Hydrodynamics Beyond the Gradient Expansion: Resurgence and Resummation

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Consistent formulations of relativistic viscous hydrodynamics involve short-lived modes, leading to asymptotic rather than convergent gradient expansions. In this Letter we consider the Müller-Israel-Stewart theory applied to a longitudinally expanding quark-gluon plasma system and identify hydrodynamics as a universal attractor without invoking the gradient expansion. We give strong evidence for the existence of this attractor and then show that it can be recovered from the divergent gradient expansion by Borel summation. This requires careful accounting for the short-lived modes which leads to an intricate mathematical structure known from the theory of resurgence.

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Introduction.—The past 15 years have witnessed the rising practical importance of relativistic viscous hydrodynamics. One reason for this is the success of hydrodynamic modeling of quark-gluon plasma (QGP) in heavy ion collision experiments at the RHIC and the LHC and the realization that QGP viscosity provides a crucial probe of QCD physics [1]. Another motivation is the relation between black holes and fluids, which originated in the 1970s as the rather mysterious black hole membrane paradigm [2]. With the advent of holography, this connection has been promoted to a precise correspondence [3] shedding light on both the physics of QGP and liquids in general, as well as on gravity.

The perfect fluid approximation is widely used in astrophysics and this theoretical description of relativistic inviscid fluids is rather well established [4]. On the other hand, relativistic viscous hydrodynamics is much less well understood. One of the recent insights is to regard hydrodynamics as a systematic gradient expansion, much in the spirit of low-energy effective field theory [5].

However, the requirement of causality leads to a framework which necessarily incorporates very large momenta (and frequencies). In all known examples, this is accompanied by the appearance of short-lived excitations—nonhydrodynamic modes. It has recently been shown, in the context of the AdS/CFT correspondence, that their presence leads to the divergence of the hydrodynamic gradient series for strongly coupled $\mathcal{N}=4$ super Yang-Mills (SYM) plasma [6]. In view of this, it is not clear whether or how a naive gradient expansion defines the theory. This is in fact a fundamental conceptual question concerning relativistic hydrodynamics as such.

In this Letter we propose a definite answer: since the nonhydrodynamic modes decay exponentially, the system relaxes to an attractor regardless of when an initial condition is set. In the following we consider a simple situation in which this can be made completely explicit: the Müller-Israel-Stewart (MIS) theory [5,7,8] specialized to a longitudinally expanding conformal fluid. We show that the attractor can be determined by relaxation from solutions which take the form of a *transseries*. The higher orders of this transseries are encoded in the divergent hydrodynamic gradient expansion, in line with expectations based on resurgence ideas [9].

Müller-Israel-Stewart theory.—The Landau-Lifschitz formulation of relativistic viscous hydrodynamics [10] asserts that the evolution equations for the hydrodynamic fields—temperature T and flow velocity u^μ—are the conservation equations of the energy-momentum tensor

$$\langle T^{\mu\nu}\rangle = \mathcal{E}u^\mu u^\nu + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}, \qquad (1)$$

where the shear stress tensor $\Pi^{\mu\nu}$ is given by

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}. \qquad (2)$$

From a modern perspective, one could contemplate including all possible terms graded by the number of derivatives of T and u^μ. If this is done to a finite order, as in Eq. (2), the resulting theory will not have a well-posed initial value problem due to superluminal signal propagation [11–14].

MIS theory resolves this problem by promoting the shear stress tensor to an independent dynamical field which satisfies a relaxation-type equation:

$$(\tau_\Pi u^\alpha \partial_\alpha + 1)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \cdots, \qquad (3)$$

where τ_Π is a phenomenological parameter (the relaxation time) and the ellipsis denotes several additional terms whose explicit form can be found in Ref. [5]. Linearization of the resulting theory is causal as long as $T\tau_\Pi \geq \eta/s$. This approach has enjoyed great success in describing the evolution of QGP [15]. It has also been obtained as the long-wavelength effective description of strongly coupled $\mathcal{N}=4$ SYM plasma in the framework of the AdS/CFT correspondence [3,5].

