Robust Characterization of Loss Rates

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Many physical implementations of qubits—including ion traps, optical lattices and linear optics—suffer from loss. A nonzero probability of irretrievably losing a qubit can be a substantial obstacle to fault-tolerant methods of processing quantum information, requiring new techniques to safeguard against loss that introduce an additional overhead that depends upon the loss rate. Here we present a scalable and platform-independent protocol for estimating the average loss rate (averaged over all input states) resulting from an arbitrary Markovian noise process, as well as an independent estimate of detector efficiency. Moreover, we show that our protocol gives an additional constraint on estimated parameters from randomized benchmarking that improves the reliability of the estimated error rate and provides a new indicator for non-Markovian signatures in the experimental data. We also derive a bound for the state-dependent loss rate in terms of the average loss rate.

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In order to build practical devices for processing and transmitting quantum information, the rate of decoherence and other errors must be below certain fault-tolerant thresholds. In particular, many experimental implementations of qubits—such as ion traps [1,2], optical lattices [3] and linear optics [4]—suffer from irretrievable loss; that is, there is a nonzero probability of the qubit vanishing (as opposed to leaking to other energy levels). Such a loss of normalization can be a substantial obstacle to many quantum information protocols, requiring different error-correction techniques to achieve fault tolerance [4–6]. For example, the surface code may not be used directly if there is any probability of losing a qubit, while for the topological cluster states, loss rates of less than 1% are required to avoid impractical overheads [6].

However, there are two substantial challenges in characterizing loss. First, the loss rate may depend on the state of the qubit, such as when a qubit is encoded in a superposition of vacuum and single-photon states. Second, the loss due to imperfect operations has to be distinguished from the inefficiency of the detector [7]. Quantum process tomography [8,9] could be used to characterize loss; however, it is inefficient in the number of qubits and is sensitive to state preparation and measurement (SPAM) errors [10] and so cannot distinguish between loss due to imperfect operations and inefficient detectors.

In this Letter, we present a robust and efficient protocol that characterizes the loss rate due to imperfect operations averaged over input states. Our protocol is platform independent, simple to implement and analyze, and only assumes that the noise is Markovian. We begin by defining survival rates and then present our protocol and derive the associated analytical decay curve under the assumption of Markovian noise. We then prove that the average loss rate estimated via our protocol provides a practical bound on the loss rate for any state. Since our protocol is robust to SPAM, the choice of state and measurement in our protocol is unconstrained. However, we discuss two particularly suitable choices. The first of these allows one of the parameters in randomized benchmarking [11] to be independently estimated, providing a new test for non-Markovian effects. The first choice also enables the unitarity metric introduced in Ref. [12] to be estimated with no additional experimental overhead. The second choice maximizes the signal, reducing the resources required to obtain a reliable fit. In addition, we demonstrate that our protocol produces reliable estimates of loss rates through a numerical simulation under an error model that has the greatest variation in loss over states. Finally, we illustrate how the analytical model breaks down when applied to systems that have reversible (coherent) leakage to an ancillary level.

Average survival rates.—In order to distinguish between inefficient detectors and lossy processes, we now define survival rates. Many methods for characterizing a process $\mathcal{E}$ (including randomized benchmarking [11,13,14]) assume it is trace preserving. However, many experimental processes are not trace preserving, but instead a state ρ has a survival rate under $\mathcal{E}$,

$$S(\rho|\mathcal{E}) = \frac{\mathrm{Tr}[\mathcal{E}(\rho)]}{\mathrm{Tr}\rho}, \tag{1}$$

that is less than 1, or, equivalently, a nonzero loss rate $L(\rho|\mathcal{E}) = 1 - S(\rho|\mathcal{E})$. Since the trace is linear and any

unnormalized density matrix is proportional to a unit-trace density matrix, the survival rate averaged over all states (hereafter the average survival rate) is simply the survival rate of the maximally mixed state, that is, $S(\mathcal{E}) \coloneqq S([1/d]\mathbb{I}|\mathcal{E})$. Correspondingly, the average loss rate is $L(\mathcal{E}) = 1 - S(\mathcal{E})$.

