Quasilocal Conserved Operators in the Isotropic Heisenberg Spin-1/2 Chain

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Composing higher auxiliary–spin transfer matrices and their derivatives, we construct a family of quasilocal conserved operators of isotropic Heisenberg spin-1/2 chain and rigorously establish their linear independence from the well-known set of local conserved charges.

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Introduction.—The Heisenberg chain of n spins $1/2$ with the Hamiltonian (known as the XXX model)

$$H = \sum_{x=0}^{n-1} (\vec{\sigma}_x \cdot \vec{\sigma}_{x+1} + \mathbb{1}), \tag{1}$$

where $\vec{\sigma}_x = (\sigma_x^x, \sigma_x^y, \sigma_x^z)$ are Pauli operators and periodic boundaries are assumed $\vec{\sigma}_n \equiv \vec{\sigma}_0$, is arguably the simplest nontrivial interacting quantum many-body model. The spectrum and eigenstates of H can be formulated in terms of the famous Bethe ansatz [1], which gave birth to the theory of quantum integrable systems [2,3]. Equation (1) has been originally proposed as the model of (anti)ferromagnetism in solids [4] and is, indeed, a very good description of the modern spin-chain materials [5]. It may also be considered as a fundamental paradigm of quantum statistical mechanics which is being used for developing theoretical mechanisms of nonequilibrium dynamics and thermalization or relaxation to the generalized Gibbs ensemble (GGE) [6–8].

The relaxation dynamics based on quantum quenches [9–12] gave firm evidence that the full set of ($\sim n$) local conserved operators, the existence of which is granted for a quantum integrable system, is *incomplete*, in the sense that it cannot describe the steady state completely through a GGE. Similarly, a numerical experiment counting the number of linearly independent time-averaged local operators [13] indicated that the set of local conserved charges should be incomplete and numerical approximations of new quasilocal operators have been put forward.

In this Letter, we explicitly construct new families of nonlocal but quasilocal operators by composition of a transfer matrix (TM)—in the sense of an algebraic Bethe ansatz but for higher integer or half-integer auxiliary spins $s > \frac{1}{2}$ —and its derivative, at a special combination of spectral parameters, which in the thermodynamic limit (TL) becomes equivalent to a logarithmic derivative of a TM. Furthermore, we prove quasilocality (in full rigor for a finite set of auxiliary spins s) as well as linear independence of these new operator families with respect to local conserved charges. Generally, we identify quasilocality with the condition of *factorizability* of the largest eigenvalue of an auxiliary TM that enters in the computation of the norm of the conserved operator, i.e., a product of a higher-spin TM and its derivative. As we facilitate finite-dimensional *unitary* representations of quantum or Lie symmetries, the new quasilocal operators are always spin-reversal symmetric unlike in alternative recent constructions in the XXZ chain [14–17] which only work at special commensurate values of anisotropy. These features promise that our technique shall be applicable for generating new quasilocal charges in other generic integrable models with Lie or quantum group symmetries. Being able to construct a complete or as-large-as-possible set of independent quasilocal conserved charges is crucial for any application in quantum statistical mechanics besides constructing the GGE, e.g., in linear response theory at finite temperatures, studies of quantum ergodicity and many-body localization, etc. Quasilocal conservation laws are also closely related to boundary-driven or dissipative quantum chains [14,18].

Transfer matrices and conserved operators.—Let $\mathcal{V}_s$, $s \in \frac{1}{2}\mathbb{Z}^+$ denote a $(2s+1)$-dimensional spin-s module, $\mathcal{V}_s \equiv \mathbb{C}^{2s+1} = \mathrm{lsp}\{|m\rangle, m = -s, -s+1, \ldots, s\}$, with lsp denoting a linear span of a set of vectors carrying the *unitary* irreducible representation of SU(2) with generators

$$\begin{aligned} \mathbf{s}^z|m\rangle &= m|m\rangle, \\ \mathbf{s}^{\pm}|m\rangle &= \sqrt{(s+1\pm m)(s\mp m)}|m\pm 1\rangle. \end{aligned} \tag{2}$$

