Nonrenormalization Theorems without Supersymmetry

Clifford Cheung* and Chia-Hsien Shen†
Walter Burke Institute for Theoretical Physics California Institute of Technology, Pasadena, California 91125, USA
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We derive a new class of one-loop nonrenormalization theorems that strongly constrain the running of higher dimension operators in a general four-dimensional quantum field theory. Our logic follows from unitarity: cuts of one-loop amplitudes are products of tree amplitudes, so if the latter vanish then so too will the associated divergences. Finiteness is then ensured by simple selection rules that zero out tree amplitudes for certain helicity configurations. For each operator we define holomorphic and antiholomorphic weights, $(w, \bar{w}) = (n - h, n + h)$, where n and h are the number and sum over helicities of the particles created by that operator. We argue that an operator $\mathcal{O}_i$ can only be renormalized by an operator $\mathcal{O}_j$ if $w_i \geq w_j$ and $\bar{w}_i \geq \bar{w}_j$, absent nonholomorphic Yukawa couplings. These results explain and generalize the surprising cancellations discovered in the renormalization of dimension six operators in the standard model. Since our claims rely on unitarity and helicity rather than an explicit symmetry, they apply quite generally.

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Introduction.—Technical naturalness dictates that all operators not forbidden by symmetry are compulsory—and thus generated by renormalization. Softened ultraviolet divergences are in turn a telltale sign of underlying symmetry. This is famously true in supersymmetry, where holomorphy enforces powerful nonrenormalization theorems.

In this Letter we derive a new class of nonrenormalization theorems for nonsupersymmetric theories. Our results apply to the one-loop running of the leading irrelevant deformations of a four-dimensional quantum field theory of marginal interactions,

$$\Delta\mathcal{L} = \sum_i c_i \mathcal{O}_i, \tag{1}$$

where $\mathcal{O}_i$ are higher dimension operators. At leading order in c_i, renormalization induces operator mixing via

$$(4\pi)^2 \frac{dc_i}{d\log\mu} = \sum_j \gamma_{ij} c_j, \tag{2}$$

where by dimensional analysis the anomalous dimension matrix γ_{ij} is a function of marginal couplings alone.

The logic of our approach is simple, making no reference to symmetry. Renormalization is induced by log divergent amplitudes, which by unitarity have kinematic cuts equal to products of on-shell tree amplitudes [1]. If any of these tree amplitudes vanish, then so too will the divergence. Crucially, many tree amplitudes are zero due to helicity selection rules, which, e.g., forbid the all minus helicity gluon amplitude in Yang-Mills theory.

For our analysis, we define the holomorphic and antiholomorphic weight of an on-shell amplitude A by

$$w(A) = n(A) - h(A), \qquad \bar{w}(A) = n(A) + h(A), \tag{3}$$

where $n(A)$ and $h(A)$ are the number and sum over helicities of the external states. (Holomorphic weight is a generalization of k charge in super Yang-Mills theory, where the N^kMHV amplitude has $w = k + 4$.) Since A is physical, its weight is field reparameterization and gauge independent. The weights of an operator $\mathcal{O}$ are then invariantly defined by minimizing over all amplitudes involving that operator: $w(\mathcal{O}) = \min\{w(A)\}$ and $\bar{w}(\mathcal{O}) = \min\{\bar{w}(A)\}$. In practice, operator weights are fixed by the leading nonzero contact amplitude built from an insertion of $\mathcal{O}$,

$$w(\mathcal{O}) = n(\mathcal{O}) - h(\mathcal{O}), \qquad \bar{w}(\mathcal{O}) = n(\mathcal{O}) + h(\mathcal{O}), \tag{4}$$

where $n(\mathcal{O})$ is the number of particles created by $\mathcal{O}$ and $h(\mathcal{O})$ is their total helicity. (By definition, all covariant derivatives D are treated as partial derivatives ∂ when computing the leading contact amplitude.) For field operators we find

$\mathcal{O}$	$F_{\alpha\beta}$	ψ_α	ϕ	$\bar{\psi}_{\dot{\alpha}}$	$\bar{F}_{\dot{\alpha}\dot{\beta}}$
h	$+1$	$+1/2$	0	$-1/2$	-1
$(w, \bar{w})$	$(0, 2)$	$(1/2, 3/2)$	$(1, 1)$	$(3/2, 1/2)$	$(2, 0)$

where all Lorentz covariance is expressed in terms of four-dimensional spinor indices, so e.g., the gauge field strength is $F_{\alpha\dot{\alpha}\beta\dot{\beta}} = F_{\alpha\beta}\bar{\epsilon}_{\dot{\alpha}\dot{\beta}} + \bar{F}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}$. The weights of all dimension five and six operators are shown in Fig. 1.

