Self-Ordered Limit Cycles, Chaos, and Phase Slippage with a Superfluid inside an Optical Resonator

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We study dynamical phases of a driven Bose-Einstein condensate coupled to the light field of a high-Q optical cavity. For high field seeking atoms at red detuning the system is known to show a transition from a spatially homogeneous steady state to a self-ordered regular lattice exhibiting superradiant scattering into the cavity. For blue atom pump detuning the particles are repelled from the maxima of the light-induced optical potential suppressing scattering. We show that this generates a new dynamical instability of the self-ordered phase, leading to the appearance of self-ordered stable limit cycles characterized by large amplitude self-sustained oscillations of both the condensate density and cavity field. The limit cycles evolve into chaotic behavior by period doubling. Large amplitude oscillations of the condensate are accompanied by phase slippage through soliton nucleation at a rate that increases in the chaotic regime. Different from a superfluid in a closed setup, this driven dissipative superfluid is not destroyed by the proliferation of solitons since kinetic energy is removed through cavity losses.

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The experimental realization of strong collective coupling between a large number of (ultra)cold atoms and the electromagnetic field of Fabry-Perot cavities [1–8], nanophotonic fibers [9–11], or photonic crystals [12] opens up new interesting routes both in the field of quantum optics and condensed matter physics. From the latter point of view, the interesting new ingredient is provided by very well controllable strong long-range photon-mediated atom-atom interactions appearing due to the backaction of even a single atom onto the light field. Strong long-range interactions can indeed lead to several intriguing phenomena [13,14] and are crucial in many intensively explored condensed matter phases, like supersolids [15] or topological states [16]. What is more, differently from typical condensed matter situations, these light-mediated interactions are in general (i) retarded, since the photon field owns intrinsic time scales that can be made comparable with atomic scales, and (ii) nonconservative, since the system is typically driven and dissipates energy through atomic spontaneous emission and photon losses through the cavity mirrors.

A striking consequence of cavity field mediated interactions is the appearance of self-ordered phases [17–24] where the particles break a translation symmetry by forming a spatial pattern determined by a characteristic interaction length scale. This phenomenon has been observed experimentally both for a thermal gas [1,6] and an ultracold Bose-Einstein condensate (BEC) [5,7] coupled to a standing-wave mode of an optical cavity as sketched in Fig. 1. In the regimes considered so far, a thermal gas and a BEC share the same qualitative behavior [25]. The self-ordering of a BEC can be closely mapped to the superradiant transition of the famous Dicke model of N two-state atoms coupled to a single cavity mode [5,26,27].

In this work, we study a new regime of quantum gas cavity QED [28], where the self-ordering phase transition is tight to dynamical instabilities. This novel behavior appears upon a rather innocent looking change of operating conditions, namely, by choosing the frequency of the driving laser larger than the atomic internal transition frequency (blue detuning). In this regime the atoms are low field seekers as opposed to high field seekers in the typically

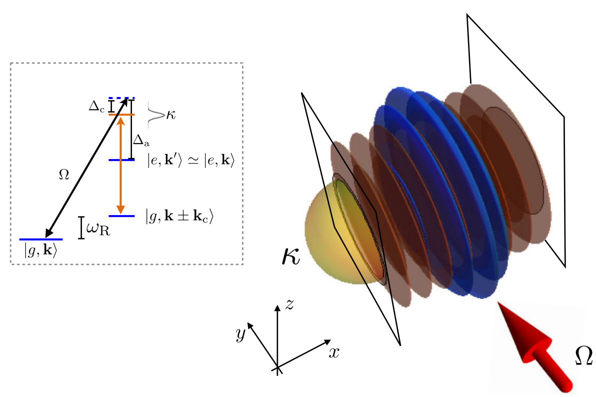

FIG. 1 (color online). A Bose-Einstein condensate (blue surface) trapped inside an optical resonator is laser driven with a Rabi frequency Ω. The laser frequency is blue detuned by Δ_a with respect to an internal atomic transition $g\leftrightarrow e$ and by Δ_c with respect a standing-wave cavity mode $\sim\cos(k_c x)$ defining the characteristic recoil frequency $\omega_R = \hbar k_c/2m$ with atomic mass m. The coupling between a single atom and the cavity mode has a strength g_0. For large $\Delta_a \gg (\Delta_c, \kappa, g_0, \omega_R)$, with κ being the cavity linewidth due to leakage out of the mirrors, the dispersive regime is reached. Atoms experience negligible spontaneous emission and an optical potential (orange surface) arising from the interfering cavity and pump fields [see Eq. (1)].