The physical Hilbert space is an n-fold tensor product of fundamental representations $\mathcal{H}_p = \mathcal{V}_{1/2}^{\otimes n}$, with $\sigma^z \equiv 2\mathbf{s}^z$, $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y) \equiv \mathbf{s}^{\pm}$. Fixing arbitrary $s \in \frac{1}{2}\mathbb{Z}^+$ and considering another *auxiliary* Hilbert space $\mathcal{H}_a = \mathcal{V}_s$, we define Lax matrices as operators over $\mathcal{H}_p \otimes \mathcal{H}_a$,

$$\mathbf{L}_{x,a}(\lambda) = \lambda \mathbb{1} + \sigma_x^z \mathbf{s}_a^z + \sigma_x^+ \mathbf{s}_a^- + \sigma_x^- \mathbf{s}_a^+ = \lambda \mathbb{1} + \vec{\sigma}_x \cdot \vec{\mathbf{s}}_a, \tag{3}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. Throughout the Letter, operators acting nontrivially over the auxiliary Hilbert space are written in bold or double-strike font if acting over multiple (tensor product of) auxiliary spaces.

As a simple consequence of Yang-Baxter equation, the (physical) TMs $T_s(\lambda) \in \mathrm{End}(\mathcal{H}_p)$,

$$T_s(\lambda) = \mathrm{tr}_a \mathbf{L}_{0,a}(\lambda)\mathbf{L}_{1,a}(\lambda)\ldots\mathbf{L}_{n-1,a}(\lambda), \tag{4}$$

where s is the auxiliary spin, form a commuting family

$$[T_s(\lambda), T_{s'}(\lambda')] = 0 \quad \forall\ s, s', \lambda, \lambda'. \tag{5}$$

The fundamental TM $T_{1/2}(\lambda)$ is generating the complete set of local conserved Hermitian operators

$$Q_k = -i\partial_t^{k-1} \log T_{1/2}\left(\frac{1}{2} + it\right)\Big|_{t=0} = \sum_{x=0}^{n-1} \hat{\mathcal{S}}^x(\mathbb{1}_{2^{n-k}} \otimes q_k), \tag{6}$$

$k \geq 2$, with $Q_2 = H/2$, where $q_k \in \mathrm{End}(\mathcal{V}_{1/2}^{\otimes k})$ is a k-point operator density, and $\hat{\mathcal{S}}$ is a cyclic lattice shift map over $\mathrm{End}(\mathcal{H}_p)$ defined by $\hat{\mathcal{S}}(\sigma_x^\alpha) = \sigma_{\mathrm{mod}(x+1,n)}^\alpha$.

Locality and quasilocality.—The 4^n-dimensional space of physical operators $\mathrm{End}(\mathcal{H}_p)$ is turned into a Hilbert space by defining a Hilbert-Schmidt (HS) inner product $(A, B) \coloneqq \langle A^\dagger B\rangle$ with respect to the infinite-temperature state $\langle A\rangle \coloneqq 2^{-n}\mathrm{tr}A$. Let $\{A\} \coloneqq A - \langle A\rangle\mathbb{1}$ denote the traceless part of an operator. One of the physically most important features of the local conservation laws Q_k is the extensivity of the HS norm $\|\{Q_k\}\|_{\mathrm{HS}}^2 \coloneqq (\{Q_k\}, \{Q_k\}) = (2^{-k}\mathrm{tr}(q_k^\dagger q_k) - |2^{-k}\mathrm{tr}q_k|^2)n \propto n$. We define (equivalently to Ref. [16]) a general traceless translationally invariant operator $A = \hat{\mathcal{S}}(A) \in \mathrm{End}(\mathcal{H}_p)$ as *quasilocal* if two conditions are met: (i) $\|A\|_{\mathrm{HS}}^2 \propto n$, and (ii) for any locally supported k-site operator $b = b_k \otimes \mathbb{1}_{2^{n-k}}$, the overlap (b, A) is asymptotically, as $n \to \infty$, *independent* of n.