The iterative solution of Eq. (3) generates the gradient expansion of the shear stress tensor leading to the appearance of an infinite number of terms on the rhs of Eq. (2). Their coefficients (the transport coefficients of all orders) are expressed in terms of η and τ_Π (and four more parameters in the general conformal case [5]).

In the language of high energy physics, MIS theory can be regarded as an UV completion of Landau-Lifschitz theory. This is in contrast to the standard way of viewing it as a phenomenological model providing an effective description of some microscopic system only at late times. By treating MIS theory as an UV completion, we mean that we consider a hypothetical physical system such that the MIS theory describes it also at early times. This could be the case for systems where the approach to equilibrium is governed by a purely damped quasinormal mode.

The setup.—To overcome the complexity of the MIS equations, we focus on the case of Bjorken flow [16], which, due to a very high degree of symmetry, reduces them to a set of ordinary differential equations. The symmetry in question, boost invariance, can be taken to mean that in proper time-rapidity coordinates τ, y (related to Minkowski coordinates t, z by $t=\tau\cosh y$ and $z=\tau\sinh y$), the energy density, flow velocity, and shear stress tensor depend only on the proper time τ. The MIS equations then take the simple form

$$\tau\dot{\epsilon}=-\frac{4}{3}\epsilon+\phi,\quad \tau_\Pi\dot{\phi}=\frac{4\eta}{3\tau}-\frac{\lambda_1\phi^2}{2\eta^2}-\frac{4\tau_\Pi\phi}{3\tau}-\phi,\qquad (4)$$

where the dot denotes a proper time derivative and $\phi\equiv-\Pi_y^y$, the single independent component of the shear stress tensor. The term involving λ_1 comes from the elided terms in Eq. (3); for details, see Ref. [5].

In a conformal theory, $\epsilon\sim T^4$ and the transport coefficients satisfy

$$\tau_\Pi=\frac{C_{\tau\Pi}}{T},\qquad \lambda_1=C_{\lambda_1}\frac{\eta}{T},\qquad \eta=C_\eta s,\qquad (5)$$

where s is the entropy density and $C_{\tau\Pi}, C_{\lambda_1}, C_\eta$ are dimensionless constants. In the case of $\mathcal{N}=4$ SYM theory their values are known from fluid-gravity duality [3]:

$$C_{\tau\Pi}=\frac{2-\log(2)}{2\pi},\qquad C_{\lambda_1}=\frac{1}{2\pi},\qquad C_\eta=\frac{1}{4\pi}.\qquad (6)$$

The hydrodynamic attractor.—From Eq. (4) one can derive a single second order equation for the energy density or, equivalently, the temperature:

$$\tau C_{\tau\Pi}\frac{\ddot{T}}{T}+3\tau C_{\tau\Pi}\left(\frac{\dot{T}}{T}\right)^2+\left(\frac{11C_{\tau\Pi}}{3T}+\tau\right)\dot{T}+ -\frac{4C_\eta}{9\tau}+\frac{4C_{\tau\Pi}}{9\tau}+\frac{1}{3}T=0.\qquad (7)$$

To simplify the presentation, we have set $C_{\lambda_1}=0$ in this equation as well as in Eqs. (9), (10), and (13) below.

To proceed further, it is crucial to rewrite Eq. (7) in first order form. Introducing the dimensionless variables w and f (as in Ref. [17]),

$$w=\tau T,\qquad f=\tau\frac{\dot{w}}{w},\qquad (8)$$

the MIS evolution equation (7) takes the form

$$C_{\tau\Pi}wff'+4C_{\tau\Pi}f^2+\left(w-\frac{16C_{\tau\Pi}}{3}\right)f+ -\frac{4C_\eta}{9}+\frac{16C_{\tau\Pi}}{9}-\frac{2w}{3}=0,\qquad (9)$$

where the prime denotes a derivative with respect to w. Equations (8) and (9) together are equivalent to Eq. (7) as long as the function $w(\tau)$ is invertible.

