Optimal Feedback Scheme and Universal Time Scaling for Hamiltonian Parameter Estimation

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(Received 15 April 2015; published 8 September 2015)

Time is a valuable resource and it is expected that a longer time period should lead to better precision in Hamiltonian parameter estimation. However, recent studies in quantum metrology have shown that in certain cases more time may even lead to worse estimations, which puts this intuition into question. In this Letter we show that by including feedback controls this intuition can be restored. By deriving asymptotically optimal feedback controls we quantify the maximal improvement feedback controls can provide in Hamiltonian parameter estimation and show a universal time scaling for the precision limit under the optimal feedback scheme. Our study reveals an intriguing connection between noncommutativity in the dynamics and the gain of feedback controls in Hamiltonian parameter estimation.

DOI: 10.1103/PhysRevLett.115.110401 PACS numbers: 03.65.Wj, 03.67.-a, 06.20.Dk

Implementations of quantum technology often require full and precise information about the parameters that govern the system evolution, which makes quantum Hamiltonian parameter estimation a crucial problem. An important task of Hamiltonian parameter estimation is to find out the ultimate achievable precision limit with given resources and design schemes to attain it [1–16]. Typically, Hamiltonian parameter estimation is achieved by preparing some initial quantum state ρ_0 and letting it evolve under the Hamiltonian $H(x)$, through the evolution $\rho_x = U_x \rho_0 U_x^\dagger$, where $U_x = e^{-iH(x)T}$, the unknown parameter in the Hamiltonian is imprinted on ρ_x; one can then estimate the parameter through measurements on ρ_x. This problem is well studied in quantum metrology when the Hamiltonian is in the multiplicative form of the parameter $H(x) = xH$; in this case it is known that the optimal strategy is to prepare the initial state as $(|\lambda_{\max}\rangle + |\lambda_{\min}\rangle/\sqrt{2})$, where $|\lambda_{\max(\min)}\rangle$ is the eigenvector of H for the maximum (minimum) eigenvalue; the standard deviation of the optimal unbiased estimator of x then scales as $1/\sqrt{nJ}$; here, n is the number of times that the process is repeated and $J = (\lambda_{\max} - \lambda_{\min})^2 T^2$ is the maximal quantum Fisher information, where T is the time that the Hamiltonian acts on initial states [3]. In this case the standard deviation of the estimation scales as $1/T$, showing that asymptotically more time always leads to better precision, which is consistent with our intuition. However, for general Hamiltonian $H(x)$, recent studies have shown different time scalings [17]; for example, if a Hamiltonian takes the form $H(x) = B[\cos(x)\sigma_1 + \sin(x)\sigma_3]$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are Pauli matrices, the maximum quantum Fisher information is $4\sin^2 BT$, which oscillates with time [17,18]. Thus, for general Hamiltonians more time may even lead to worse precision; this is against our intuition. In this Letter we will show that this intuition can be restored when feedback controls are included.

Previous studies have obtained some lower bounds on the precision limit with independent noises under feedback schemes [8,14]. However, it was largely unknown when feedback controls can actually help improve the precision limit and how to actually design optimal feedback controls to achieve better precision. In this Letter we present an optimal feedback scheme that provides the maximal improvement in the precision limit for general Hamiltonian parameter estimation, we show that under the optimal feedback scheme the precision limit displays a universal time scaling $1/T$, which is independent of the form of the Hamiltonian. This restores the intuition that more time always lead to better precision. Our study also shows that the gain of feedback control is intriguingly connected to the noncommutativity in the dynamics, while noncommutativity was previously thought to only play a role at the measurement stage. We focus on single parameter estimation; generalization to multiple parameters is possible, but it is beyond the scope of this Letter.

Methods developed previously for computing the precision limit of general Hamiltonian parameter estimation are quite involved and it is hard to incorporate feedback controls [17,18]. We first introduce a tool which is computationally efficient and it is convenient to include feedback controls.