One should stress that quasilocality only makes sense in TL $n \to \infty$, as it is the property of an infinite sequence of operators labeled by n rather than operators for any fixed size n. More intuitively, a quasilocal operator Q can be thought of as a convergent sum of local operators $Q = \sum_{r=1}^\infty Q^{(r)}$, where $Q^{(r)}$ includes only terms supported on r contiguous sites, and the sum $\|Q\|_{\mathrm{HS}}^2 = \sum_{r=1}^\infty \|Q^{(r)}\|_{\mathrm{HS}}^2$ is rapidly (typically exponentially) converging. Usually [14–16], quasilocality can be detected by inspecting the leading eigenvalue of a certain auxiliary transfer matrix, whose rth power yields the partial norm of the r-site terms $\|Q^{(r)}\|_{\mathrm{HS}}^2$. The effect of quasilocal conserved operators to statistical mechanics is arguably *as important* as that of local operators. In particular, quasilocal charges can be understood as those conserved operators of one-dimensional systems which can influence equilibrated (steady-state) values, say, after a quantum quench, of strictly local observables. Our central result is the following.

Theorem: Traceless operators $X_s(t)$, $s \in \frac{1}{2}\mathbb{Z}^+, t \in \mathbb{R}$ defined over the physical Hilbert space $\mathcal{H}_p$ as

$$X_s(t) = [\tau_s(t)]^{-n}\left[T_s\left(-\frac{1}{2} + it\right)T_s'\left(\frac{1}{2} + it\right)\right], \tag{7}$$

$$\tau_s(t) = -t^2 - \left(s + \frac{1}{2}\right)^2, \tag{8}$$

where $T_s'(\lambda) \equiv \partial_\lambda T_s(\lambda)$ are quasilocal for all s, t and linearly independent from $\{Q_k; k \geq 2\}$ for $s > \frac{1}{2}$.

The fact that $X_s(t)$ are exactly conserved and $[X_s(t), X_{s'}(t')] = [X_s(t), Q_k] = 0$ follows directly from Eq. (5). The form of our ansatz (7) is inspired from an observation [see Eq. (6) or, e.g., Ref. [19]] that at $s = \frac{1}{2}$, TM becomes in TL $n \to \infty$, a unitary operator

$$T_{1/2}\left(\frac{1}{2} + it\right) \simeq \exp\left(i\sum_{k=1}^\infty \frac{t^k}{k!}Q_{k+1}\right), \tag{9}$$

and, hence, Eq. (7) can be associated with a logarithmic derivative via $T_s^\dagger(\lambda) \equiv T_s^T(\bar{\lambda}) = (-1)^n T_s(-\bar{\lambda})$, where the last equality is due to spin-reversal symmetry $\mathbf{s}^z \to -\mathbf{s}^z$, $\mathbf{s}^\pm \to -\mathbf{s}^\mp$.

Proof of quasilocality.—First, we write a matrix product form of a general product of a pair of TMs [20]

$$T_s(\mu)T_s(\lambda) = \mathrm{tr}_{a_1,a_2}\prod_{x=0}^{n-1}\left(\sum_{\alpha\in\mathcal{J}}\mathbb{L}_s^\alpha(\mu,\lambda)\sigma_x^\alpha\right), \tag{10}$$

where the operators $\mathbb{L}_s^\alpha(\mu, \lambda)$, $\alpha \in \mathcal{J} \coloneqq \{0, x, y, z\}$ act over a pair of auxiliary spaces $\mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2} \equiv \mathcal{V}_s \otimes \mathcal{V}_s$,

$$\mathbb{L}_s^0(\mu, \lambda) = \lambda\mu\mathbb{1} + \vec{\mathbf{s}}_{a_1} \cdot \vec{\mathbf{s}}_{a_2}, \tag{11}$$

$$\vec{\mathbb{L}}_s(\mu, \lambda) = i\vec{\mathbf{s}}_{a_1} \times \vec{\mathbf{s}}_{a_2} + \lambda\vec{\mathbf{s}}_{a_1} + \mu\vec{\mathbf{s}}_{a_2}. \tag{12}$$