As we will prove, an operator $\mathcal{O}_i$ can only be renormalized by an operator $\mathcal{O}_j$ at one loop if the corresponding weights $(w_i, \bar{w}_i)$ and $(w_j, \bar{w}_j)$ satisfy the inequalities

$$w_i \geq w_j \quad \text{and} \quad \bar{w}_i \geq \bar{w}_j, \tag{5}$$

and all Yukawa couplings are of a "holomorphic" form consistent with a superpotential. This implies a new class of nonrenormalization theorems,

$$\gamma_{ij} = 0 \quad \text{if } w_i < w_j \quad \text{or} \quad \bar{w}_i < \bar{w}_j, \tag{6}$$

which impose mostly zero entries in the matrix of anomalous dimensions. The resulting nonrenormalization theorems for all dimension five and six operators are shown in Tables I and II.

Because our analysis hinges on unitarity and helicity rather than off-shell symmetry principles, the resulting

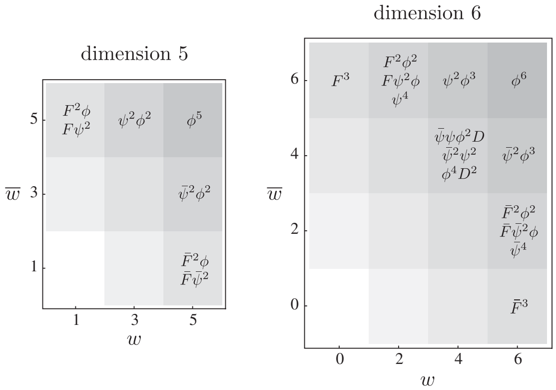

FIG. 1. Weight lattice for dimension five and six operators, suppressing flavor and Lorentz structures, e.g., on which fields the derivatives act. Our nonrenormalization theorems permit mixing of operators into operators of equal or greater weight. Pictorially, this forbids transitions down or to the left.

nonrenormalization theorems are general. Moreover, they explain the ubiquitous and surprising cancellations [2] in the one-loop renormalization of dimension six operators in the standard model [4–7]. Absent an explanation from power counting or spurions, the authors of Ref. [2] conjectured a hidden "holomorphy" enforcing nonrenormalization among holomorphic and antiholomorphic operators. We show here that this classification simply corresponds to $w < 4$ and $\bar{w} < 4$, so these cancellations follow immediately from Eq. (6), as shown in Table II.

Weighing tree amplitudes.—To begin, we compute the holomorphic and antiholomorphic weights $(w_n, \bar{w}_n)$ of a general n-point on-shell tree amplitude in a renormalizable theory of massless particles. We start at lower-point amplitudes and apply induction to extend to higher-point amplitudes.

The three-point amplitude is

$$A(1^{h_1}2^{h_2}3^{h_3}) = g \times \begin{cases} \langle 12\rangle^{r_3}\langle 23\rangle^{r_1}\langle 31\rangle^{r_2}, & \sum_i h_i \le 0 \\ [12]^{\bar{r}_3}[23]^{\bar{r}_1}[31]^{\bar{r}_2}, & \sum_i h_i \ge 0 \end{cases} \tag{7}$$

where g is the coupling and each case corresponds to maximally helicity violating (MHV) and $\overline{\text{MHV}}$ kinematics, $|1] \propto |2] \propto |3]$ and $|1\rangle \propto |2\rangle \propto |3\rangle$. Lorentz invariance fixes the exponents to be $r_i = -\bar{r}_i = 2h_i - \sum_j h_j$ and $\sum_i r_i = \sum_i \bar{r}_i = 1 - [g]$ by dimensional analysis [8]. According to Eq. (7), the corresponding weights are

$$(w_3, \bar{w}_3) = \begin{cases} (4-[g], 2+[g]), & \sum_i h_i \le 0 \\ (2+[g], 4-[g]), & \sum_i h_i \ge 0. \end{cases} \tag{8}$$

In a renormalizable theory, $[g] = 0$ or 1, so we obtain

$$w_3, \bar{w}_3 \ge 2, \tag{9}$$

for the three-point amplitude.