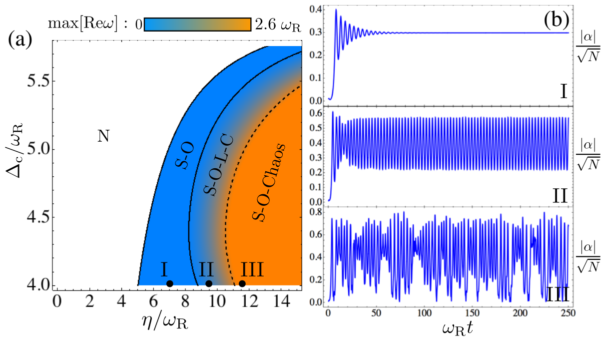

FIG. 2 (color online). (a) Nonequilibrium phase diagram as a function of the effective pump strength $\eta = \sqrt{N} g_0 \Omega / \Delta_a$ and the detuning Δ_c, for $\kappa = 10\omega_R$, $U_0 N = 12.1\omega_R$, and $g_{aa} = 0$. The color scale indicates the growth rate $\mathrm{Re}\,\omega$ of the most unstable collective excitation mode above the steady state. The time evolution of the cavity mode amplitude α is shown in (b), where panels I, II, and III correspond to the points marked in the phase diagram of (a). Four different phases emerge. In the normal phase (N) the steady state is an empty cavity with a homogeneous condensate. In the self-ordered phase (S-O) the steady state is stable and has a finite α accompanied by the corresponding condensate density modulation in space. In the self-ordered limit cycle (S-O-L-C) phase the steady state is unstable and evolves into periodic self-sustained oscillations of large amplitude about a finite value of α. At every time during the oscillation the condensate is self-ordered, i.e., has chosen the density modulation giving rise to the instantaneous cavity amplitude. In the self-ordered chaotic phase (S-O-Chaos) the collective oscillations lose their periodicity. The transition from limit cycles to chaos takes place by period doubling, as illustrated in Fig. 3.

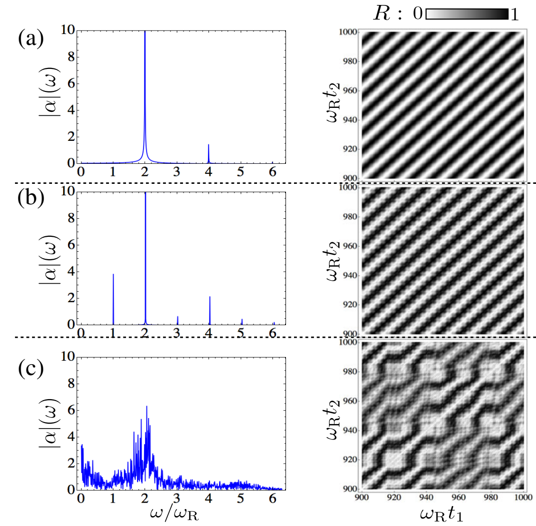

FIG. 3 (color online). Transition from limit cycles to chaos. The Fourier transform of the cavity amplitude time oscillations (left column) is shown together with the condensate wave function recurrence [29] $R(t_1, t_2) = \int dx |\psi(x, t_1) - \psi(x, t_2)|^2$ (middle column). For the same parameters as in Fig. 2, results are shown for three different pump strengths: (a) $\eta = 9.5\omega_R$, (b) $\eta = 10\omega_R$, and (c) $\eta = 11\omega_R$ at $\Delta_c = 4\omega_R$. The spectra clearly show period doubling [between (a) and (b)] before the onset of the noisy background in (c). This indication for the appearance of chaos is confirmed by the large-scale structures in the recurrence R, as opposed to the stripes characterising a periodic behavior, visible in (a) and (b). In the latter, period doubling shows up as further small-scale structures perpendicular to the stripes in R.

considered red detuning case (see Fig. 1). Naively, one would expect that this prevents any self-ordering as the atoms are pushed towards field minima, where light scattering is suppressed. Surprisingly, a closer look reveals that the complex interplay of collective coherent scattering and optical dipole forces still can generate a self-ordered phase at sufficient pump strength. However, the particles are now localized at cavity field nodes and this order gets dynamically unstable again at only a somewhat higher critical pump intensity, as illustrated in Fig. 2.