The identity component can be written with the Casimir operator $\mathbf{C} = (\vec{\mathbf{s}}_{a_1} + \vec{\mathbf{s}}_{a_2})^2$ as $\mathbb{L}_s^0 = \mu\lambda\mathbb{1} + \frac{1}{2}(\mathbf{C} - \vec{\mathbf{s}}_{a_1}^2 - \vec{\mathbf{s}}_{a_2}^2)$; hence, its spectrum reads $\tau_s^j(\mu, \lambda) = [j(j+1)/2] - s(s+1) + \mu\lambda$, $j = 0, 1, \ldots, 2s$. Placing the spectral parameters along one of the two lines

$$\mathcal{D}^\pm = \{(\mu_t^\pm, \lambda_t^\pm);\ t \in \mathbb{R}\} \subset \mathbb{C}^2,$$
$$\mu_t^\pm \coloneqq \mp\frac{1}{2} + it, \qquad \lambda_t^\pm \coloneqq \pm\frac{1}{2} + it, \tag{13}$$

we define the restricted auxiliary operators as $\mathbb{L}_s^{\pm\alpha}(t) \coloneqq \mathbb{L}_s^\alpha(\mu_t^\pm, \lambda_t^\pm)$. The *dominating* eigenvalue of Hermitian operator $\mathbb{L}_s^{+0}(t) \equiv \mathbb{L}_s^{-0}(t)$ is $\tau_s(t) = \tau_s^0(\mu_t^\pm, \lambda_t^\pm)$, Eq. (8), corresponding to the *singlet* eigenstate

$$|\psi_0\rangle = (2s+1)^{-1/2}\sum_{m=-s}^{s}(-1)^{s-m}|m\rangle\otimes|-m\rangle, \qquad (14)$$

with a finite gap to the subleading eigenvalue $\tau_s'(t)$, $\delta = \log|\tau_s(t)/\tau_s'(t)| > 0$, for any t. The condition $(\vec{\mathbf{s}}_{a_1}+\vec{\mathbf{s}}_{a_2})|\psi_0\rangle = 0$ and the SU(2) algebra $\vec{\mathbf{s}}_{a_k}\times\vec{\mathbf{s}}_{a_k} = i\vec{\mathbf{s}}_{a_k}$ imply the following useful identities:

$$\vec{\mathbb{L}}_s^-(t)|\psi_0\rangle = 0, \qquad \langle\psi_0|\vec{\mathbb{L}}_s^-(t) = -2\langle\psi_0|\vec{\mathbf{s}}_{a_1},$$
$$\langle\psi_0|\vec{\mathbb{L}}_s^+(t) = 0, \qquad \vec{\mathbb{L}}_s^+(t)|\psi_0\rangle = 2\vec{\mathbf{s}}_{a_1}|\psi_0\rangle. \qquad (15)$$

We proceed by constructing a TM over a four-spin auxiliary space $\mathcal{H}_a = \otimes_{k=1}^4\mathcal{H}_{a_k}$, $\mathcal{H}_{a_{1,2}}\equiv\mathcal{V}_s$, $\mathcal{H}_{a_{3,4}}\equiv\mathcal{V}_{s'}$,

$$\mathbb{T}_{s,s'}(\mu,\lambda,\mu',\lambda') = \sum_{\alpha\in\mathcal{J}}\mathbb{L}_s^\alpha(\mu,\lambda)\otimes\mathbb{L}_{s'}^\alpha(\mu',\lambda'), \qquad (16)$$

which helps us to compute a general inner product of the form $K_{s,s'}(t,t') \coloneqq (X_s(t), X_{s'}(t'))$. The Hilbert-Schmidt kernel (HSK) then immediately follows after differentiating traces of powers of suitable TMs

$$K_{s,s'}(t,t') = [\tau_s(t)\tau_{s'}(t')]^{-n}\partial_{\lambda_t^-}\partial_{\lambda_{t'}^+}(\mathrm{tr}[\mathbb{T}_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+)]^n - \mathrm{tr}[\mathbb{L}_s^0(\mu_t^-,\lambda_t^-)]^n\mathrm{tr}[\mathbb{L}_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)]^n). \qquad (17)$$

