Theory of the Sea Ice Thickness Distribution

Srikanth Toppaladoddi[1,2] and J. S. Wettlaufer[1,2,3,*]
[1]*Yale University, New Haven, Connecticut 06520-8109, USA*
[2]*Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom*
[3]*Nordita, Royal Institute of Technology and Stockholm University, SE-10691 Stockholm, Sweden*
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We use concepts from statistical physics to transform the original evolution equation for the sea ice thickness distribution $g(h)$ from Thorndike *et al.* into a Fokker-Planck-like conservation law. The steady solution is $g(h) = \mathcal{N}(q)h^q e^{-h/H}$, where q and H are expressible in terms of moments over the transition probabilities between thickness categories. The solution exhibits the functional form used in observational fits and shows that for $h \ll 1$, $g(h)$ is controlled by both thermodynamics and mechanics, whereas for $h \gg 1$ only mechanics controls $g(h)$. Finally, we derive the underlying Langevin equation governing the dynamics of the ice thickness h, from which we predict the observed $g(h)$. The genericity of our approach provides a framework for studying the geophysical-scale structure of the ice pack using methods of broad relevance in statistical mechanics.

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Earth's climate system is a complex nonlinear dynamical system [1]. Three main research approaches in climate science are common. (a) Observation of the past and present state of the system and extrapolation to the future. (b) Numerical simulations using global circulation models, which treat the system with the deterministic approach of weather forecasting by modeling the processes on a coarse-grained scale. (c) Constructing a low-order description of the system or subsystem of the climate, in the vein of theoretical physics. There are substantial cultural and technical differences between these approaches. All have value and all have limitations. Evidently, the ready availability of computing power has made approach (c) less favorable. Here, we show that one of the key variables in polar climate, the sea ice thickness distribution $g(h)$, can be fruitfully examined quantitatively with a core set of tools in statistical physics.

Although there are high-fidelity measurements of the area of the ice cover during the satellite era [2], the key quantity reflecting the climatological state of the sea ice cover is its volume, making $g(h)$ the central state variable of the system. The thickness distribution underlies and reflects ice melting or freezing due to the thermodynamic forcing of the ocean and the atmosphere, and mechanical deformation: rafting, ridging, and the formation of open water [3]. Nonetheless, although the theory of ice thickness distribution has been with us for 40 years, we still seek a basic understanding of its components in order to test its predictions [2].

The theory of Thorndike *et al.* [3] is described by a continuous, deterministic partial differential equation that contains the principal physical processes mentioned above and is given by

$$\frac{\partial g}{\partial t} = -\nabla \cdot (\mathbf{u}g) - \frac{\partial}{\partial h}(fg) + \psi, \tag{1}$$

where $\mathbf{u}$ is the velocity of the ice pack and f is its growth or melting rate. The principal reason for the difficulty in testing the theory arises from the so-called redistribution function, ψ, intended to capture mechanical processes. Although from observational, theoretical, and numerical perspectives we have gained a quantitative explanation for many aspects of the redistribution function (e.g., Refs. [4,5], and references therein), a closed mathematical analysis of the original theory is still principally limited by this term. In what follows, by viewing ψ within the framework of kinetic theory, we show that the original theory can be rewritten as a Fokker-Planck-type equation. In so doing, a number of useful advantages arise. First, we can determine the steady solution analytically. Second, we provide access to the full range of methods and approaches of nonequilibrium statistical physics. In this Letter we describe just two: (1) by comparison with observations, we deduce the transport coefficients in the new evolution equation, which allows for its full numerical solution in a geophysically relevant setting, and (2) we derive the corresponding Langevin equation for the evolution of the ice thickness itself.

A central ansatz in stochastic dynamics is to consider the "microscopic noise" underlying a "macroscopic process" as decorrelating on a time scale far faster than the macroscopic displacement. The classical test is Brownian motion, wherein the inertial macroscopic displacement of a pollen grain in water evolves slowly relative to the collisional events driving its motion. This is embodied in the fluctuation-dissipation theorem (e.g., Ref. [6]).