At large times (which translate to large w), we expect universal hydrodynamic behavior [17]. In phenomenological analyses of heavy ion experiments, usually based on MIS theory, hydrodynamic codes are initialized typically at $w\approx 0.5$, which corresponds roughly to a time $\tau=0.5$ fm after the collision, with the temperature $T=350$ MeV at the center of the fireball at the RHIC (see, e.g., Ref. [18]). Equation (9) indeed possesses a unique stable solution which can be presented as a series in powers of $1/w$:

$$f(w)=\frac{2}{3}+\frac{4C_\eta}{9w}+\frac{8C_\eta C_{\tau\Pi}}{27w^2}+O\left(\frac{1}{w^3}\right).\qquad (10)$$

This is, in fact, the hydrodynamic gradient expansion.

It is easy to see that linear perturbations around this formal solution decay exponentially on a time scale set by τ_Π:

$$\delta f(w)\sim\exp\left(-\frac{3}{2C_{\tau\Pi}}w\right)w^{[(C_\eta-2C_{\lambda_1})/(C_{\tau\Pi})]}\left[1+O\left(\frac{1}{w}\right)\right].\qquad (11)$$

This is precisely the short-lived mode introduced by the MIS prescription. In the language of the gravity dual to $\mathcal{N}=4$ SYM theory, this would be an analog of a quasinormal mode [6,19] whose frequency is purely imaginary.

The presence of this exponentially decaying mode suggests that Eq. (9) possesses an attractor solution. We propose that this attractor constitutes the definition of hydrodynamic behavior. As discussed below, the presence of this attractor can be inferred without reference to the gradient expansion, which, as shown in the following section, is in fact divergent.

The existence of the hydrodynamic attractor is supported by examining the behavior of generic solutions of Eq. (8), with initial conditions set at various values of w. As seen in Fig. 1, a generic solution rapidly decays to the attractor. Furthermore, the attractor appears to persist even at very small values of w, where hydrodynamics of finite order

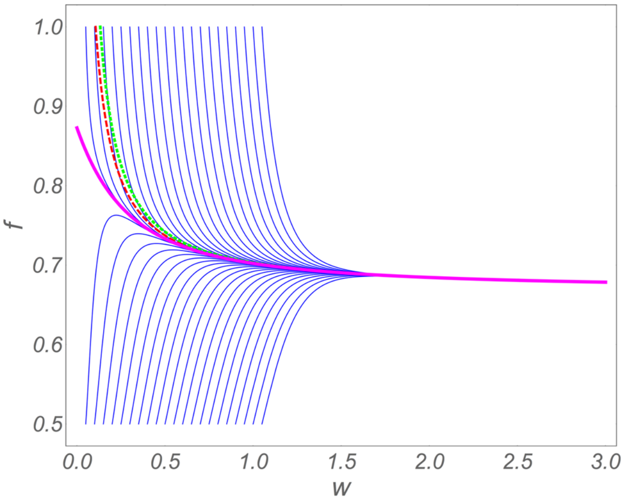

FIG. 1 (color online). The blue lines are numerical solutions of Eq. (8) for various initial conditions; the thick magenta line is the numerically determined attractor. The red dashed and green dotted lines represent first and second order hydrodynamics.

becomes ill defined. Perhaps unsurprisingly, truncating Eq. (10) at first or second order gives results distinctly different from the attractor at very small w. The magnitude of this difference depends on the values of the transport coefficients (this point is discussed further in the Supplemental Material [20]). Assuming $\mathcal{N} = 4$ SYM parameter values, we see that adopting just the viscous hydrodynamics constitutive relations provides a remarkably good approximation of the attractor for a wide range of w. In particular, this holds with an error smaller than 10% for $w > 0.5$.