Interestingly, this instability does not simply lead to heating and disintegration of the order, but we find the emergence of limit cycles, whereby the condensate performs large periodic self-sustained oscillations between different ordered patterns. The atomic density oscillations are tightly coupled to the oscillation of the cavity field with the same frequency, as shown in Fig. 4. This provides a built-in nondestructive monitoring tool of the nonlinear dynamics. By further increase of the drive strength the limit cycles turn into chaotic dynamics by doubling their period (see Fig. 3).

Dynamical instabilities toward limit cycles evolving into chaos have been observed with nanomechanical oscillators coupled to light [30–32]. Limit cycles have been studied within the open Dicke model [33], where chaos appears in the closed-system limit [34–36]. Self-sustained oscillations of a BEC inside a driven cavity have been observed [37] and shown to be very well described through an optomechanical model. In the same setup, a transition to chaos has also been predicted [38], analogous to the one appearing with a nonlinear dielectric medium [39]. Recently, superfluid Josephson dynamics of a BEC coupled to a driven cavity have also been theoretically studied [40]. Self-organized criticality and chaos have been observed with ^{4}He [41].

Here, we show how the dynamical self-ordered regimes of large nonlinear excitations let the peculiarities of the BEC emerge clearly and prevent an understanding of the system via (generalized) optomechanical models. In particular, once direct short-range atom-atom interactions are taken into account, the superfluid nature of the BEC manifests itself through the onset of phase slippage dynamics [42–45], whereby the condensate lowers the kinetic energy stored in the phase of the macroscopic wave function by creating phase singularities in the form of nonlinear dispersive waves (solitons in our one-dimensional model). While phase slips take place periodically in the limit-cycle phase, they appear irregularly and at a much faster rate in the chaotic regime, as illustrated in Fig. 4. Interestingly, different from a superfluid in a closed system where phase-slip proliferation eventually destroys the superfluid, here the dissipation through cavity losses counteracts this heating process by subtracting energy from the system.