The majority of four-point tree amplitudes satisfy $w_4, \bar{w}_4 \ge 4$ because $w_4 < 4$ and $\bar{w}_4 < 4$ require a nonzero

TABLE I. Anomalous dimension matrix for dimension five operators in a general quantum field theory. The shaded entries vanish by our nonrenormalization theorems.

	$(w,\bar{w})$	$F^2\phi$ $(1,5)$	$F\psi^2$ $(1,5)$	$\psi^2\phi^2$ $(3,5)$	$\bar{F}\bar{\psi}^2$ $(5,1)$	$\bar{F}^2\phi$ $(5,1)$	$\bar{\psi}^2\phi^2$ $(5,3)$	ϕ^5 $(5,5)$
$F^2\phi$	$(1,5)$							
$F\psi^2$	$(1,5)$							
$\psi^2\phi^2$	$(3,5)$							
$\bar{F}^2\phi$	$(5,1)$							
$\bar{F}\bar{\psi}^2$	$(5,1)$							
$\bar{\psi}^2\phi^2$	$(5,3)$							
ϕ^5	$(5,5)$							

total helicity which is typically forbidden by helicity selection rules. To see why, we enumerate all possible candidate amplitudes with $w_4 < 4$. Analogous arguments will apply for $\bar{w}_4 < 4$.

Most four-point tree amplitudes with $w_4 = 1$ or 3 vanish since they have no Feynman diagrams, so

$$\begin{aligned} 0 &= A(F^+F^+F^\pm\phi) = A(F^+F^+\psi^\pm\psi^\pm) \\ &= A(F^+F^-\psi^+\psi^+) = A(F^+\psi^+\psi^-\phi) \\ &= A(\psi^+\psi^+\psi^+\psi^-). \end{aligned}$$

Furthermore, most amplitudes with $w_4 = 0$ or 2 vanish due to helicity selection rules, so

$$\begin{aligned} 0 &= A(F^+F^+F^+F^\pm) = A(F^+F^+\psi^+\psi^-) \\ &= A(F^+F^+\phi\phi) = A(F^+\psi^+\psi^+\phi). \end{aligned}$$

While Feynman diagrams exist, they vanish on shell for the chosen helicities. This leaves a handful of candidate nonzero amplitudes,

$$0 \ne A(\psi^+\psi^+\psi^+\psi^+), \quad A(F^+\phi\phi\phi), \quad A(\psi^+\psi^+\phi\phi),$$

with $w_4 = 2, 3, 3$, respectively. These "exceptional amplitudes" are the only four-point tree amplitudes with $w_4 < 4$ that do not vanish identically.

The exceptional amplitudes all require internal or external scalars, so they are absent in theories with only gauge bosons and fermions, e.g., QCD. The second and third amplitudes involve superrenormalizable cubic scalar interactions, which we do not consider here. The first amplitude arises from Yukawa couplings of nonholomorphic form: that is, $\phi\psi^2$ together with $\bar{\phi}\psi^2$, which in a supersymmetric theory would violate holomorphy of the superpotential. In the standard model, Higgs doublet exchange generates an exceptional amplitude proportional to the product up-type and down-type Yukawa couplings. This diagram will be important later when we consider the standard model. In summary,

$$w_4, \bar{w}_4 \ge 4, \tag{10}$$

for the four-point amplitude, modulo exceptional amplitudes.

Finally, consider a general higher-point tree amplitude, A_i, which on a factorization channel equals a product of amplitudes, A_j and A_k,

TABLE II. Anomalous dimension matrix for dimension six operators in a general quantum field theory. The shaded entries vanish by our nonrenormalization theorems, in full agreement with [2]. Here y^2 and $\bar{y}^2$ label entries that are nonzero due to nonholomorphic Yukawa couplings, $\times$ labels entries that vanish because there are no diagrams [3], and $\times^*$ labels entries that vanish by a combination of counterterm analysis and our nonrenormalization theorems.

