a=1}^{d}\mathrm{Tr}\left(|\phi_x^a\rangle\langle\phi_x^a|\sum_{\lambda}q_\lambda p_\lambda(a|x)\sigma_\lambda\right)\\ &\leq \sum_{\lambda}q_\lambda\sum_{x=1}^{n}(\sup_a\mathrm{Tr}(|\phi_x^a\rangle\langle\phi_x^a|\sigma_\lambda))\left(\sum_{a=1}^{d}p_\lambda(a|x)\right)\\ &\leq n\sup_a\xi_{\vec{a}} \leq \frac{n}{d}\left(1 + \frac{d-1}{\sqrt{n}}\right).\end{aligned} \tag{13}$$

Since the above inequality holds for any $\sigma \in \mathcal{L}$, we get

$$S_{\mathrm{LHS}}(F) \leq \frac{n}{d}\left(1 + \frac{d-1}{\sqrt{n}}\right). \tag{14}$$

Furthermore, $S_Q(F) = n$ by Lemma 1. Thus, we get the following lower bound:

$$V(F) \geq \frac{d\sqrt{n}}{\sqrt{n} + d - 1}. \tag{15}$$

Still, if the dimension d is an integer power of a prime number, the violation is lower bounded by $O(\sqrt{d})$, which coincides with the result of Theorem 1. The authors of Ref. [26] conjectured that the MUBs will give the most uncertain measurement results for the special uncertainty relations considered in the same article.

Unbounded violation: Clifford observables.—Now we will focus on the dichotomic case, where there are only two outcomes for each input setting. Let us consider operators $A_i \in \mathcal{B}(\mathbb{C}^{2^n})$, $i = 1, 2, \ldots, 2^n$, with the following properties: (i) $A_i^\dagger = A_i$, (ii) $A_iA_j + A_jA_i = 2\delta_{ij}\mathbb{1}_{2^n}$, and (iii) the set $\mathcal{A} = \{A_i, i = 1, \ldots, 2^n\}$ forms a linear basis of $\mathcal{B}(\mathbb{C}^{2^n})$.

The algebra that is generated by these A_i's is the Clifford algebra. A representation of this algebra can be constructed by tensor products of Pauli matrices [27]. Now choose arbitrary n operators $A_x, x = 1, \ldots, n$ from the set $\mathcal{A}$. We will consider the following projectors $P_x^a\colon x = 1, \ldots, n, a = 1, 2$, where

$$P_x^1 = \frac{1}{2}(\mathbb{1} + A_x), \qquad P_x^2 = \frac{1}{2}(\mathbb{1} - A_x), \qquad x = 1, \ldots, n. \tag{16}$$

By the above projectors, we can define a steering functional $F = \{F_x^a = P_x^a - \mathbb{1}/2\colon x = 1, \ldots, n, a = 1, 2\}$; i.e.,

$$F_x^1 = \frac{1}{2}A_x, \qquad F_x^2 = -\frac{1}{2}A_x, \qquad x = 1, \ldots, n. \tag{17}$$

As before, first, we would like to estimate the quantity $S_Q(F)$; direct calculation shows that $S_Q(F) = n/2$. And as before, to estimate the bound of $S_{\mathrm{LHS}}(F)$ we use the operator norm estimation (see Appendix B in Supplemental Material [25]).

Theorem 2.—If F is a steering functional defined in Eq. (17), then we have

$$V(F) \geq \sqrt{\frac{n}{2}}. \tag{18}$$

The proof of the above theorem is in Appendix B in Supplemental Material [25].

There is an alternative way to explain this unbounded violation. By using the notion in Ref. [28], we can define a traceless operator F_x corresponding to P_x^a as $F_x = P_x^1 - P_x^2 = A_x$. On the other hand, if we only consider projective measurement, we can define a dichotomic assemblage $\sigma_x = \sigma_x^1 - \sigma_x^2$. Hence, if the LHS model exists, then

$$\sigma_x = \sum_\lambda p_\lambda I(x, \lambda)\sigma_\lambda, \tag{19}$$

where $I(x, \lambda) = p(1|x, \lambda) - p(2|x, \lambda) \in [-1, 1]$. So we can define a dichotomic steering functional F^{dicho} as

$$|\langle F^{\mathrm{dicho}}, \sigma\rangle| = \mathrm{Tr}\left(\sum_{x=1}^n F_x\sigma_x\right). \tag{20}$$

The quantum and the LHS bound can be similarly defined as before. The following corollary holds.

Corollary 1.—Let F^{dicho} be the dichotomic steering functional corresponding to the one in Theorem 2, i.e., $F^{\mathrm{dicho}} = \{A_x\colon x = 1, \ldots, n\}$, then

$$V(F^{\mathrm{dicho}}) \geq \sqrt{\frac{n}{2}}. \tag{21}$$

The proof is the same as the proof in Appendix B in Supplemental Material [25]. We have

$$S_{\mathrm{LHS}}(F^{\mathrm{dicho}}) \leq \sup_\lambda \left\| \sum_{x=1}^n I(x, \lambda)A_x \right\|_\infty \leq \sqrt{2n}. \tag{22}$$

For the quantum bound, we use the dichotomic assemblage $\sigma_x = (1/2^n)A_x$. Thus, $S_Q(F^{\mathrm{dicho}}) = n$.

Conclusions.—In this Letter, we have provided two steering inequalities with the unbounded violation. One is derived from MUBs with violation $O(\sqrt{d})$ in the scheme $(d+1, d, d)$, where d is an integer power of the prime number. We obtain this result using a much simpler method than the operator space theory approach. Interestingly, this violation is connected to the fine-grained uncertainty relations for MUBs. The question, do stronger uncertainty relations exist?, appeared here naturally. Another is constructed by using the basis of Clifford algebra with violation $O(\sqrt{n/2}) \simeq O(\sqrt{\log d})$ in the scheme $(n, 2, 2^n = d)$. Our result shows an interesting property of quantum steering, since there does not exist a bipartite-correlation-type Bell inequality with unbounded violation [28–30]. The mathematical reason for our unbounded violation was explained in a companion paper [13], by means of the operator space theory. It shows a different property in quantum steering compared to Bell nonlocality. The question, is there large violation in Bell-type correlation in dichotomic case?, seems to be a natural conclusion of this result. Recently, the positive answer for the above question was found [31]. It would be most intriguing and interesting to find the fine-grained uncertainty relations using anticommuting observables as a follow-up to this work. We hope that the results we obtained will allow a better understanding of correlations that exist in quantum systems.

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*Corresponding author.
fizar@ug.edu.pl

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