The precision of estimating x from quantum states ρ_x is related to the Bures distance between ρ_x and its neighboring states ρ_{x+dx} [5],

$$d^2_{\text{Bures}}(\rho_x, \rho_{x+dx}) = \frac{1}{4} J(\rho_x) dx^2, \tag{1}$$

the Bures distance d_{Bures} is defined here as

$$d_{\text{Bures}}(\rho_1, \rho_2) = \sqrt{2 - 2F_B(\rho_1, \rho_2)}, \tag{2}$$

where $F_B(\rho_1, \rho_2) = \text{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}$ is the fidelity. Maximizing the quantum Fisher information $J(\rho_x)$ is then equivalent to maximizing the Bures distance between ρ_x and its neighboring states. If the evolution is governed by $U_x = e^{-iH(x)T}$ with a general Hamiltonian $H(x)$, then $\rho_x = U_x\rho_0 U_x^\dagger$ and $\rho_{x+dx} = U_{x+dx}\rho_0 U_{x+dx}^\dagger$; thus,

$$\begin{aligned}&\max_{\rho_0} d^2_{\text{Bures}}(U_x\rho_0 U_x^\dagger, U_{x+dx}\rho_0 U_{x+dx}^\dagger)\\ &\quad = \max_{\rho_0}[2 - 2F_B(U_x\rho_0 U_x^\dagger, U_{x+dx}\rho_0 U_{x+dx}^\dagger)]. \end{aligned}\tag{3}$$

Denote $B_\theta(U_x, U_{x+dx})$ as

$$B_\theta(U_x, U_{x+dx}) = \arccos \min_{\rho_0} F_B(U_x\rho_0 U_x^\dagger, U_{x+dx}\rho_0 U_{x+dx}^\dagger),$$

which we call the Bures angle between U_x and U_{x+dx}, then $\max_{\rho_0} d^2_{\text{Bures}}(\rho_x, \rho_{x+dx}) = 2 - 2\cos B_\theta(U_x, U_{x+dx})$. From Eq. (1) we get

$$\max_{\rho_0} J = \lim_{dx\to 0} \frac{8[1 - \cos B_\theta(U_x, U_{x+dx})]}{dx^2}. \tag{4}$$

Since $F_B(U_x\rho_0U_x^\dagger, U_{x+dx}\rho_0 U_{x+dx}^\dagger) = F_B(\rho_0, U'\rho_0 U'^\dagger)$, where $U' = U_x^\dagger U_{x+dx}$, we have $B_\theta(U_x, U_{x+dx}) = B_\theta(I, U')$. Let $e^{-i\theta_j^{U'}}$ be eigenvalues of U', where $\theta_j^{U'} \in (-\pi, \pi]$ for $1 \le j \le d$; here, d denotes the dimension of U'. We call $\theta_j^{U'}$, $1 \le j \le d$ the eigenangles of U' and arrange $\theta_{\max}^{U'} = \theta_1^{U'} \ge \theta_2^{U'} \ge \cdots \ge \theta_d^{U'} = \theta_{\min}^{U'}$ in decreasing order. Then, if $\theta_{\max}^{U'} - \theta_{\min}^{U'} \le \pi$ we have $\min_{\rho_0} F_B(\rho_0, U'\rho_0 U'^\dagger) = \cos[(\theta_{\max}^{U'} - \theta_{\min}^{U'})/2]$ [19–23].

If U_x is continuous with respect to x, then when $dx \to 0$, $U' = U^\dagger(x)U(x+dx) \to I$; thus, $(\theta_{\max}^{U'} - \theta_{\min}^{U'})/2 \to 0$. Denote $C_\theta(U) = (\theta_{\max}^U - \theta_{\min}^U)/2$ for a given unitary operator; then, for continuous dynamics when dx is sufficiently small, $B_\theta(U_x, U_{x+dx}) = C_\theta(U_x^\dagger U_{x+dx})$. From Eq. (4) we get

$$\begin{aligned}\max_{\rho_0} J &= \lim_{dx\to 0} \frac{8[1 - \cos C_\theta(U_x^\dagger U_{x+dx})]}{dx^2}\\ &= \lim_{dx\to 0} \frac{16\sin^2 \frac{C_\theta(U_x^\dagger U_{x+dx})}{2}}{dx^2}\\ &= \lim_{dx\to 0} 16\left(\frac{\sin\frac{C_\theta(U_x^\dagger U_{x+dx})}{2}}{\frac{C_\theta(U_x^\dagger U_{x+dx})}{2}}\right)^2 \frac{\frac{C_\theta^2(U_x^\dagger U_{x+dx})}{4}}{dx^2}\\ &= \lim_{dx\to 0} \frac{4C_\theta^2(U_x^\dagger U_{x+dx})}{dx^2}, \end{aligned}\tag{5}$$