		F^3	$F^2\phi^2$	$F\psi^2\phi$	ψ^4	$\psi^2\phi^3$	$\bar{F}^3$	$\bar{F}^2\phi^2$	$\bar{F}\bar{\psi}^2\phi$	$\bar{\psi}^4$	$\bar{\psi}^2\phi^3$	$\bar{\psi}^2\psi^2$	$\bar{\psi}\psi\phi^2 D$	$\phi^4 D^2$	ϕ^6
	$(w,\bar{w})$	$(0,6)$	$(2,6)$	$(2,6)$	$(2,6)$	$(4,6)$	$(6,0)$	$(6,2)$	$(6,2)$	$(6,2)$	$(6,4)$	$(4,4)$	$(4,4)$	$(4,4)$	$(6,6)$
F^3	$(0,6)$			$\times$	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$F^2\phi^2$	$(2,6)$				$\times$	$\times$				$\times$	$\times$	$\times$			$\times$
$F\psi^2\phi$	$(2,6)$									$\times$				$\times$	$\times$
ψ^4	$(2,6)$	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	y^2		$\times$	$\times$
$\psi^2\phi^3$	$(4,6)$	$\times^*$									y^2				$\times$
$\bar{F}^3$	$(6,0)$			$\times$	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$\bar{F}^2\phi^2$	$(6,2)$				$\times$	$\times$				$\times$	$\times$	$\times$			$\times$
$\bar{F}\bar{\psi}^2\phi$	$(6,2)$				$\times$									$\times$	$\times$
$\bar{\psi}^4$	$(6,2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$			$\times$	$\bar{y}^2$		$\times$	$\times$
$\bar{\psi}^2\phi^3$	$(6,4)$					$\bar{y}^2$	$\times^*$								$\times$
$\bar{\psi}^2\psi^2$	$(4,4)$		$\times$		$\bar{y}^2$	$\times$		$\times$		y^2	$\times$			$\times$	$\times$
$\bar{\psi}\psi\phi^2 D$	$(4,4)$														$\times$
$\phi^4 D^2$	$(4,4)$				$\times$					$\times$		$\times$			$\times$
ϕ^6	$(6,6)$	$\times^*$		$\times$	$\times$		$\times^*$		$\times$	$\times$		$\times$			

$$\text{fact}[A_i] = \frac{i}{\ell^2}\sum_h A_j(\ell^h)A_k(-\ell^{-h}), \tag{11}$$

depicted in Fig. 2. If the total numbers and helicities of A_i, A_j, and A_k, are (n_i, h_i), (n_j, h_j), and (n_k, h_k), then $n_i = n_j + n_k - 2$ and $h_i = h_j + h_k$, since either side of the factorization channel carries equal and opposite helicity. Thus, the corresponding weights, $(w_i, \bar{w}_i)$, $(w_j, \bar{w}_j)$, and $(w_k, \bar{w}_k)$, satisfy the following tree selection rule:

$$\begin{aligned} w_i &= w_j + w_k - 2 \\ \bar{w}_i &= \bar{w}_j + \bar{w}_k - 2. \end{aligned} \tag{12}$$

We have already shown that $w_3, \bar{w}_3 \geq 2$ and $w_4, \bar{w}_4 \geq 4$ modulo the exceptional diagrams. Since all five-point amplitudes factorize into three and four-point amplitudes, Eq. (12) implies that $w_5, \bar{w}_5 \geq 4$. Induction to a higher-point amplitude then yields the main result of this section,

$$w_n, \bar{w}_n \geq \begin{cases} 2, & n = 3 \\ 4, & n > 3 \end{cases} \tag{13}$$

which, modulo exceptional amplitudes, is a lower bound on the weights of n-point tree amplitudes in a theory of massless particles with marginal interactions. Note that even when exceptional amplitudes exist, $w_n, \bar{w}_n \geq 2$.