As a consequence of the boundary condition given by Eq. (15), we obtain that $\tau_{s,s'}(t,t') \coloneqq \tau_s(t)\tau_{s'}(t)$ is always an eigenvalue of $\mathbb{T}_{s,s'}(t,t') \coloneqq \mathbb{T}_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+)$ with a product-singlet eigenvector $|\Psi_0\rangle = |\psi_0\rangle\otimes|\psi_0\rangle$. One can further show that it is always a dominating and nondegenerate eigenvalue by demonstrating that $\mathbb{T}_{s,s'}(t,t') - \tau_s(t)\tau_{s'}(t)\mathbb{1}$ is a negative definite operator on $\mathcal{H}_a\backslash\mathbb{C}|\Psi_0\rangle$ (see Secs. A and B of Ref. [21] for details). (We note though that we managed to rigorously prove negativity only for $s \le s_0 = 3/2$ and further succeeded to confirm it with exact numerical computations up to much larger maximal auxiliary spin s_0, while for *any* s it formally remains a conjecture.) Denoting by $\tau_{s,s'}(\mu,\lambda,\mu',\lambda')$ the continuation of the dominating eigenvalue in the proximity of the domain $\mathcal{D}^-\times\mathcal{D}^+$ and using Hellmann-Feynman theorem to evaluate its first derivatives $\partial_{\lambda_t^-}\tau_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+) = \partial_{\lambda_t^-}\tau_s^0(\mu_t^-,\lambda_t^-)\tau_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)$, $\partial_{\lambda_{t'}^+}\tau_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+) = \tau_s^0(\mu_t^-,\lambda_t^-)\partial_{\lambda_{t'}^+}\tau_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)$, the HSK can be computed as

$$K_{s,s'}(t,t') = n[\tau_s(t)\tau_{s'}(t')]^{-1}\partial_{\lambda_t^-}\partial_{\lambda_{t'}^+}[\tau_{s,s'}(\mu_t^-,\lambda_t^-,\mu_{t'}^+,\lambda_{t'}^+) - \tau_s^0(\mu_t^-,\lambda_t^-)\tau_{s'}^0(\mu_{t'}^+,\lambda_{t'}^+)] + O(e^{-\gamma n}). \qquad (18)$$

Remarkably, the n^2 term exactly cancels, while the finite-size corrections are exponentially small in the gap $\gamma = \log|\tau_{s,s'}(t,t')/\tau'| > 0$ to subleading eigenvalue τ' of $\mathbb{T}_{s,s'}(t,t')$. We shall later derive an explicit expression for the HSK.

What remains to be shown is that $X_s(t)$ has well-defined expansions in terms of local operators in TL $n\to\infty$. For any k-local basis operator $\sigma_{1:k}^{\underline{\alpha}} \coloneqq \sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\ldots\sigma_k^{\alpha_k}$, $\alpha_{1,k}\neq 0$, we write the component of Eq. (7) as $[\tau_s(t)]^{-n}\partial_{\lambda_t^+}(\sigma_{1:k}^{\underline{\alpha}}, T_s(\mu_t^+)\times T_s(\lambda_t^+))$. For treating $n\to\infty$ asymptotics, we substitute $[\mathbb{L}_s^{+0}(t)/\tau_s(t)]^{n-k} = |\psi_0\rangle\langle\psi_0| + O(e^{-\delta n})$ and take into account the fact that the λ derivative should always hit the first factor, producing $\partial_\lambda\vec{\mathbb{L}}_s = \vec{\mathbf{s}}_{a_1}$; otherwise, the whole term would vanish due to Eq. (15). Thus, we find a compact matrix product formula for the components (with the $k=1$ component vanishing)