where for the last equality we used the fact that when $dx \to 0$, $[C_\theta(U_x^\dagger U_{x+dx})/2] \to 0$ and $\lim_{y\to 0}(\sin y/y) = 1$. The ultimate precision limit is then given by

$$\delta\hat{x} \ge \frac{1}{\sqrt{n\max_{\rho_0} J}} = \frac{1}{\lim_{dx\to 0}\frac{2C_\theta(U_x^\dagger U_{x+dx})}{|dx|}\sqrt{n}}, \tag{6}$$

where n is the number of the times that the procedures are repeated.

If the Hamiltonian takes the multiplicative form $H(x) = xH$, $U_x = e^{-ixHT}$, $U_x^\dagger U_{x+dx} = e^{-iHTdx}$; then,

$$C_\theta(U_x^\dagger U_{x+dx}) = \frac{(\lambda_{\max} - \lambda_{\min})T|dx|}{2}.$$

Equation (6) then recovers the well-known formula [3]

$$\delta\hat{x} \ge \frac{1}{\sqrt{n}(\lambda_{\max} - \lambda_{\min})T}. \tag{7}$$

For general Hamiltonians, this also provides a straightforward way of calculating maximum quantum Fisher information. We demonstrate it through an example, which will also be used later to show the gain of feedback controls. Consider the Hamiltonian $H(x) = B[\cos(x)\sigma_1 + \sin(x)\sigma_3]$, where x is the interested parameter, representing the direction of a magnetic field in the XZ plane [17,18]. The Hamiltonian can be written compactly as $H(x) = B[\vec{a}(x)\cdot\vec{\sigma}]$, where $a_1(x) = \cos(x), a_2(x) = 0, a_3(x) = \sin(x)$. If it evolves with time T, then $U_x = e^{-iH(x)T} = e^{-iBT[\vec{a}(x)\cdot\vec{\sigma}]}$. In this case,

$$U' = U_x^\dagger U_{x+dx} = e^{iBT[\vec{a}(x)\cdot\vec{\sigma}]}e^{-iBT[\vec{a}(x+dx)\cdot\vec{\sigma}]}. \tag{8}$$

With a simple calculation, one gets

$$e^{iBT[\vec{a}(x)\cdot\vec{\sigma}]}e^{-iBT[\vec{a}(x+dx)\cdot\vec{\sigma}]} = e^{iB'[\vec{a'}\cdot\vec{\sigma}]}, \tag{9}$$

where $\vec{a'}$ is a unit vector and

$$\begin{aligned}\cos B' &= \cos^2(BT) + \cos(dx)\sin^2(BT) = \cos^2(BT)\\ &\quad + \left(1 - \frac{dx^2}{2}\right)\sin^2(BT) + O(dx^3)\\ &= 1 - \sin^2(BT)\frac{dx^2}{2} + O(dx^3). \end{aligned}\tag{10}$$

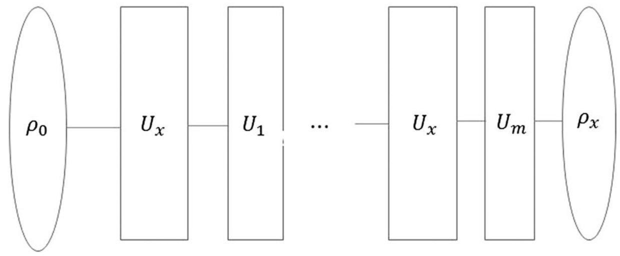

FIG. 1. Hamiltonian parameter estimation with feedback controls.