An important consequence of Eq. (12) is that attaching renormalizable interactions to an arbitrary amplitude A_j—perhaps involving irrelevant interactions—can only produce an amplitude A_i of greater or equal weight. To see why, note that A_i factorizes into A_j and an amplitude A_k composed of only renormalizable interactions, where $w_k, \bar{w}_k \geq 2$ by Eq. (13). Equation (12) then implies that $w_i \geq w_j$ and $\bar{w}_i \geq \bar{w}_j$, so the minimum weight amplitude involving a higher dimension operator is the contact amplitude built from a single insertion of that operator.

Weighing one-loop amplitudes.—The weights of one-loop amplitudes are obtained from generalized unitarity and the tree-level results of the previous section. The leading order renormalization of higher dimension operators is encoded in the anomalous dimension matrix γ_{ij} describing how $\mathcal{O}_i$ is radiatively generated by $\mathcal{O}_j$ and loops of marginal interactions. In practice, γ_{ij} is extracted from the one-loop amplitude A_i^{loop} built around an insertion of $\mathcal{O}_j$ with the same external states as the tree amplitude A_i built around an insertion of $\mathcal{O}_i$. Any ultraviolet divergence in A_i^{loop} must then be absorbed by the counterterm A_i, which implies nonzero γ_{ij}. By dimensional analysis, a necessary condition for renormalization is that $\mathcal{O}_i$ and $\mathcal{O}_j$ have equal mass dimension, but as we will see, this is not a sufficient condition because of our nonrenormalization theorems.

The Passarino-Veltman (PV) reduction [9] of the one-loop amplitude A_i^{loop} is

$$A_i^{\text{loop}} = \sum_{\text{box}} d_4 I_4 + \sum_{\text{triangle}} d_3 I_3 + \sum_{\text{bubble}} d_2 I_2 + \text{rational},$$

which sums over topologies of scalar box, triangle, and bubble integrals, I_4, I_3, and I_2. Tadpole integrals vanish for massless particles. The integral coefficients d_4, d_3, and d_2 are rational functions of external kinematic data. Ultraviolet log divergences arise from the scalar bubble integrals in

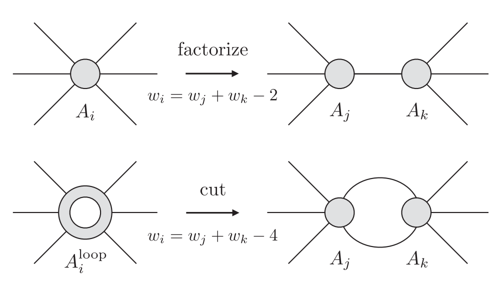

FIG. 2. Diagrams of tree factorization and one-loop unitarity, with the weight selection rules from Eqs. (12) and (18).

the PV reduction, where in dimensional regularization, $I_2 \to 1/(4\pi)^2\epsilon$. Separating ultraviolet divergent and finite terms, we find

$$A_i^{\rm loop} = \frac{1}{(4\pi)^2\epsilon}\sum_{\rm bubble} d_2 + \text{finite}, \tag{14}$$

which implies a counterterm tree amplitude,

$$A_i = -\frac{1}{(4\pi)^2\epsilon}\sum_{\rm bubble} d_2, \tag{15}$$

so $A_i^{\rm loop} + A_i$ is finite.

Generalized unitarity [1] fixes integral coefficients by relating kinematic singularities of the one-loop amplitude to products of tree amplitudes. The two-particle cut in a particular channel is

$$\text{cut}[A_i^{\rm loop}] = \sum_{h_1,h_2} A_j(\ell_1^{h_1},\ell_2^{h_2})A_k(-\ell_1^{-h_1},-\ell_2^{-h_2}), \tag{16}$$

where ℓ_1, ℓ_2 and h_1, h_2 are the momenta and helicities of the cut lines and A_j and A_k are on-shell tree amplitudes corresponding to the cut channel, as depicted in Fig. 2.