Examining the behavior of f for w close to zero, one finds two solutions, one of which is stable:

$$f(w) = \frac{2\sqrt{C_{\tau\Pi}} + \sqrt{C_\eta}}{3\sqrt{C_{\tau\Pi}}} + O(w). \qquad (12)$$

By setting the initial value of f at $w \approx 0$ arbitrarily close to Eq. (12), the attractor can be determined numerically with the result shown in Fig. 1.

Another way of characterizing the attractor is to expand Eq. (9) in derivatives of f—this is an analog of the slow-roll expansion in theories of inflation (see, e.g., Ref. [21]). At leading order one finds

$$f(w) = \frac{2}{3} - \frac{w}{8C_{\tau\Pi}} + \frac{\sqrt{64C_\eta C_{\tau\Pi} + 9w^2}}{24C_{\tau\Pi}}. \qquad (13)$$

Continuing this to second order gives an analytic representation of the attractor which matches the numerically computed curve even for w as small as 0.1.

Finally, one can also construct the attractor in an expansion around $w = 0$ starting with the $f(w)$ given by Eq. (12). It turns out that the radius of convergence of this series is finite.

All three expansion schemes described above are consistent with the numerically determined attractor.

Hydrodynamic gradient expansion at high orders.—In what follows we focus on the hydrodynamic expansion, the expansion in powers of $1/w$. It is straightforward to generate the gradient expansion up to essentially arbitrarily high order (in practice, we chose to stop at 200). The coefficients f_n of the series solution

$$f(w) = \sum_{n=0}^{\infty} f_n w^{-n} \qquad (14)$$

show factorial behavior at large n, as seen in Fig. 2. This is analogous to the results obtained in Ref. [6] for the case of $\mathcal{N} = 4$ SYM theory.

In view of the divergence of the hydrodynamic expansion, we turn to the Borel summation technique. The Borel transform of f is given by

$$f_B(\xi) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \xi^n \qquad (15)$$

and results in a series which has a finite radius of convergence. Note that in Eq. (15) large w corresponds to small ξ. To invert the Borel transform, it is necessary to know the analytic continuation of series (15), which we denote by $\tilde{f}_B(\xi)$. The inverse Borel transform

$$f_R(w) = \int_C d\xi e^{-\xi} \tilde{f}_B(\xi/w) = w \int_C d\xi e^{-w\xi} \tilde{f}_B(\xi), \qquad (16)$$

where C denotes a contour in the complex plane connecting 0 and ∞, is interpreted as a resummation of the original divergent series (14). To carry out the integration, it is essential to know the analytic structure of $\tilde{f}_B(\xi)$.

We perform the analytic continuation using diagonal Padé approximants [22], given by the ratio of two polynomials of order 100. This function has a dense sequence of poles on the real axis, starting at $\xi_0 = 7.21187$, which signals the presence of a cut originating at that point [23].

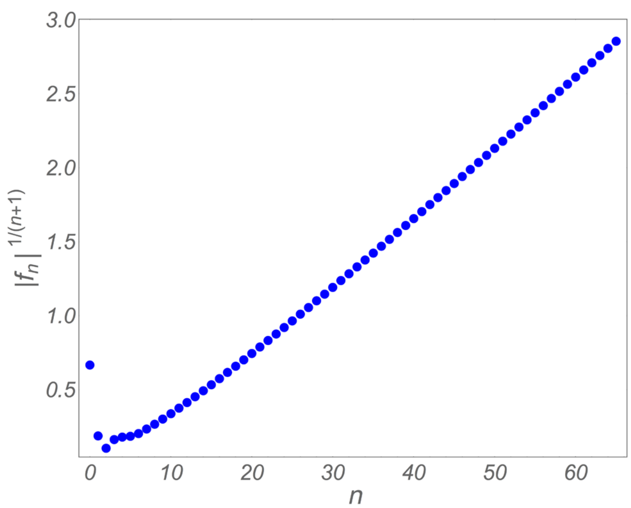

FIG. 2 (color online). The large order behavior of the hydrodynamic series. The slope is consistent with the location of the singularity nearest the origin, as given by Eq. (17).