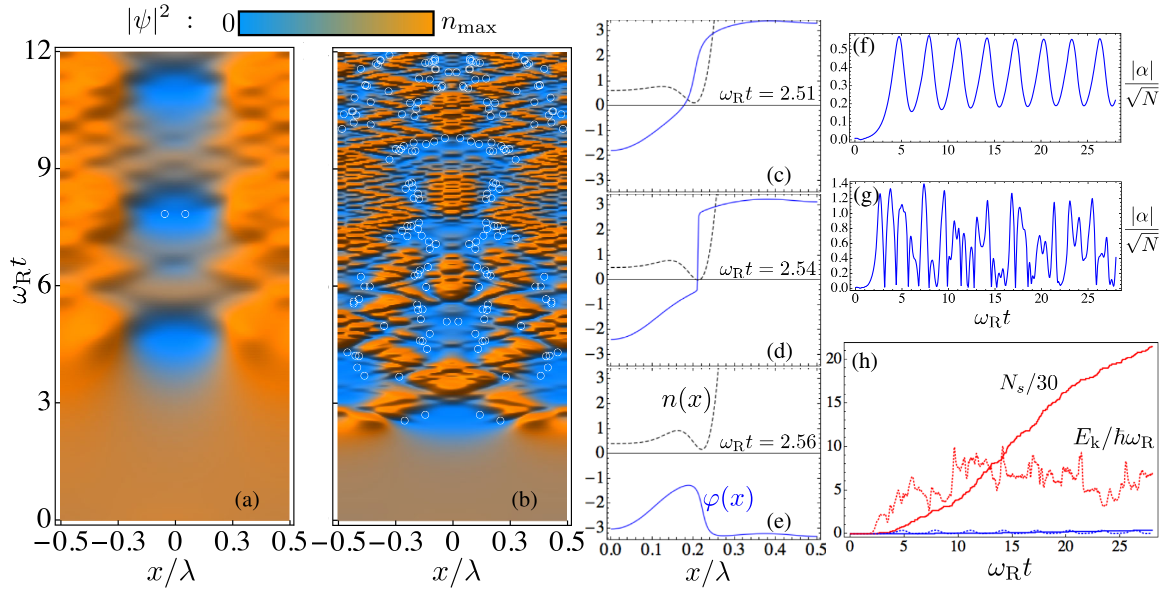

FIG. 4 (color online). Phase slippage through soliton nucleation for a finite value of the atom-atom interaction $g_{aa} = 1.0\hbar\omega_R$. The condensate density (color scale) as a function of position and time is shown in panels (a) and (b). The time evolution of the cavity amplitude is shown in panels (f) and (g). Panels (a) and (f) correspond to limit cycle dynamics at $\eta = 12.0\omega_R$ while panels (b) and (g) correspond to chaotic dynamics at $\eta = 22.0\omega_R$. In panels (a) and (b), white circles indicate the space-time coordinate of phase slips, i.e., phase singularities in the form of solitons having a zero-density core together with a π phase difference across it. The dynamics of a single phase slip is illustrated in panels (c)–(e) for $\eta = 22.0\omega_R$. The phase singularity is present in (d) as a π-phase jump localized at the dark $[n(x) = 0]$ soliton position. The latter has a size of the order of the healing length $\xi = 1/\sqrt{2mg_c n} \sim 0.1\lambda$. The phase slip consists in changing the sign of the phase jump $\pi \to -\pi$ [they are equivalent at the point $n(x) = 0$]. Subsequently, in (e) the soliton increases its minimum density from zero and moves away, carrying the energy subtracted from the initial phase gradient in (c). Panel (h) shows the total number of phase slips $N_s(t)$ (solid line) generated up to time t together with the condensate kinetic energy E_k (dotted line) as a function of time, for $\eta = 12.0\omega_R$ (in blue) and $\eta = 22.0\omega_R$ (in red).

Our findings introduce a new scenario where nonlinear chaotic dynamics, self-ordering, and superfluidity appear together in a driven or dissipative system, bridging between the optomechanics or nonlinear optics and condensed matter communities.

As sketched in Fig. 1, we consider an ultracold atomic gas trapped along the axis of a single mode of a Fabry-Perot cavity. The gas is illuminated with coherent light that is blue detuned to an atomic transition. Corresponding experiments with BECs of ultracold atoms [5,7] have been shown to be well described by coupled classical field equations [18] describing the mean-field dynamics of the BEC with the Gross-Pitaevskii equation [46]

$$i\hbar\partial_t\psi(x,t) = \left[-\frac{\hbar^2\partial_{xx}^2}{2m} + g_{aa}|\psi(x,t)|^2 + \hbar U_0|\alpha(t)|^2\cos^2(k_c x) + 2\hbar(\eta/\sqrt{N})\mathrm{Re}[\alpha(t)]\cos(k_c x)\right]\psi(x,t) \qquad (1)$$

and the cavity field by its coherent component dynamics $\alpha(t)$:

$$i\partial_t\alpha(t) = \left[-\Delta_c - i\kappa + U_0\int dx|\psi(x,t)|^2\cos^2(k_c x)\right]\alpha(t) + (\eta/\sqrt{N})\int dx|\psi(x,t)|^2\cos(k_c x). \qquad (2)$$

The BEC wave function $\psi(x,t)$ is normalized to N and the motion of the particles with mass m is restricted along the cavity axis x upon assuming additional trapping in the other directions. This can be easily achieved in ultracold atom experiments [47]. Correspondingly, the direct atom-atom interaction strength g_{aa} is the effective coupling for the one-dimensional problem [46]. $U_0 = g_0^2/\Delta_a$ is the potential depth per photon felt by an atom as well as the energy shift of the cavity resonance per atom [see Eq. (1)]. The terms containing $\eta/\sqrt{N} = g_0\Omega/\Delta_a$ introduce a further optical potential for the atoms and an effective pump term for the cavity field. Finally, the loss of photons through the cavity mirrors is reflected by the field damping term $-i\kappa$.

The phase diagram of Fig. 2 is obtained by solving Eqs. (1) and (2) and analyzing their long-time behavior. As initial conditions we choose a homogeneous BEC: $\psi(x,0) = \sqrt{n}$ with $n = N/L$ the system's density in one dimension and an infinitesimally occupied cavity $\alpha(0) \ll 1$, which is needed as a seed since Eqs. (1) and (2) do not include noise. A transition toward stable self-ordering is observed at a critical pump strength [48] $\hbar^2\eta_{\mathrm{crit}}^2 = (\hbar\omega_R + 2g_{aa}n)(\delta_c + \kappa^2/\delta_c)/2$, with the dispersively shifted cavity detuning $\delta_c = -\Delta_c + U_0 N/2$. This transition is triggered by unstable density modulations at the cavity wavelength λ_c. The latter indeed tend to optimize light scattering into the cavity, which in turn enhances the density modulation in a runaway process. This is stabilized by losses, providing convergence toward a steady state with finite cavity amplitude $\alpha \neq 0$ and density modulation. This happens both for high ($U_0 < 0$) and low ($U_0 > 0$) field-seeking atoms. However, as we show here, in the latter case the stable self-ordered steady state exists only up to a second critical pump strength, above which it becomes dynamically unstable and no time independent steady state can be found anymore. The system performs instead self-sustained oscillations (limit cycles) between self-ordered configurations of the condensate [see Fig. 4(a) and the cavity field, shown in Figs. 2(b), II 3(a), 3(b), and 4(f)]. The onset of superradiance occupation of the cavity mode pushes the low-field seeking atoms to the field nodes, which then lowers the cavity occupation, letting the self-ordered solution evolve back toward a state with smaller density modulations. By the chosen pump strength the latter is however unstable toward superradiance with large modulations, which starts the whole cycle again.