Since the eigenvalues of $e^{iB'(\vec{a}'\cdot\vec{\sigma})}$ are $e^{\pm iB'}$, we have $\theta^{U'}_{\max} = B'$ and $\theta^{U'}_{\min} = -B'$; thus, $(\theta^{U'}_{\max} - \theta^{U'}_{\min})/2 = B'$, i.e., $C_\theta(U^\dagger_x U_{x+dx}) = B'$. From Eq. (5) we then get

$$\max J = \lim_{dx\to 0} 8\frac{1-\cos B'}{dx^2} = 4\sin^2(BT).$$

This is consistent with previous studies [17,18]; however, our method makes the computation much simpler.

We now include feedback controls. Under the feedback scheme, the evolution is interspersed with feedback controls, as shown in Fig. 1. The whole evolution is described by

$$U_{mt}(x) = U_m U_t(x)\ldots U_2 U_t(x) U_1 U_t(x).$$

Here, $U_t(x) = e^{-iH(x)t}$, with $t = T/m$, $U_1, U_2, \ldots, U_m$ are coherent controls. We assume the controls take negligible time, which is a valid assumption in many physical settings. For example, in most quantum metrology applications, the field to be measured is usually very weak, while the controls can be executed by relative strong fields thus take negligible time.

We first derive optimal controls for the case of $m = 2$, same strategy works in the general case. When $m = 2$, $U_{2t}(x) = U_2 U_t(x) U_1 U_t(x)$, then

$$\begin{aligned}
&U^\dagger_{2t}(x)U_{2t}(x+dx)\\
&= U^\dagger_t(x)U^\dagger_1 U^\dagger_t(x)U^\dagger_2 U_2 U_t(x+dx)U_1 U_t(x+dx)\\
&= U^\dagger_t(x)U^\dagger_1[U^\dagger_t(x)U_t(x+dx)]U_1[U_t(x)U^\dagger_t(x)]U_t(x+dx)\\
&= [U^\dagger_t(x)U^\dagger_1][U^\dagger_t(x)U_t(x+dx)][U_1 U_t(x)]\\
&\quad\times[U^\dagger_t(x)U_t(x+dx)].
\end{aligned}$$

Since $C_\theta[U^\dagger_{2t}(x)U_{2t}(x+dx)]$ is determined by the eigenvalues of $U^\dagger_{2t}(x)U_{2t}(x+dx)$ and U_2 does not change the eigenvalues in this case, so it can be chosen as any unitary. This is reasonable, as the last control only rotates the final state but does not change the maximal information that can be extracted with measurements, as rotating the final state is basically equivalent to taking the measurements in another basis. We divide $U^\dagger_{2t}(x)U_{2t}(x+dx)$ into two parts, $[U^\dagger_t(x)U^\dagger_1]$ $[U^\dagger_t(x)U_t(x+dx)][U_1 U_t(x)]$ and $U^\dagger_t(x)U_t(x+dx)$. Then,

$$\begin{aligned}
C_\theta[U^\dagger_{2t}(x)U_{2t}(x+dx)] &\le C_\theta[U^\dagger_t(x)U_t(x+dx)]\\
&\quad + C_\theta\{[U^\dagger_t(x)U^\dagger_1][U^\dagger_t(x)U_t(x+dx)][U_1 U_t(x)]\}\\
&= 2C_\theta[U^\dagger_t(x)U_t(x+dx)],
\end{aligned} \tag{11}$$

where for the first inequality we used the following property of $C_\theta(U)$: $C_\theta(U_1U_2) \le C_\theta(U_1) + C_\theta(U_2)$, if $C_\theta(U_1) + C_\theta(U_2) \le \pi/2$ (see the Supplemental Material [24]); for the second equality, we used the fact that $[U^\dagger_t(x)U^\dagger_1]$ $[U^\dagger_t(x)U_t(x+dx)][U_1 U_t(x)]$ has the same eigenangles as $U^\dagger_t(x)U_t(x+dx)$. One obvious choice of control that saturates the equality is $U_1 = U^\dagger_t(x)$, as it aligns the eigenvalues of the two parts and the corresponding maximal and minimal eigenangles add up. In this case, $U^\dagger_{2t}(x)U_{2t}(x+dx) = [U^\dagger_t(x)U_t(x+dx)]^2$, $C_\theta[U^\dagger_{2t}(x)U_{2t}(x+dx)] = 2C_\theta[U^\dagger_t(x)U_t(x+dx)]$.