This can be corroborated by applying the ratio method [22], which allows the estimation of the location and order of the leading branch-cut singularity by examining the series coefficients. Specifically, if the function approximated by Eq. (15) has the leading singularity of the form $(\xi_0 - \xi)^\gamma$, then for large n,

$$\frac{f_n}{f_{n+1}} = \frac{\xi_0}{n+1}\left[1 + \frac{\gamma+1}{n+1} + O\left(\frac{1}{n^2}\right)\right]. \qquad (17)$$

Applying this formula, one finds $\xi_0 = 7.21181$ (which is consistent with the pattern of poles) and $\gamma = 1.1449$.

As we argue in the Supplemental Material [20], the analytic structure of $\tilde{f}_B$ must involve further singularities on the real axis precisely in the following form:

$$\tilde{f}_B(\xi) = h_0(\xi) + (\xi_0 - \xi)^\gamma h_1(\xi) + (2\xi_0 - \xi)^{2\gamma} h_2(\xi) + \cdots, \qquad (18)$$

where the functions $h_k(\xi)$ are analytic and the ellipsis denotes further singularities at integer multiples of ξ_0.

Such branch-cut singularities of $\tilde{f}_B$ lead to ambiguities in the inverse Borel transform. Indeed, such a series is not Borel summable in the usual sense. It is, however, known that even in such cases a resummation is possible (see, e.g., Ref. [24]), but it requires a nontrivial choice of integration contour. The freedom in the choice of integration contour leads to complex ambiguities:

$$\delta f_R(w) = e^{i\pi k p\gamma} w \int_{k\xi_0}^{\infty} d\xi e^{-w\xi} (\xi - k\xi_0)^{k\gamma} h_k(\xi), \qquad (19)$$

where p is an odd integer reflecting the choice of Riemann sheet. For large w this becomes

$$\delta f_R(w) \approx e^{i\pi k p\gamma} \Gamma(k\gamma + 1) h_k(k\xi_0)(w^{-\gamma} e^{-w\xi_0})^k. \qquad (20)$$

This ambiguity is a feature of the hydrodynamic series and its presence is an indication of physics outside the gradient expansion. We saw in the previous section that there are nonanalytic, exponentially suppressed corrections to the hydrodynamic series following from the presence of the nonhydrodynamic MIS mode. These have precisely the correct structure to eliminate the $k = 1$ ambiguity in inverting the Borel transform. Indeed, comparing Eq. (20) with Eq. (11), we are led to identify ξ_0 with $3/2C_{\tau\Pi}$ and $-\gamma$ with $(C_\eta - 2C_{\lambda_1})/C_{\tau\Pi}$. Evaluating these combinations with parameter values appropriate for $\mathcal{N} = 4$ SYM theory [Eq. (6)] gives agreement to five significant digits. Both Eq. (20) and Eq. (11) receive corrections in $1/w$, and we expect them to match also.

The nonlinear structure of Eq. (9) suggests the presence of an infinite series of exponential corrections, which are matched by further branch cuts in Eq. (18). In the following section we calculate these corrections and give strong evidence that they conspire to yield an unambiguous, finite, and real answer for f_R, up to a real constant of integration (see also the Supplemental Material [20]).

Resurgence.—The results presented so far suggest that Eq. (9) should possess a solution in the form of a transseries [25]:

$$f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n}, \qquad (21)$$

where $\Omega \equiv w^{-\gamma} \exp(-w\xi_0)$, while c and $a_{m,n}$ are coefficients to be determined by the equation. By direct substitution, one can check up to high order that all of the coefficients $a_{m,n}$ in Eq. (21) are fixed uniquely apart from $a_{1,0}$, which can be absorbed into the constant c.