Applying this cut to the PV reduction, we find

$$\text{cut}[A_i^{\rm loop}] = d_2 + \text{terms depending on } \ell_1, \ell_2, \tag{17}$$

where the ℓ_1, ℓ_2 dependent terms correspond to two-particle cuts of triangle and box integrals. Famously, the divergence of the one-loop amplitude is related to the two-particle cut [10–12]. However, a kinematic singularity is present only if A_j and A_k are four-point amplitudes or higher, corresponding to "massive" bubble integrals. When A_j or A_k are three-point amplitudes, the associated "massless" bubble integrals are scaleless and vanish in dimensional regularization. We ignore these subtle contributions for now but revisit them later.

Equations (15), (16), and (17) imply that the total numbers and helicities (n_i, h_i), (n_j, h_j), (n_k, h_k) of A_i, A_j and A_k satisfy $n_i = n_j + n_k - 4$ and $h_i = h_j + h_k$, and thus the one-loop selection rule,

$$w_i = w_j + w_k - 4,$$
$$\bar{w}_i = \bar{w}_j + \bar{w}_k - 4, \tag{18}$$

where $(w_i, \bar{w}_i)$, $(w_j, \bar{w}_j)$, and $(w_k, \bar{w}_k)$ are the corresponding amplitude weights. For each γ_{ij} we identify A_i and A_j with tree amplitudes built around insertions of $\mathcal{O}_i$ and $\mathcal{O}_j$, and A_k with a tree amplitude of the renormalizable theory. As noted earlier, the amplitudes on both sides of the cut must be four-point or higher for a nontrivial unitarity cut, so Eq. (13) implies that $w_k, \bar{w}_k \geq 4$, absent exceptional amplitudes. Equation (18) then implies that $w_i \geq w_j$ and $\bar{w}_i \geq \bar{w}_j$, which is the nonrenormalization theorem of Eq. (5). If exceptional amplitudes with $w_k, \bar{w}_k = 2$ are present from nonholomorphic Yukawas, then Eq. (5) is violated by exactly two units.

Figure 1 shows the weight lattice for all dimension five and six operators in a general quantum field theory. We employ the operator basis of Ref. [13], so redundant operators, e.g., those involving $\Box\phi$, are eliminated by equations of motion. Our nonrenormalization theorems imply that operators can only renormalize operators of equal or greater weight, which in Fig. 1 forbids transitions that move down or to the left. The form of the anomalous dimension matrix for all dimension five and six operators is shown in Tables I and II.

Infrared divergences.—We now return to the issue of massless bubble integrals. While these contributions formally vanish in dimensional regularization, this is potentially misleading because ultraviolet and infrared divergences enter with opposite sign $1/\epsilon$ poles. Thus, an ultraviolet divergence may be present if there is an equal and opposite virtual infrared divergence [10–12]. Crucially, the Kinoshita-Lee-Nauenberg theorem [14] maintains that all virtual infrared divergences are canceled by an inclusive final state sum incorporating tree-level real emission of an unresolved soft or collinear particle. Inverting the logic, if real emission is infrared finite, then there can be no virtual infrared divergence and thus no ultraviolet divergence. As we will see, this is true of the massless bubble contributions which were discarded but could *a priori* violate Eq. (5).

To diagnose potential infrared divergences in $A_i^{\rm loop}$, we analyze the associated amplitude for real emission, $A_{i'}^{\rm real}$. In the infrared regime, the singular part of this amplitude factorizes: $A_{i'}^{\rm real} \to A_i S_{i\to i'} + A_j S_{j\to i'}$, where A_i and A_j are tree amplitudes built around insertions of $\mathcal{O}_i$ and $\mathcal{O}_j$, and $S_{i\to i'}$ and $S_{j\to i'}$ are soft-collinear functions describing the emission of an unresolved particle. The soft-collinear functions from marginal interactions diverge as $1/\omega$ and $1/\sqrt{1-\cos\theta}$ in the soft and collinear limits, respectively, where ω and θ are the energy and splitting angle characterizing the emitted particle. By dimensional analysis, irrelevant interactions have additional powers of soft or collinear momentum rendering them infrared finite—a fact we have verified explicitly for all dimension five and six operators. Since the phase-space measure is $\int d\omega\omega \int d\cos\theta$, infrared divergences require that $S_{i\to i'}$ and $S_{j\to i'}$ both arise from soft and/or collinear marginal interactions.

