

Emergent Supersymmetry from Strongly Interacting Majorana Zero Modes

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We show that a strongly interacting chain of Majorana zero modes exhibits a supersymmetric quantum critical point corresponding to the $c = \frac{7}{10}$ tricritical Ising model, which separates a critical phase in the Ising universality class from a supersymmetric massive phase. We verify our predictions with numerical density-matrix-renormalization-group computations and determine the consequences for tunneling experiments.

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Supersymmetry (SUSY) is a theoretical framework, which predicts a fermionic superpartner for every bosonic elementary particle, explaining long-standing puzzles (see, e.g., Ref. [1] and the references therein). The experimental verification of SUSY, however, has remained elusive. Another important milestone of physics is Majorana's prediction of fermionic particles that are their own antiparticle. Neutrinos were Majorana's original candidate for these so-called Majorana fermions, but their experimental status remains unclear [2,3].

While both Majorana fermions and SUSY are yet to be observed in high energy physics, Majorana fermions are predicted to emerge as *collective* excitations in many-body systems of electrons [4–11]. Of particular interest are unpaired Majorana fermions, localized to topological defects such as vortices or domain walls, that occur at zero energy. These Majorana zero modes (MZMs) provide a promising candidate platform for topological quantum computing [7]. There has been significant recent experimental progress toward the detection of the condensed-matter incarnations of MZMs [12–18]. Strongly interacting Majorana fermions may serve as building blocks for novel phases of matter, which remain relatively unexplored [19–24].

On the other hand, there have been few works on the realization of SUSY in condensed matter physics [25–31]. The canonical example of emergent SUSY in statistical physics is the tricritical Ising (TCI) model in $1+1$ dimensions [32–34]. This model is the second simplest unitary minimal conformal field theory (CFT) in $1+1$ dimensions. It has central charge $c = \frac{7}{10}$ (Ising model with $c = \frac{1}{2}$ being the simplest). It is also the only such CFT that exhibits SUSY.

Similar to the Ising model, the TCI CFT has two realizations [32]: (i) spin models such as the Blume-Capel model [35–37] in which all local operators are bosonic, and (ii) fermionic models in which both fermionic and bosonic local operators are present. Systems with Majorana fermions as local degrees of freedom provide promising candidates for realizing the fermionic models of

TCI CFT [38]. The local operator content of the TCI CFT, which determines the experimentally accessible correlation functions, is directly related to the finite-size spectrum with periodic (antiperiodic) boundary conditions for the spin (fermionic) model.

In the TCI CFT, there is only one relevant operator allowed by symmetry (independent of the realization), which must be fine-tuned to zero. This relevant operator can destabilize the TCI CFT, causing a phase transition either to a doubly degenerate gapped phase or the Ising CFT phase (depending on the sign of the corresponding coupling constant). This operator preserves the supersymmetry of the Hamiltonian [32]. SUSY is thus preserved in the gapped phase but spontaneously broken in the gapless Ising phase [39]. While nonsupersymmetric irrelevant operators could perturb the supersymmetric spectrum deep inside the gapped phase, in the vicinity of the critical point, the gapped phase is supersymmetric, with possible experimental signatures in tunneling experiments.

Finding experimental realizations of the TCI CFT is of great interest. Important progress was made recently by constructing models of coupled bosonic degrees of freedom and fermionic Majorana modes, with the intuition that these local bosonic degrees of freedom can serve as superpartners to the Majoranas [40]. Purely fermionic field theories can also give rise to the TCI CFT [38,41]. Here we show that indeed the simplest lattice model of interacting Majorana fermions, which may be realized in a superconducting vortex lattice and has clear experimental signatures in scanning tunneling microscopy (STM) experiments, gives rise to the TCI CFT and emergent SUSY. Similar to Ref. [40], our model has the advantage that the TCI critical point can be reached by tuning only *one* parameter (thanks to translation invariance of the model). The two models, however, differ in an important aspect. The model of Ref. [40] can be thought of as a model of interacting Majoranas upon integrating out the bosons. However, at the critical point, these boson-mediated interactions have a long-range character, whereas our interactions are strictly local.