Previous results showing feedback controls do not help when $H(x) = xH$ [4] can be easily understood in our framework, as in that case $U_t(x)$ commutes with $U_t(x+dx)$. Thus, even when $U_1 = I$, i.e., without adding any control, $C_\theta[U^\dagger_{2t}(x)U_{2t}(x+dx)]$ still achieves its maximal value, as $[U^\dagger_t(x)U^\dagger_1][U^\dagger_t(x)U_t(x+dx)][U_1 U_t(x)]$ already equals to $U^\dagger(x)U(x+dx)$.

This analysis can be extended to general m straightforwardly with

$$C_\theta[U^\dagger_{mt}(x)U_{mt}(x+dx)] \le mC_\theta[U^\dagger_t(x)U_t(x+dx)], \tag{12}$$

where the equality can be saturated with the controls $U_1 = U_2 = \cdots = U_{m-1} = U^\dagger_t(x)$ and an arbitrary U_m, and under the optimal scheme $U^\dagger_{mt}(x)U_{mt}(x+dx) = [U^\dagger_t(x)U_t(x+dx)]^m$. This scheme was also used as a practical heuristic in recent studies of quantum Hamiltonian learning [29–32].

In practice, the true value of x is *a priori* unknown. Instead, an estimated value $\hat{x}$ has to be used for the feedback controls; the controls $U_1 = U_2 = \cdots = U_{m-1} = U^\dagger_t(\hat{x})$ need to be updated adaptively with the estimated value. The maximum quantum Fisher information is achievable asymptotically when $\hat{x} \to x$ [33–39]. For example, with $H(x) = B[\cos(x)\sigma_1 + \sin(x)\sigma_3]$, considering the feedback scheme with a total evolution time $T = mt$, and letting the controls $U_1 = U_2 = \cdots = U_{m-1} = U^\dagger_t(\hat{x})$, $\hat{x}$ here is the estimated value of x, which we write as $\hat{x} = (1+\beta)x$, where β calibrates the error in the estimation. In Fig. 2 we plotted the Fisher information with different values of β. It can be seen that for a broad range of β, the feedback scheme gains over the uncontrolled scheme. At the asymptotical limit $\beta = 0$, the feedback scheme attains the maximum quantum Fisher information, which equals to $4m^2\sin^2(BT/m)$. This value is much higher than the maximum Fisher information without feedback controls. When m is sufficiently large, $\sin(BT/m) \doteq (BT/m)$, the

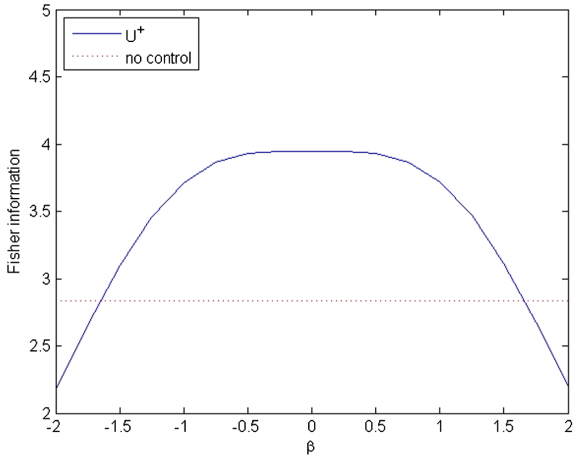

FIG. 2 (color online). Quantum Fisher information for parameter estimation of the Hamiltonian $H(x)=B[\cos(x)\sigma_1+\sin(x)\sigma_3]$, with $B=1$, $T=1$, and $t=1/5$; i.e., the evolution is interspersed with five controls. The controls are taken as $U_t^\dagger(\hat{x})$, with $\hat{x}=(1+\beta)x$, and the real value of x is assumed to be 1. In this case the feedback controls gain, as long as $\beta\in(-1.66, 1.66)$.

maximum quantum Fisher information is then $4B^2T^2$. The ultimate precision limit thus scales as $1/T$.