$$(\sigma_{1:k}^{\underline{\alpha}}, X_s(t)) = \langle\psi_{\alpha_1}|\mathbb{X}^{\alpha_2}\cdots\mathbb{X}^{\alpha_{k-1}}|\psi_{\alpha_k}\rangle + O(e^{-\delta n}), \qquad (19)$$

where $\mathbb{X}^\alpha \coloneqq \mathbb{L}_s^{+\alpha}(t)/\tau_s(t)$, $|\psi_\alpha\rangle \coloneqq \sqrt{2}\mathbf{s}_{a_1}^\alpha|\psi_0\rangle/\tau_s(t)$. The HS norm of $X_s(t)$ projected onto ℓ sites in the limit $n-\ell\to\infty$ can be written analogously to Eq. (17)

$$\lim_{n\to\infty}\sum_{k=2}^{\ell}(\ell-k+1)\sum_{\underline{\alpha}}|(\sigma_{1:k}^{\underline{\alpha}}, X_s(t))|^2 = \frac{1}{[\tau_s(t)]^{2\ell}}\partial_{\lambda_t^-}\partial_{\lambda_t^+}\Big(\langle\Psi_0|(\mathbb{T}_{s,s}(\mu_t^-,\lambda_t^-,\mu_t^+,\lambda_t^+))^\ell|\Psi_0\rangle - \langle\psi_0|(\mathbb{L}_s^0(\mu_t^-,\lambda_t^-))^\ell|\psi_0\rangle\langle\psi_0|(\mathbb{L}_s^0(\mu_t^+,\lambda_t^+))^\ell|\psi_0\rangle\Big), \qquad (20)$$

thus, resulting in expression $\propto\ell$, cf. Eq. (18), without any finite-size (ℓ-dependent) corrections as $|\Psi_0\rangle$ is an exact eigenstate. We have, thus, shown that the expansion

$$X_s(t) = \lim_{\ell\to\infty}\lim_{n\to\infty}\sum_{k=2}^{\ell}\sum_{\underline{\alpha}}(\sigma_{1:k}^{\underline{\alpha}}, X_s(t))\sum_{x=0}^{n-1}\hat{\mathcal{S}}^x(\sigma_{1:k}^{\underline{\alpha}}) \qquad (21)$$

is complete in the HS norm. ■

Equations (20) and (21) have two useful implications: (i) As the state $|\Psi_0\rangle$ is a spin singlet (in four-spin auxiliary space), the only relevant part of the SU(2) invariant TM $\mathbb{T}_{s,s'}(t,t') = \sum_\alpha\mathbb{L}_s^{-\alpha}(t)\otimes\mathbb{L}_{s'}^{+\alpha}(t')$, is the $(2J+1)$-dimensional block $J=\min\{s,s'\}$ constituting the spin singlet subspace of $\mathcal{H}_a$, where it can be written explicitly as a tridiagonal matrix (see Sec. A of Ref. [21]). (ii) The HSK can be compactly written in terms of the resolvent of the TM, similar as in Ref. [16], namely, $K_{s,s'}(t,t') = n\sum_{k=0}^\infty\langle\Psi|[\tilde{\mathbb{T}}_{s,s'}(t,t')]^k|\Psi\rangle$, where $\tilde{\mathbb{T}}_{s,s'}(t,t') = \mathbb{T}_{s,s'}(t,t')/[\tau_s(t)\tau_{s'}(t')]$ and $|\Psi\rangle = \sum_{\alpha\in\{x,y,z\}}|\psi_\alpha\rangle\otimes|\psi_\alpha\rangle$, e.g., via solving a system of $2J$ linear equations

$$K_{s,s'}(t,t') = n\langle\Psi|\Phi\rangle, \qquad [\mathbb{1} - \tilde{\mathbb{T}}_{s,s'}(t,t')]|\Phi\rangle = |\Psi\rangle. \tag{22}$$