We begin by writing the Hamiltonian

$$H = it \sum_j \gamma_j \gamma_{j+1} + g \sum_j \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3}, \quad (1)$$

where $\gamma_j = \gamma_j^\dagger$ (with $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$) is the annihilation (and creation) operator for a MZM at position j in a one-dimensional lattice. This model can describe a vortex lattice in a narrow strip of a two dimensional topological superconductor [23]. Throughout this Letter we focus on $g > 0$, for which supersymmetric phases emerge (setting $g = 1$ without loss of generality) [42]. Positive g corresponds to attractive interactions between the underlying Dirac fermions, which may indeed appear in the presence of superconductivity. The model exhibits a rich and complex phase diagram for $g < 0$, which is discussed elsewhere [43,44].

Our main result is the phase diagram shown in Fig. 1. For $|t| > t_c$, the system is described by the Ising CFT with central charge $c = \frac{1}{2}$, while for $0 < |t| < t_c$, we have a gapped phase with broken symmetry. At $|t| = t_c$ the system realizes the $c = \frac{7}{10}$ TCI model. As we will see, t_c is extremely small (relative to the interaction strength g). However, the regime of strong interactions is accessible in experiments due to a chiral symmetry (at chemical potential $\mu = 0$ in vortex realization of Majoranas), which forbids hopping processes $it\gamma_j\gamma_{j'}$ [23,45]. By tuning μ , we can then make t arbitrarily small without changing the interactions.

It is convenient [46] for the analysis of the problem to break the translation invariance of the system and write a more general Hamiltonian

$$H = it_1 \sum_j \alpha_j \beta_j + it_2 \sum_j \beta_j \alpha_{j+1} + g_1 \sum_j \alpha_j \beta_j \alpha_{j+1} \beta_{j+1} + g_2 \sum_j \beta_j \alpha_{j+1} \beta_{j+1} \alpha_{j+2}, \quad (2)$$

where $\alpha_j \equiv \gamma_{2j}$ and $\beta_j \equiv \gamma_{2j+1}$. Each pair of Majoranas can be written in terms of one Dirac fermion $c_j = (\alpha_j + i\beta_j)/2$. The product of two Majoranas is then related to the occupation number of a Dirac fermion through $i\alpha_j\beta_j = 2n_j - 1$, where $n_j = c_j^\dagger c_j$. Note that the pairing of MZMs into Dirac fermions is rather arbitrary and we could have introduced another set of Dirac fermions $d_j = (\beta_j + i\alpha_{j+1})/2$.

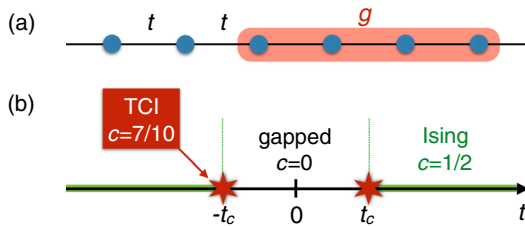


FIG. 1 (color online). (a) The model with nearest-neighbor hopping and interactions between four nearest Majoranas. (b) The phase diagram of Hamiltonian (1) for $g = 1$ as a function of t .

In the limit of $t \rightarrow \infty$ ($g/t \rightarrow 0$), we see from Eq. (1) that the system is described by a free massless Majorana theory corresponding to the critical phase of the transverse field Ising model:

$$H_I \approx iv \int dx (\gamma_R \partial_x \gamma_R - \gamma_L \partial_x \gamma_L), \quad (3)$$

where $\alpha = 2(\gamma_R + \gamma_L)$, $\beta = 2(\gamma_R - \gamma_L)$, and v is a velocity related to the renormalized hopping t . No mass term $im \int dx \gamma_R \gamma_L$ is present if we have translation symmetry. To incorporate the interactions into the effective theory, we Taylor expand the Majorana fields and obtain $H_g = -256g \int dx \gamma_L (\partial_x \gamma_L) \gamma_R (\partial_x \gamma_R) + \dots$, where the dots indicate terms of third and higher order in derivatives. From simple power counting, we find that the perturbation above is irrelevant in the renormalization-group sense. This implies that the Ising critical phase should extend *at least* to a finite g/t [41].