Within this framework we recast ψ as follows. Each of the "microscopic" mechanical processes that influence the ice thickness distribution—rafting, ridging, and the formation of open water—occur over a time scale that is very rapid relative to the geophysical-scale changes of $g(h)$. We thus view these processes as the collisions of solvent molecules with a Brownian particle; the individual collisions have a probability of displacing the particle, but their phase space is so enormous that we do not study them individually. In the same vein, we do not study the individual floe-floe interactions in the ice pack. Rather, we write ψ as

$$\psi = \int_0^\infty [g(h+h')w(h+h',h') - g(h)w(h,h')]dh'. \qquad (2)$$

Here, the first and second terms represent the processes by which (i) ice floes of thickness $h+h'$ become ice floes of thickness h and (ii) ice floes of thickness h become ice floes of thickness $h-h'$, respectively, with wdh' being the transition probability per unit time for these events.

Taylor expanding the right-hand side of Eq. (2) and substituting this is into Eq. (1), we obtain

$$\frac{\partial g}{\partial t} = -\nabla\cdot(\mathbf{u}g) - \frac{\partial}{\partial h}(fg) + \frac{\partial}{\partial h}(k_1 g) + \frac{\partial^2}{\partial h^2}(k_2 g), \qquad (3)$$

where

$$k_1 = \int_0^\infty h' w(h,h')dh' \quad \text{and} \quad k_2 = \int_0^\infty \frac{1}{2}h'^2 w(h,h')dh'. \qquad (4)$$

Thus, we have transformed the original theory into a Fokker-Planck-type of evolution equation. Note that in the absence of ice motion, Eq. (3) is exactly the Fokker-Planck equation, an advection-diffusion equation for the probability density [1,6]. Here, the coefficients [Eq. (4)] are the first and second order moments of the transition probability between ice thickness categories.

Choosing L as the horizontal scale, H_{eq} as the vertical scale and U_0 as the velocity scale, we find three time scales in the problem: (1) the thermal diffusion time scale, $t_D = H_{\mathrm{eq}}^2/\kappa$, with κ being the thermal diffusivity of ice; (2) the time scale associated with the horizontal motion of ice floes, $t_m = L/U_0$; and (3) the relaxation time scale of the ice floes when they are involved in collisions, $t_R \sim 1/\dot{\gamma}$, where $\dot{\gamma}$ is the collisional strain rate. These time scales are such that $t_m \approx t_R$ and $\tau \equiv t_R/t_D \ll 1$. Hence, we have $f_0 = H_{\mathrm{eq}}/t_D$, $\tilde{k}_1 = H_{\mathrm{eq}}/t_R$, and $\tilde{k}_2 = H_{\mathrm{eq}}^2/t_R$ as the scales for the remaining terms. Maintaining the prescaled notation, the dimensionless equation can be written in one spatial dimension as

$$\frac{\partial g}{\partial t} = -\frac{\partial}{\partial x}(ug) - \tau\frac{\partial}{\partial h}(fg) + \frac{\partial}{\partial h}(k_1 g) + \frac{\partial^2}{\partial h^2}(k_2 g). \qquad (5)$$

Now, we obtain the steady solely h-dependent solution of Eq. (5) with boundary conditions $g(0) = g(\infty) = 0$. The growth rate f in the original theory of Thorndike *et al.* [3] was determined numerically from the climatologically forced Stefan problem for the ice thickness. If ΔT is the temperature difference over a solid layer of thickness h, we take a standard analytical solution for its diffusive growth into an isothermal liquid (e.g., Ref. [7]). Here, one balances heat conduction through the layer ($\propto \Delta T/h$) against latent heat production at the interface ($\propto dh/dt \equiv f$) giving

$$f = \frac{1}{S}\left(\frac{1}{h}\right), \qquad (6)$$

with $S = L/c_p\Delta T$ being the Stefan number, in which L is the latent heat of fusion and c_p is the specific heat at constant pressure. Ignoring the advection term leads to

$$\frac{\partial g}{\partial t} = -\frac{\partial}{\partial h}\left[\left(\frac{\epsilon}{h} - k_1\right)g\right] + \frac{\partial^2}{\partial h^2}(k_2 g), \qquad (7)$$

where $\epsilon \equiv \tau/S \ll 1$ because $S \gg 1$ and $\tau \ll 1$. Because the small parameter multiplies regular singularities, which become $O(1)$ when $h = O(\epsilon)$, we keep all terms in Eq. (7), to which we seek the stationary solution and rewrite it as