This time scaling, $1/T$, actually holds for any $H(x)$ under the optimal feedback scheme, which we will now show. Assume the evolution is interspersed with m controls,

$$U_{mt}(x) = U_m U_t(x)\ldots U_2 U_t(x) U_1 U_t(x).$$

Here, $U_t(x)=e^{-iH(x)t}$ with $t=T/m$. Under the optimal strategy $U_1=U_2=\cdots=U_{m-1}=U_t^\dagger(x)$, $C_\theta[U_{mt}^\dagger(x)U_{mt}(x+dx)]$ attains its maximal value $mC_\theta[U_t^\dagger(x)U_t(x+dx)]$. If m is taken as sufficiently large, i.e., $t=T/m$ is sufficiently small, then

$$U_t^\dagger(x)U_t(x+dx) = e^{iH(x)t}e^{-iH(x+dx)t} \doteq e^{-i\{[H(x+dx)-H(x)]T/m\}}. \quad (13)$$

Thus,

$$C_\theta[U_t^\dagger(x)U_t(x+dx)] \doteq \frac{[\lambda_{\max}(x,dx)-\lambda_{\min}(x,dx)]T}{2m},$$

here $\lambda_{\max(\min)}(x,dx)$ denotes the maximum (minimum) eigenvalue of $H(x+dx)-H(x)$. Thus, when m is sufficiently large,

$$mC_\theta[U_t^\dagger(x)U_t(x+dx)] \doteq \frac{[\lambda_{\max}(x,dx)-\lambda_{\min}(x,dx)]T}{2},$$

which gives the maximal quantum Fisher information

$$\max J = \lim_{dx\to 0}\frac{[\lambda_{\max}(x,dx)-\lambda_{\min}(x,dx)]^2}{dx^2}T^2. \quad (14)$$

The ultimate precision limit is then given by

$$\delta\hat{x} \geq \frac{1}{\sqrt{nJ}} = \frac{1}{\sqrt{n}\lim_{dx\to 0}\frac{\lambda_{\max}(x,dx)-\lambda_{\min}(x,dx)}{|dx|}T},$$

which displays the universal time scaling $1/T$.

Note that without feedback controls, the precision limit is determined by $C_\theta[U_T^\dagger(x)U_T(x+dx)]$, where $U_T^\dagger(x)U_T(x+dx)=e^{iH(x)T}e^{-iH(x+dx)T}$. Using the Baker-Campbell-Hausdorff formula [40], it can be expanded as

$$e^{iH(x)T}e^{-iH(x+dx)T} = e^{-i[H(x+dx)-H(x)]T+\frac{1}{2}[H(x)T,H(x+dx)T]+\cdots}. \quad (15)$$

With optimal feedback controls, the precision limit is determined by $C_\theta\{[U_{T/m}^\dagger(x)U_{T/m}(x+dx)]^m\}$. When m is sufficiently large, $[U_{T/m}^\dagger(x)U_{T/m}(x+dx)]^m \doteq e^{-i[H(x+dx)-H(x)]T}$. Thus the gain of the optimal feedback scheme is achieved by eliminating the commutators due to noncommutativity in the dynamics.

Besides measuring the direction of a magnetic field, which is closely related to determining the orientation of an object, noncommutativity arises frequently in practical experiments. The standard quantum metrology model assumes that the Hamiltonian takes the multiplicative form $H(x)=xH$, which only includes the interaction between the probe and the interested field. In real experiments, however, there are often other interactions. For example, for the nitrogen-vacancy (NV)-center spin, the Hamiltonian is

$$H_{\rm NV} = D(S_z^2-2/3)+E(S_x^2-S_y^2)+g\mu_B\vec{B}\cdot\vec{S}, \quad (16)$$

where $D\sim 2.87$ GHz is the crystal field splitting between the $m_s=0$ and $m_s=\pm1$ sublevels, $\vec{S}$ is the spin-1 operator,