Because of the large value of g/t_c , the strong coupling limit of the Hamiltonian provides a good qualitative understanding of the gapped phase. It can be understood in terms of the occupation numbers of the c and d Dirac fermions as shown in Fig. 2. First, we consider the case of $t = 0$. A dominant positive g_1 (g_2) gives ferromagnetic [all empty or all occupied] states for the occupation number of the c (d) fermions. The phase transition at $t = 0$ and $g_1 = g_2$ (between phases with dominant g_1 and g_2) is expected to be *first order* so (i) all of these ferromagnetic states are present in the ground state manifold of Hamiltonian (1) for $t = 0$ and (ii) there is a gap to excitations. As discussed in the Supplemental Material [47], the presence of a first-order transition for $g_1 = g_2$ and $t = 0$ follows from the connection of our model to spin chains [49] and, in particular, a generalized Ising spin chain with multispin interactions, which bears a strong similarity to the eight-state Potts model [50–53]. Despite being a small perturbation, the hopping terms (depending on their sign) lift the degeneracy between the all-occupied and all-empty states leading to a doubly degenerate (instead of fourfold) gapped phase for $0 < |t| \ll g$.

The analytical arguments for a gapped phase at strong coupling are not rigorous. However, if the strong coupling

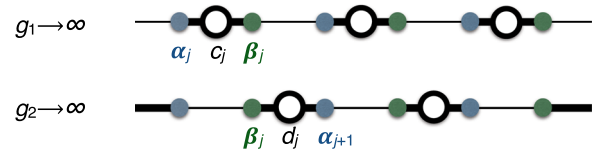


FIG. 2 (color online). The symmetry broken states at strong coupling. The thick lines indicate pairs of Majorana operators combined to form Dirac operators. The empty circles symbolize that the resulting Dirac levels are empty. For both of the states shown, there is another degenerate state, where all Dirac levels are filled.

phase is a doubly degenerate gapped phase, and assuming there is only one phase transition between the Ising phase and this gapped phase, then the TCI CFT is the most natural theory of this phase transition [47]. The $c = 1/2$ Ising ($c = 0$ gapped) phase can be thought of as the line of a 2nd-order (1st-order) transition at $t_1 = t_2$ ($g_1 = g_2$) in a regime dominated by hopping (interaction). Similar to the Ising model with vacancies [54], the critical point, at which the 2nd-order transition changes to the 1st-order transition, naturally corresponds (at the mean-field level) to the Φ^6 Landau-Ginzburg theory at its tricritical point, whose universality class is given by the TCI CFT.

This scenario needs numerical verification. As discussed below, we found that the picture is indeed correct and the value of t_c in the phase diagram of Fig. 1 is $t_c/g = 0.00405$. This small value in turn implies a gapped phase whose shortest correlation length is thousands of lattice sites. Establishing the nature of the phases and determining the value of t_c is therefore exceedingly challenging with most numerical diagnostics such as extrapolation of gaps and order parameters as well as entanglement entropy. Despite this, we found that universal ratios in the finite-size spectrum provide a powerful numerical diagnostic for determining the phase diagram even though the system sizes we are able to reach are significantly smaller than the correlation length of the gapped phase in our model.

Our evidence for the fermionic TCI CFT is the excellent agreement between the theoretical predictions for several universal ratios at the critical point shown in Table I, and the numerically computed values of these ratios.