For each value of m in Eq. (21), the series over n is expected to be divergent—we have checked this for $m \leq 2$. Applying the Padé-Borel techniques discussed earlier leads to complex resummation ambiguities for each of these series. To obtain a meaningful answer, it must be possible to choose the single complex constant c in such a way that the result does not depend on the choice of integration contours and that the imaginary parts cancel.

The key observation is that the ambiguity at the leading order of the transseries is proportional to Ω, so it can only be canceled by terms of order $m = 1$ or higher. This cancellation determines the constant c,

$$c = r - e^{i\pi\gamma p} \Gamma(\gamma + 1) h_1^{(0)}(\xi_0), \qquad (22)$$

up to an arbitrary real number r, which is the expected integration constant for the first order differential equation (9).

Resummation.—Having provided strong evidence for the existence of an unambiguous and physically sensible result encoded in the transseries, we now invert the Borel sums for $m \leq 2$. For these calculations, we used extended precision arithmetic (keeping one thousand digits).

Inverting the Borel transform at each order of the transseries requires performing the integration in Eq. (16). The analytic continuation by Padé approximants works well in regions of the complex plane away from branch-cut singularities, so we take all of the integration contours to be straight lines at $\arg(\xi) = \pi/4$ (this is discussed further in the Supplemental Material [20]). The integrals computed in this way are complex. The findings of the previous section suggest that by taking the sum as in Eq. (21) one should be able to choose the imaginary part of the constant c so that the result is real for some range of w. This is indeed the case and gives a value for $\mathrm{Im}(c)$ consistent with Eq. (22) (with $p = -1$). A combined measure of error is the imaginary part of the result of the resummation—it remains very small (below 0.01% relative to the real part) for $w > 0.25$.

We compared the result of the resummation with the numerically computed attractor, which required fitting the integration constant $r = 0.049$ [see Eq. (22)]. As illustrated in the Supplemental Material [20], the generalized Borel sum of the gradient series indeed follows the attractor. Note that to match the attractor we need to choose the coefficient $r \neq -\mathrm{Re}(c)$, while naively one might expect that the attractor should correspond to omitting the exponential terms. This suggests that the resummation of the gradient series contains exponentially decaying terms which are not canceled by the exponential terms from higher orders in the transseries. Consequently, instead of thinking about nonhydrodynamic modes in terms of perturbations about the gradient expansion, one should more properly think of them as perturbations around the hydrodynamic attractor.

Conclusions.—Recent advances in applying the theory of resurgence to quantum theories [26–31] have motivated us to rethink the foundational aspects of relativistic hydrodynamics. The root of the problem is that causality precludes us from regarding hydrodynamics as a truncated gradient expansion, yet the series itself is divergent. We propose to view hydrodynamics as an attractor which governs the late time behavior of systems in their approach to equilibrium. In the context of boost-invariant flow in MIS theory, we have constructed such an attractor in several ways which are all consistent with each other.

By identifying the structure of singularities of the analytic continuation of the Borel transform of the hydrodynamic series in terms of nonhydrodynamic degrees of freedom, we hope that this Letter will provide a useful road map for understanding the meaning of higher order gradients in the case of $\mathcal{N} = 4$ SYM theory [6]. From the point of view of AdS/CFT, the exploratory studies described here suggest that the geometry constructed in the gradient expansion of fluid-gravity duality should be viewed as the leading term of a transseries containing the effects of quasinormal modes.

From a phenomenological perspective, MIS theory includes explicitly transport coefficients for terms up to second order in gradients, but it generates a gradient expansion to all orders. The transport coefficients can only match real QCD plasma up to second order. One may then wonder about the effects of all of the higher order terms [32] which cannot be matched by MIS theory. Our findings suggest that the attractor which governs its late time behavior is not very sensitive to the higher order terms (even when the gradient series is resummed). This makes it less surprising to learn that MIS theory can describe QGP evolution so well.

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