$$\frac{d^2 g}{dh^2} + \frac{d}{dh}\left[\left(\frac{1}{H} - \frac{q}{h}\right)g\right] = 0, \qquad (8)$$

where $H = k_2/k_1$ and $q = \epsilon/k_2$. The first integral is

$$\frac{dg}{dh} + \left(\frac{1}{H} - \frac{q}{h}\right)g = B, \qquad (9)$$

where B is the integration constant. We solve Eq. (9) using an integrating factor $e^{h/H - q\ell n(h)}$, which requires $B = 0$ to satisfy $g(0) = g(\infty) = 0$, and we find the solution

$$g(h) = \mathcal{N}(q)h^q e^{-h/H}. \qquad (10)$$

The prefactor is determined by the normalization condition $\int_0^\infty g(h)dh = 1$ and is $\mathcal{N}(q) = [H^{1+q}\Gamma(1+q)]^{-1}$, with $\Gamma(x)$ being the Euler gamma function. Hence, $\mathcal{N}(q)$ is unique and single valued for $\mathbb{R}(q) > -1$ and $\mathbb{R}(H) > 0$. Note that q and H have an independent interpretation within the framework of the theory *and* are the sole fitting parameters. Clearly, for $h \ll 1$, $g(h)$ is controlled by both thermodynamics and mechanics, whereas for $h \gg 1$, $g(h)$ is controlled solely by mechanical interactions.

A Fokker-Planck equation describes the evolution of the probability density of a random process, but to study the

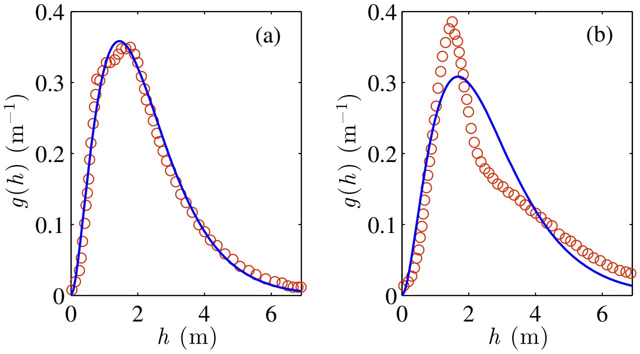

FIG. 1 (color online). Comparison of our theory with satellite measurements for February through March (F-M) of (a) 2008 and (b) 2004. Circles are the distribution functions from ICESat [9] and lines are the fits using Eq. (10). In (a), $q = 1.849$ and $H = 0.783$ m, and in (b), $q = 1.848$ and $H = 0.910$ m.

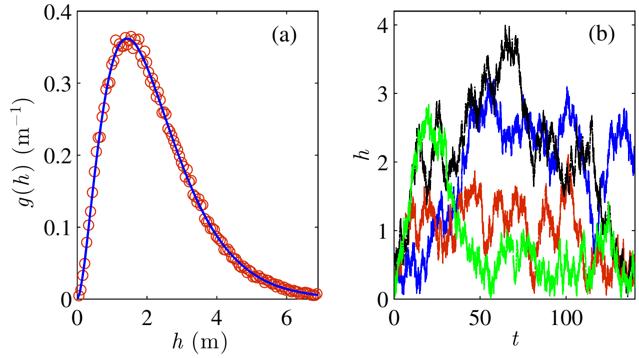

FIG. 2 (color online). Solution to the Langevin equation (11) for F-M 2008. Here, $\epsilon = 0.046$, $k_1 = 0.048$, and $k_2 = 0.025$. The ensemble size used is $N_{\mathrm{en}} = 10^5$ and the time step for the Euler-Maruyama scheme [10] is $\Delta t = 10^{-5}$. The total nondimensional integration time is $T = 140$. Panels (a) and (b) show the thickness distribution and four realizations from the ensemble, respectively. In (a), the circles represent $g(h)$ from the Langevin equation and the solid curve is the fit using Eq. (10), which gives $q = 1.838$ and $H = 0.777$ m.

random process itself (h in our case) we study the Langevin equation corresponding to Eq. (7), which we write as

$$\frac{dh}{dt} = \left(\frac{\epsilon}{h} - k_1\right) + \sqrt{2k_2}\,\xi(t), \tag{11}$$

where $[(\epsilon/h) - k_1]$ and $\sqrt{2k_2}$ are the drift and diffusion terms, respectively, and $\xi(t)$ is Gaussian white noise (e.g., Ref. [8]). Clearly, our assumption of k_2 being a constant in the determination of the solution for $g(h)$ in Eq. (10) translates into additive noise in the corresponding Langevin equation shown in Eq. (11).