The subscripts A and P , respectively, indicate antiperiodic (APBC) and periodic (PBC) boundary conditions, E_0 and E_1 represent the energy of the ground state and the first excited state in a given fermion-parity sector (denoted by the superscripts even and odd), and ϵ_0 is the thermodynamic-limit energy density in the ground state. The fourth universal ratio we use provides direct access to central charge c through the general finite-size dependence of the ground-state energy (which is in the even parity sector for APBC): $E_{A,0}^{\text{even}} = \epsilon_0 L - (2\pi v/L)(c/12)$, where L is the length of the system. [55].

We briefly outline the derivation of the above results based on the relationship between the operator content and the finite-size spectrum of the two CFTs. The results can be obtained from the formalism developed in Refs. [56,57] as discussed in the Supplemental Material [47]. We start with the Ising model, for which the predictions can be verified exactly in a free-fermion model. The Ising CFT has three

primary fields $\mathbb{1}$ (identity), σ (spin), and ϵ (energy) with conformal dimensions $h = \bar{h} = 0, \frac{1}{16}, \frac{1}{2}$, respectively [a field with conformal dimension (h, \bar{h}) has scaling correlators $\langle \phi_{h,\bar{h}}(x, t) \phi_{h,\bar{h}}(0, 0) \rangle = (x - vt)^{-2h} (x + vt)^{-2\bar{h}}$]. An integer (half-odd-integer) conformal spin $h - \bar{h}$ corresponds to a bosonic (fermionic) excitation. Now in a fermionic theory, we have APBC in the imaginary time direction, which implies that modular invariance [58] can be most easily satisfied if we also impose APBC in the spatial direction. The analog of the fermionic model in the Ising case is the free-Majorana model of Eq. (3), which has the conformal towers $(\mathbb{1}, \mathbb{1})$, $(\mathbb{1}, \epsilon)$, $(\epsilon, \mathbb{1})$, and (ϵ, ϵ) with APBC (due to modular invariance [58] as shown in the Supplemental Material [47]), while the analog of the spin model has only diagonal conformal towers with bosonic excitations: $(\mathbb{1}, \mathbb{1})$, (σ, σ) , and (ϵ, ϵ) .

In the fermionic model with APBC, the ground state and the first excited state of the even parity sector have operator content $(\mathbb{1}, \mathbb{1})$ and (ϵ, ϵ) , respectively (note that the conformal spin vanishes implying even fermion parity). Similarly, the ground state in the odd parity sector is doubly degenerate with operator content $(\mathbb{1}, \epsilon)$ and $(\epsilon, \mathbb{1})$. For the Ising CFT, we then obtain $E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}} = (2\pi v/L)(\frac{1}{2} + \frac{1}{2})$ and $E_{A,0}^{\text{odd}} - E_{A,0}^{\text{even}} = (2\pi v/L)(0 + \frac{1}{2})$, which lead to the first universal ratio shown in Table I.

The TCI model has 6 primary fields $\mathbb{1}$, ϵ , ϵ' , ϵ'' , σ , and σ' , with scaling dimensions $h = \bar{h} = 0, \frac{1}{10}, \frac{3}{5}, \frac{3}{2}, \frac{3}{80}, \frac{7}{16}$, respectively. Similar to the Ising case, we have $E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}} = (2\pi v/L)(\frac{1}{10} + \frac{1}{10})$. However, in this case, $(\mathbb{1}, \epsilon)$ does not appear in the modularly invariant conformal towers of the fermionic realization of TCI CFT (notice that a conformal spin of $\frac{1}{10}$ is neither an integer nor half integer). Here, the spin model also has 6 diagonal conformal towers, while, as shown in the Supplemental Material [47], the fermion model has 8 such towers with APBC, which include (ϵ, ϵ') , corresponding to the ground state of the odd sector with APBC [47]. This has a conformal spin $\frac{3}{5} - \frac{1}{10} = \frac{1}{2}$. We then find $E_{A,0}^{\text{odd}} - E_{A,0}^{\text{even}} = (2\pi v/L)(\frac{3}{5} + \frac{1}{10})$, leading to the first universal ratio in Table I. The spectrum with periodic boundary conditions is a bit more involved but can be similarly derived using CFT methods (see Supplemental Material [47]).