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},$$

$$S_y = \frac{1}{\sqrt{2}i}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&1\end{pmatrix},$$

$$S_z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},$$

E is the crystal strain parameter which arises from the coupling of the NV spin to the lattice strain (usually in the megahertz range), $g=2$ is the electron g factor, μ_B is the Bohr magneton, and $\vec{B}$ is magnetic field [41]. If the magnetic field is along the Z direction, then the optimal probe state should take a superposition of $m_s=\pm1$ sublevels. The Hamiltonian can be effectively restricted to the subspace of these sublevels,

$$H_{\mathrm{eff}} = \begin{pmatrix} B_z & E \\ E & -B_z \end{pmatrix} = E\sigma_1 + B_z\sigma_3, \tag{17}$$

where we have absorbed $g\mu_B$ into B_z. Using the tools introduced, it is easy to calculate the maximal quantum Fisher information for measuring B_z: without feedback controls, the maximal quantum Fisher information is

$$4\left[\frac{B_z^2}{B_z^2+E^2}T^2 + \frac{E^2}{(B_z^2+E^2)^2}\sin^2(\sqrt{B_z^2+E^2}T)\right],$$

with feedback controls (in this case it can also be achieved by applying relative strong π pulses along the σ_3 direction), it can reach $4T^2$. The gain of feedback control can reach $(B_z^2+E^2)/B_z^2$ when T gets large. For a small B_z, which the NV center is mostly used for, this can be significant. Similarly, feedback controls can also gain when NV centers are used to measure temperature, pressure, and mechanical force [42–44] due to the noncommutativity in the dynamics.

Noncommutativity can also arise when multiple spins are used, as couplings between the spins may not commute with the interested field [45,46]. Besides quantum metrology, in other applications of Hamiltonian parameter estimation—for example, in quantum process tomography [47]—the structure of the Hamiltonian is often known and one needs to estimate some parameter in the Hamiltonian, where noncommutativity is usually generic.

With the presence of noise, even if the Hamiltonian takes the multiplicative form, the noisy part can make the dynamics noncommuting. For example, consider the following two dynamics:

$$\dot{\rho} = -i\left[\frac{\sigma_3}{2}x, \rho\right] + \frac{\gamma}{2}(\sigma_3\rho\sigma_3 - \rho),$$
$$\dot{\rho} = -i\left[\frac{\sigma_3}{2}x, \rho\right] + \frac{\gamma}{2}(\sigma_1\rho\sigma_1 - \rho). \tag{18}$$

The first represents a dynamics with both the magnetic field and the noise along the Z direction and thus commuting with each other; the second represents a dynamics with the magnetic field along the Z direction but noise along the X direction and thus not commuting with each other. It turns out that for the first dynamics, the feedback control $U_t^\dagger(x)$ cannot improve the precision limit, while for the second dynamics it does improve. Actually, recent studies have shown that by using quantum error correction techniques, feedback controls can extend the time scaling to $1/T$ for the second dynamics, as in that case the noise is perpendicular to the Hamiltonian and thus correctable with the aid of an auxiliary system [48–53]. Our numerical simulation also shows that even when the noise is not perpendicular to the Hamiltonian—for example, when the noise is along the direction $(\sigma_1+\sigma_3)/\sqrt{2}$—feedback controls still help to improve the precision limit, as in this case the dynamics still does not commute. We note that with the presence of noise, both $U_t^\dagger(x)$ and quantum error correction techniques may not be optimal. Finding the optimal feedback control will be a future research subject. Previous studies have given some upper bounds on the maximum quantum Fisher information under the feedback scheme when the dynamics are full rank [8,14], where it was shown that, generally, there is only a constant improvement over the the standard quantum limit. The time scaling thus cannot be extended to $1/T$ for those cases. However, in general—including those cases where the dynamics are of full rank—little was known when and how feedback controls can actually help to improve the precision limit, which is of practical importance. By revealing the intriguing connection between the gain of feedback scheme and noncommutativity in the dynamics, our study has significantly advanced the understanding of these problems.

Summary.—We derived an asymptotically optimal feedback scheme for Hamiltonian parameter estimation and showed that, under this scheme, the ultimate precision limit has a universal time scaling. This restored the intuition that time is always a valuable resource when the evolution is unitary. Our study also revealed that the gain of feedback schemes is intriguingly connected to the noncommutativity in the dynamics. The efficient tool developed here for computing the maximal quantum Fisher information can be extended to the noisy case [54], which is expected to have wide applications in quantum parameter estimation.

H. D. Yuan acknowledges the financial support from RGC of Hong Kong with Grant No. 538213.

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