We now compare our theory with the thickness distributions obtained during the ICESat mission [9]. Figure 1 shows fits of our solution (10) to the distribution functions for the period of February through March (F-M) for (a) 2008 and (b) 2004, for which we have chosen to demonstrate both the typical fit (2008) and the worst fit (2004) of our solution to the observations (cf. Fig. 6 of Ref. [9]). The key reasons for deviations are (1) the observations span the ice cover and yet—near landmasses and depending on wind direction—it is possible that k_1 and k_2 will differ locally, (2) we neglected the advection term when determining the solution, (3) the form of f used in Eq. (3) is the solution of the ideal Stefan problem for *growth* only, and (4) we are comparing the steady solution to the data. Incorporating these and related issues are key aspects of a thorough numerical study of Eq. (5) which are part of a longer treatment.

Now, from the values of q and H, we obtain k_1 and k_2 and use them in Eq. (11) to evolve h itself. The solution to the Langevin equation corresponding to Fig. 1(a) is shown in Fig. 2. Invoking ergodicity, Eq. (11) clearly qualitatively reproduces the observed $g(h)$ and thus acts as an ideal and simple model to study the thickness distribution.

We have transformed the original evolution equation for the sea ice thickness distribution, $g(h)$, from Thorndike *et al.* [3], to a Fokker-Planck-like equation by recasting the redistribution function, ψ, using an analogy with the theory of Brownian motion. The idea is that the mechanical processes embodied in ψ (rafting, ridging, and the formation of open water) are thought of like the collisions of solvent molecules with a Brownian particle—the individual events that change the ice thickness occur on time and space scales that are short relative to the geophysical-scale changes of $g(h)$. Thus, we do not treat the individual floe-floe interactions in the ice pack, but rather only the moments of the transition probabilities for these events. That the integrals describing these moments rapidly converge to take constant values is borne out by comparison with observations; the stationary solution [Eq. (10)] of the new evolution equation captures the basin-scale measurements of the distribution. Finally, the corresponding Langevin equation (11) is evolved with observationally constrained parameters to study the evolution of h itself. The associated agreement of the $g(h)$ obtained from this approach with the observations is consistent with an ergodic thickness field. The simplicity of the approach and its immediate connection with the edifice of nonequilibrium statistical mechanics make it appealing for a wide range of reasons, from a context for comparison with observations to the simplification of models.

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*john.wettlaufer@yale.edu

[1] H. Dijkstra, *Nonlinear Climate Dynamics* (Cambridge University Press, Cambridge, England, 2013).
[2] R. Kwok and N. Untersteiner, Phys. Today **64**, 36 (2011).
[3] A. S. Thorndike, D. A. Rothrock, G. A. Maykut, and R. Colony, J. Geophys. Res. **80**, 4501 (1975).
[4] A. S. Thorndike, J. Geophys. Res.-Oceans **97**, 12601 (1992).
[5] A. S. Thorndike, J. Geophys. Res. **105**, 1311 (2000).
[6] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).
[7] M. G. Worster, in *Perspectives in Fluid Dynamics: A Collective Introduction to Current Research*, edited by G. Batchelor, H. Moffatt, and M. Worster (Cambridge University Press, Cambridge, England, 2000), p. 393.
[8] R. Stratonovich, *Topics in the Theory of Random Noise*, Vol. 1 (Gordon and Breach, New York, 1969).
[9] R. Kwok, G. Cunningham, M. Wensnahan, I. Rigor, H. Zwally, and D. Yi, J. Geophys. Res. **114**, C07005 (2009).
[10] D. J. Higham, SIAM Rev. **43**, 525 (2001).