We numerically computed the four gap ratios above with the density matrix renormalization group (DMRG) method. For the Ising and TCI CFT, the ratios above exhibit remarkable independence from the system size for large enough systems. On the other hand, in the gapped phase ($c = 0$), at least one of the above ratios, namely, $[(E_{A,0}^{\text{odd}} - E_{A,0}^{\text{even}})/(E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}})]$ grows with system size for large enough systems (it has a linear dependence on system size for $t = 0$ as shown in Fig. 3). This gap ratio was used to detect the value of t_c at the tricritical point. It plateaus at $7/2$ for t_c and approaches the Ising value of $1/2$ for larger t .

TABLE I. Several universal ratios in the Ising and TCI CFTs.

CFT	c	$\frac{E_{A,0}^{\text{odd}} - E_{A,0}^{\text{even}}}{E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}}}$	$\frac{E_{P,0}^{\text{even}} - E_{A,0}^{\text{even}}}{E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}}}$	$\frac{E_{P,1}^{\text{even}} - E_{A,0}^{\text{even}}}{E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}}}$	$\frac{E_{A,0}^{\text{even}} - \epsilon_0 L}{E_{A,1}^{\text{even}} - E_{A,0}^{\text{even}}}$
Ising	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
TCI	$\frac{7}{10}$	$\frac{7}{2}$	$\frac{3}{8}$	$\frac{35}{8}$	$\frac{7}{24}$

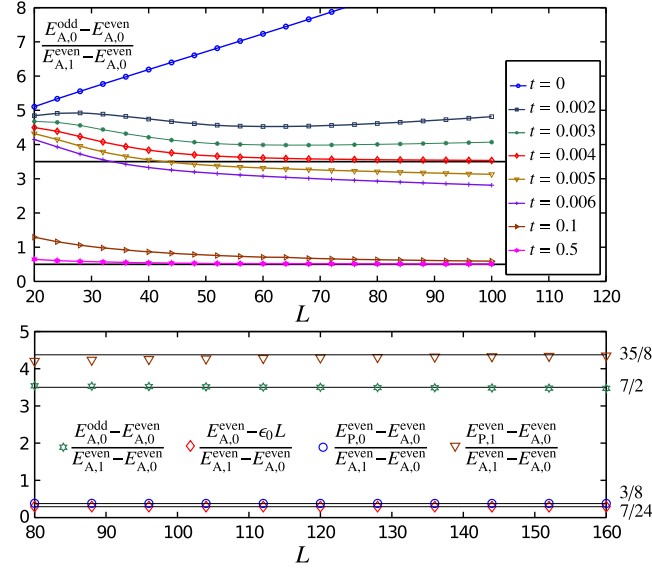


FIG. 3 (color online). Top: The detection of the critical point through one of the universal ratios. Bottom: The values of four universal gap ratios at the tricritical point $t/g = 0.00405$ as a function of L (for systems with $2L$ Majoranas). The numerical results (data points) show excellent agreement with the CFT predictions (black lines at $7/2$, $7/24$, $3/8$, and $35/8$). The DMRG truncation errors are insignificant (up to 1000 states were kept in the computations).

Having found the value of t_c , we then tested all four gap ratios for larger systems (see the bottom panel) and found excellent agreement with the theoretical predictions as seen in Fig. 3.

We finally discuss the experimental signatures of the TCI CFT. STM provides a powerful tool to probe local density of states. Tunneling into MZMs can effectively probe the critical exponent of the fermionic Green's function. We start with the Ising CFT (3). The nonvanishing fermionic correlators are $\langle \gamma_{L,R}(t, x), \gamma_{L,R}(0, 0) \rangle = \{i/[4\pi(vt \pm x + i\delta)]\}$, where δ is a positive infinitesimal number. The equal-time fermionic Green's function then decays as $1/x$. A closely related quantity is the tunneling current from an STM tip into a MZM, which goes as $I_1 \propto V$, where V is the bias voltage.