As illustrated in Fig. 3, the limit cycles involve several well defined frequency components, whose number is doubled before the system dynamics turns eventually chaotic [49]. The onset of chaos by period doubling is a well known phenomenon in nonlinear systems and has been observed with nanomechanical oscillators coupled to light [31]. Here, like in these systems, chaos develops at zero temperature and without external periodic modulations or (delayed) feedback control, with or without atom-atom interactions. The characteristic time scale is given by the recoil frequency ω_R, which is an intrinsic property of the BEC-cavity system.

Different from what is so far studied with nanomechanical media, nonlinear dielectrics, and nondirectly driven BECs [37,38], the oscillations found here have a large (nonperturbative) amplitude where the light intensity scales with N due to the self-ordering of the atoms. The latter generates nontrivial time-dependent spatial structures. While spatial structures with a length scale set by the cavity wavelength λ_c would be present had we considered any (e.g., thermal) ensemble of driven polarizable particles, the peculiar nature of their size and dynamics that we observe here originates from the combination of macroscopic phase coherence (encoded in the BEC wave function) together with short-range atom-atom interactions [inducing the term $g_{\mathrm{aa}}|\psi|^2$ in Eq. (1)]. These two properties in turn are at the core of the superfluid behavior of the BEC. They indeed provide the condensate with a finite "phase rigidity", i.e., a finite energy cost $\propto \int dx(\partial_x\varphi)^2$ of creating a phase gradient in the wave function $\psi = |\psi|\exp(i\varphi)$, which is associated with a finite flow velocity of the superfluid $v = \hbar\partial_x\varphi/m$ [50].

As illustrated in Fig. 4, the driven BEC coupled to the lossy cavity, once entering the nonlinear oscillating regime, shows large phase gradients and thus accumulates the corresponding energy. The most efficient way for the superfluid to get rid of this extra energy is to convert it into nonlinear dispersive waves through the process of phase slippage [42]. In one dimension these waves are solitons, for which phase slips take place as described in Fig. 4(c)–(e). We observe indeed phase slippage as the system enters the dynamically unstable regime. During the limit cycles, phase slips take place periodically and are synchronized with the oscillations. The rate of slippage is slow [see Fig. 4(h)] and the solitons are nucleated always at the same position, shown in Fig. 4(a). On the contrary, once the system has entered the chaotic regime, phase slippage takes place at a faster rate and the nucleation of solitons becomes irregular in space and time. Interestingly, this fast proliferation of solitons does not eventually destroy the BEC as usually happens in an isolated superfluid. In the latter case the large number of solitons causes the phase of the wave function to change fast and over short length scales, thereby destroying phase coherence. In addition, solitons contain a large kinetic energy, which can be converted into the thermal component. However, in our driven or dissipative system the proliferation of solitons is counteracted by light scattering processes transferring energy into the cavity field and expelling it through photon losses. In general, this cavity cooling [17,48,51–56] compensates the heating due to the dynamical instabilities (already in the limit cycle regime quasiparticles should be resonantly excited out of the condensate [57–62]) and can ensure the validity of the Gross-Pitaevskii equation approach (1). The ratio of the heating to cooling rate can be estimated to scale like κ/ω_R [49]. We observe that even though solitons are produced at a fast rate, the BEC kinetic energy $E_k = \int dx\hbar^2|\partial_x\psi|^2/2m$, after an initial increase, fluctuates around a finite value of the order of the recoil energy $\hbar\omega_R$, as shown in Fig. 4. The extension of the present study to two dimensions should open up new directions in the field of quantum turbulence [63] in driven or dissipative condensates [64].

We finally point out that the system studied here is already available experimentally since it involves a change from red to blue detuning of the driving laser with respect to the atomic resonance in the setups used in the ETH [5] and Hamburg [7] laboratories.

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