For soft emission, the hard process is unchanged [15]. Since $A_i S_{i\to i'}$ and $A_j S_{j\to i'}$ contribute to the same process, A_i and A_j must have the same external states and thus equal weight, $w_i = w_j$. While massless bubbles do contribute infrared and ultraviolet divergences not previously accounted for, this is perfectly consistent with the nonrenormalization theorem in Eq. (5), which allows for operator mixing when $w_i = w_j$. Violation of Eq. (5) instead requires the presence of infrared divergences when $w_i < w_j$. However, the corresponding soft emission would induce a hard particle helicity flip and thus be subleading in the soft limit and finite upon $\int d\omega$ integration.

Similarly, collinear emission is divergent for $w_i = w_j$ but finite for $w_i < w_j$. Since $A_i S_{i\to i'}$ and $A_j S_{j\to i'}$ have the same external states and weight, restricting to $w_i < w_j$ means that $w(S_{i\to i'}) > w(S_{j\to i'})$. Equation (8) then implies that $S_{i\to i'}$

and $S_{j\to i'}$ are collinear splitting functions generated by on-shell $\overline{\text{MHV}}$ and MHV amplitudes. As a result, the interference term $S^*_{j\to i'}S_{i\to i'}$ carries net little group weight with respect to the mother particle initiating the collinear emission. Rotations of angle ϕ around the mother particle axis act as a little group transformation on $S^*_{j\to i'}S_{i\to i'}$, yielding a net phase $e^{2i\phi}$ in the differential cross section. Integrating over this angle yields $\int_0^{2\pi} d\phi e^{2i\phi} = 0$, so the collinear singularity vanishes upon phase-space integration.

In summary, since real emission is infrared finite for $w_i < w_j$, there are no corresponding ultraviolet divergences from massless bubbles. The nonrenormalization theorems in Eq. (5) apply despite infrared subtleties.

Application to the standard model.—Our results apply to the standard model and its extension to higher dimension operators [2,4–7]. A tour de force calculation of the full one-loop anomalous dimension matrix of dimension six operators [5] unearthed a string of miraculous cancellations not enforced by a manifest symmetry and visible only after the meticulous application of equations of motion [2]. Lacking an explicit Lagrangian symmetry, the authors of Ref. [2] conjectured an underlying "holomorphy" of the standard model effective theory.

The cancellations in Ref. [2] are a direct consequence of the nonrenormalization theorems in Eq. (5) and Eq. (6), based on a classification of holomorphic ($w < 4$), antiholomorphic ($\bar{w} < 4$), and nonholomorphic operators ($w, \bar{w} \geq 4$), and violated only by exceptional amplitudes ($w, \bar{w} = 2$) generated by nonholomorphic Yukawas. The shaded entries in Table II denote zeroes enforced by our nonrenormalization theorems. Entries marked with $\times$ trivially vanish because there are no associated Feynman diagrams, while entries marked with $\times^*$ vanish because the expected divergences in $\psi^2\phi^3$ and ϕ^6 are accompanied by a counterterm of the form $\phi^4 D^2$ [5] which is forbidden by our nonrenormalization theorems.

The superfield formalism offers an enlightening albeit partial explanation of these cancellations [16] and analogous effects in chiral perturbation theory [17]. These results are clearly connected to our own via the "effective" supersymmetry of tree-level QCD [18], and merit further study.

Conclusions.—We have derived a new class of one-loop nonrenormalization theorems for higher dimension operators in a general four-dimensional quantum field theory. Since our arguments follow from unitarity and helicity, they are broadly applicable and explain the peculiar cancellations observed in the dimension six renormalization of the standard model.

Nonrenormalization at higher loop orders remains an open question. However, Eq. (5) will likely fail at two-loop level since helicity selection rules are violated by finite one-loop corrections [19]. Another avenue for future study is higher dimensions, where helicity is naturally extended [20] and dimensional reduction offers a bridge to massive theories. Finally, it would be interesting to link our results to conventional symmetry arguments like those of Ref. [16]. Indeed, our definition of weight is reminiscent of both R symmetry and twist, which relate to existing nonrenormalization theorems.

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*clifford.cheung@caltech.edu
†chshen@caltech.edu

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