In the TCI case, on the other hand, the leading fermionic operator χ corresponds to (ϵ, ϵ') with $(h, \bar{h}) = (3/5, 1/10)$, which gives $\langle \chi(t, x)\chi(0, 0) \rangle = (i/[2\pi(vt - x + i\delta)][(vt + i\delta)^2 - x^2]^{1/5})$, leading to equal-time Green's functions, which decay as $|x|^{-7/5}$. The tunneling current then goes as [47]

$$I_{\text{TCI}} \propto \text{sgn}(V)|V|^{7/5}. \quad (4)$$

In Fig. 4, we show the scaling behavior of the equal-time Green's function of our model computed for TCI ($t = t_c$) and Ising ($t = 10^3 \sim \infty$), where the predicted exponents are easily observed ($g = 1$ in both cases). The only relevant

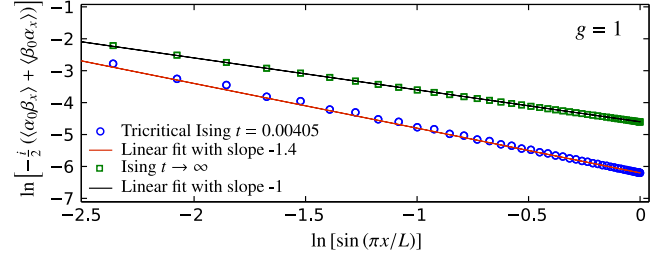


FIG. 4 (color online). The scaling behavior of the fermionic Green's function (obtained with DMRG keeping 800 states) for the Ising and the TCI CFTs observed in our microscopic model.

operator (that induces a transition to the gapped phase from the TCI critical point) is (ϵ', ϵ') with dimension $3/5 + 3/5 = 6/5$. We then expect a gap in the symmetry-broken phase that scales as $(t_c - t)^{5/4}$ near the critical point [47].

As mentioned before, an important property of the relevant (ϵ', ϵ') operator is that it is supersymmetric [38,41,57]. Therefore the SUSY of the critical point should extend into the gapped phase at least in the vicinity of the critical point. In the gapped phase, the power-law dependence of the tunneling current on V changes to exponential dependence, from which the gap to the leading fermionic excitation can be extracted. SUSY implies that the leading bosonic excitation has the same gap as the leading fermionic one. It should be possible to experimentally determine this bosonic gap from Cooper-pair tunneling via a superconducting tip or other bosonic probes such as coupling to photons or phonons.

Considering the effect of disorder on the rich physics of interacting Majoranas adds a new dimension to the problem [59]: a recent manuscript, which appeared shortly after the present Letter, examines the effects [43]. Our theory applies to a translationally invariant system. Experimentally, it is common to form Abrikosov vortex lattices with translation invariance due to energetic reasons. Spontaneous dimerization of the vortex lattice might occur, which would indeed gap the system and destroy the TCI point. However, even in that case, if the dimerization is weak, some signatures of the critical point survive in a crossover regime.

In summary, we studied the phase diagram of the simplest model of strongly interacting Majorana zero modes in one dimension for attractive underlying interactions (which may be realized in the presence of superconductivity). Supported by extensive numerical calculations, we demonstrated that our model provides one of the few examples of emergent spacetime SUSY in condensed matter physics and the first lattice model realization of TCI SUSY in a purely fermionic system with local interactions (a different type of SUSY has been predicted in a lattice model with six-fermion interactions [25]). The vortex lattice experimental realization of our model fosters the thus far elusive observation of SUSY, with clear

signatures in the behavior of the tunneling current into a Majorana mode.

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