By deriving the explicit form of matrix elements of $\mathbb{T}_{s,s'}(t,t')$ and solving Eq. (22), we can encode the HSK explicitly in terms of a superposition of Cauchy-Lorentz distributions (assuming $s \le s'$) (see Sec. A of Ref. [21])

$$K_{s,s'}(t,t') = n\frac{\kappa_{s,s'}(t-t')}{\tau_s(t)\tau_{s'}(t')},$$
$$\kappa_{s,s'} = \sum_{l=1}^{2s}\frac{l[l+2(s'-s)](2s+1-l)(2s'+1+l)}{(2s+1)(2s'+1)}c_{s'-s+l},$$
$$\text{where } c_s(\tau) := \frac{s}{s^2+\tau^2}. \tag{23}$$

Note that the HSK is symmetric $K_{s,s'}(t,t') = K_{s',s}(t',t)$ and strictly positive $K_{s,s'}(t,t') > 0$, $\forall\ s,s',t,t'$.

We would like to remind the reader that in the TL $n\to\infty$, the $s=\frac{1}{2}$ family $X_{1/2}(t)$ is equivalent to the family of local charges Q_k, as follows from Eqs. (7) and (9)

$$X_{1/2}(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}Q_{k+2}, \quad Q_{k+2} = \partial_t^k X_{1/2}(t)|_{t=0}. \tag{24}$$

Equation (19), thus, generates also a handy explicit matrix product representations of the standard local conservation laws Q_k or their densities q_k.

Proof of linear independence.—Let us first show that $X_1(t)$ are linearly independent from $X_{1/2}(t)$, i.e., from Q_k. We define an operator

$$\tilde{X}_1(t) = X_1(t) - \int_{-\infty}^{\infty} dt' f_t(t') X_{1/2}(t'), \tag{25}$$

where the function $f_t(t')$ is determined by minimizing the HS norm $\|\tilde{X}_1(t)\|^2_{\rm HS}$, i.e., by the variation

$$\frac{\delta}{\delta f_t(t')}(\tilde{X}_1(t),\tilde{X}_1(t)) = 0, \tag{26}$$

resulting in the Fredholm equation of the first kind

$$\int_{-\infty}^{\infty} dt'' K_{1/2,1/2}(t',t'') f_t(t'') = K_{1/2,1}(t',t). \tag{27}$$

Using the fact that the kernels (23) are related to Cauchy-Lorentz distributions $c_s(t)$ up to trivial rescalings, we make an ansatz $f_t(t') = [\tau_{1/2}(t')/\tau_1(t)]\varphi(t-t')$ which maps Eq. (27) to a linear convolution equation $\frac{3}{4}c_1 * \varphi = \frac{4}{3}c_{3/2}$, which, using the well-known convolution identity $c_s * c_{s'} = \pi c_{s+s'}$, results in $\varphi = (16/9\pi)c_{1/2}$, or

$$f_t(t') = \frac{8}{9\pi}\frac{1+t'^2}{[(3/2)^2+t^2][(1/2)^2+(t-t')^2]}. \tag{28}$$

The conclusion of this analysis is that a family $\tilde{X}_1(t)$ is (a) *quasilocal* (see Sec. D of Ref. [21] for a numerical example) as its HSK computed via Eqs. (23), (25), and (28) is extensive $(\tilde{X}_1(t),\tilde{X}_1(t')) = [n/\tau_1(t)\tau_1(t')][\frac{8}{9}c_1(t-t') - \frac{4}{27}c_2(t-t')]$ and (b) is orthogonal to (and, hence, linearly independent from) all known local operators contained in the $s=1/2$ family $X_{1/2}(t)$; namely, we have $(\tilde{X}_1(t), Q_k) = (\tilde{X}_1(t), X_{1/2}(t')) = 0$, for all t,t',k. More generally, one can orthogonalize $X_s(t)$ for higher s to all previous $X_{s'}(t')$ for $s'<s$ by making an ansatz $\tilde{X}_s(t) = X_s(t) - \int_{-\infty}^{\infty} dt'(f^t_{s,s-1/2}(t')X_{s-1/2}(t') + f^t_{s,s-1}(t')X_{s-1}(t'))$, with explicit expressions for bounded integrable functions $f^t_{s,s-1}(t'), f^t_{s,s-1/2}(t')$. These families are HS orthogonal for different auxiliary spins, namely, $(\tilde{X}_s(t),\tilde{X}_{s'}(t')) = 0$ for $s\neq s'$, while $(\tilde{X}_s(t),\tilde{X}_s(t)) > 0$, i.e., $\tilde{X}_s(t)\neq 0$, for all s,t (for details, see Sec. C of Ref. [21]). This implies that $X_s(t)$ are linearly independent from all previous $X_{s'}(t')$ for $s'<s$ and, in particular, from $X_{1/2}(t')$ or Q_k. ■

Discussion.—We have proposed a direct extension of local conserved operators derived from the logarithm of the fundamental TM [2,3,22] to higher-spin auxiliary spaces. We have proved that in such a case, the resulting operators are quasilocal. An interesting side result of our statement is an asymptotic (thermodynamic) $n\to\infty$ inversion formula [23] $T_s^{-1}(\frac{1}{2}+it) \simeq [\tau_s(t)]^{-1}T_s(-\frac{1}{2}+it)$, valid for any $s\in\frac{1}{2}\mathbb{Z}^+$, which can be proven by implementing our matrix product formula (19) together with the gap statements (Sec. B of Ref. [21]) to show that $T_s(\mu_t^{\pm})T_s(\lambda_t^{\pm}) \simeq \tau_s(t)\mathbb{1}$. Our quasilocal operators $X_s(t)$ (7) can, thus, be understood as *logarithmic derivatives* of $T_s(\lambda_t^+)$. In TL $n\to\infty$, they become Hermitian operators for any $t\in\mathbb{R}$. For $s=\frac{1}{2}$, the Taylor expansion coefficients in t turn out to be *local* operators, while for $s>\frac{1}{2}$, they remain nonlocal but *quasilocal*. One could, thus, equivalently work with a discrete series of quasilocal operators $Q_{s,k+2} = (1/k!)\partial_t^k X_s(t)|_{t=0}$, $s\in\frac{1}{2}\mathbb{Z}^+, k\in\mathbb{Z}^+$, rather than with a series of continuous families $X_s(t)$. As a double index suggests, the number of relevant quasilocal charges in a large finite system may grow as n^2, rather than n as in the ultralocal case, although this question cannot be made precise with the results at hand.

Our results promise a number of timely applications and generalizations. The new quasilocal families should be included in order to correctly describe the $k\to 0, \omega\to 0$ limit of dynamical structure factors and general Drude weights at finite temperatures [24–26] or the GGE in quantum quench protocols [8]. For computing stationary expectations of local observables after a quench from a nonthermal initial state, such as, e.g., the Néel state $|{\rm N}\rangle$, one can readily demonstrate extensivity $\langle \mathrm{N}|X_s(t)|\mathrm{N}\rangle \propto n$ by extracting the leading eigenvalue of an associated transfer matrix, essentially proceeding along the lines of the calculation done in Ref. [19] for the fundamental ($s = 1/2$) TM. Appropriate q deformations of the concepts developed in this Letter should provide additional quasilocal operator families for the anisotropic Heisenberg model (XXZ chain). Extensions to SU(N) symmetric integrable spin chains seem straightforward, whereas a generalization to continuous quantum integrable systems and field theories (such as Lieb-Liniger or sine-Gordon models) should be a challenge for the future. We close by stressing an important point of distinction with respect to spin-reversal symmetry breaking quasilocal conserved operators in the XXZ model [14–17]. Quasilocality, as abstractly formulated here, requires a finite-dimensional (but nonfundamental) representation of a quantum TM and a factorizability condition for the leading eigenvalue of the associated auxiliary TM. This can happen either for irreducible unitary representations of the symmetry group but will result in operators which are always even under spin reversal, as is the case here, or due to the root-of-unity (commensurability) condition for the anisotropy, where highest-weight-type nonunitary representations become reducible to finite-dimensional ones, such as in the XXZ model.

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