Experimental protocol.—We now present a protocol for characterizing the average survival rate $S(\mathcal{E})$ in the experimental implementations $\{\mathcal{E}_g\}$ of a set of gates $\mathcal{G} = \{g_1, \ldots, g_{|\mathcal{G}|}\}$ that are at least a unitary 1-design (e.g., the Pauli or Clifford groups) [15]. For simplicity, we assume the noise is time- and gate-independent Markovian noise, so that $\mathcal{E}_g = g \circ \mathcal{E}$ for some fixed map $\mathcal{E}$, where $\circ$ denotes composition (i.e., apply $\mathcal{E}$, then g). This approach can be extended to accommodate time- and gate-dependent noise and a model of non-Markovian noise by applying the approaches of Refs. [16–18].

Our protocol for estimating $S(\mathcal{E})$ is as follows. (1) Choose a sequence length $m \in \mathbb{N}$. (2) Choose a random sequence $\mathbf{k} = (k_1, \ldots, k_m)$ of m integers uniformly at random, where $k_j \in \{1, \ldots, |\mathcal{G}|\}$. (3) Prepare a state ρ. (4) Apply the sequence of gates $g_{k_m} \circ \ldots \circ g_{k_1}$. (5) Measure some operator Q (e.g., a self-adjoint operator or positive-operator valued measure element). (6) Repeat steps 3–5 to estimate

$$Q_{\mathbf{k}} = \mathrm{Tr}[Q g_{k_m} \circ \mathcal{E} \circ \ldots \circ g_{k_1} \circ \mathcal{E}(\rho)] \tag{2}$$

to a desired precision. (7) Repeat steps 2–6 to estimate

$$\mathbb{E}_{\mathbf{k}} Q_{\mathbf{k}} = |\mathcal{G}|^{-m} \sum_{\mathbf{k}} Q_{\mathbf{k}} \tag{3}$$

to a desired precision (see, e.g., Ref. [17] for methods to bound the number of sequences required to obtain a given precision). (8) Repeat steps 1–7 for multiple m and fit to the decay curve

$$\mathbb{E}_{\mathbf{k}} Q_{\mathbf{k}} = D(Q) S(\rho|\mathcal{E}) S^{m-1}(\mathcal{E}), \tag{4}$$

derived below, to obtain estimates of $S(\mathcal{E})$ and $S(\rho|\mathcal{E})D(Q)$, where $D(Q) = \mathrm{Tr}Q/d$. (Note that the above protocol differs from the randomized benchmarking protocol of Ref. [11] in that no inversion gate is applied prior to the measurement.)

Results of a numerical simulation of our protocol for a specific loss model are illustrated in Fig. 1, demonstrating the robust performance of our protocol.

For the numerical simulation, the set of operations $\mathcal{G}$ is the set of single-qubit Pauli operators, and we modeled the error by the channel

$$\mathcal{E}(\rho) = (|0\rangle\langle 0| + \alpha|1\rangle\langle 1|)\rho(|0\rangle\langle 0| + \alpha|1\rangle\langle 1|), \tag{5}$$

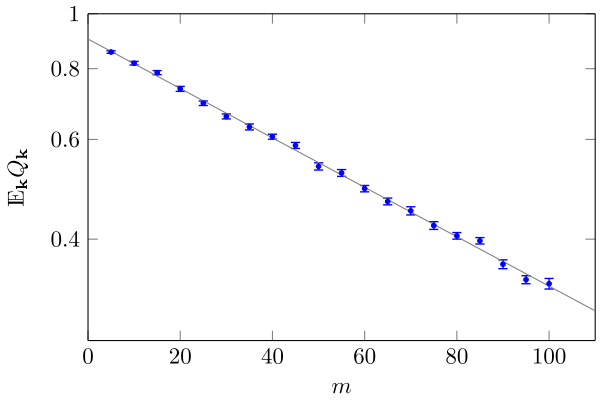

FIG. 1 (color online). Semilog plot of numerical data for our protocol demonstrating robust identification of the average loss rate. The numerical data is obtained for the loss model described by Eq. (8). The data points are the estimates of $\mathbb{E}_{\mathbf{k}} Q_{\mathbf{k}}$ for $m = 5, 10, \ldots, 100$ obtained by sampling 30 random sequences of single-qubit Pauli operators (unitary 1-design) and the error bars are the standard errors of the mean. The grey line is the fit to the model in Eq. (4), obtained using MATLAB's NLINFIT package, which gave $S(\mathcal{E}) = 0.9900(2)$ and $D(Q) = 0.902(8)$, compared to the theoretical values $S(\mathcal{E}) = 0.9901$ and $D(Q) = 0.910$, respectively.

where $\alpha = 0.99$. The channel $\mathcal{E}$ corresponds to loss from the $|1\rangle$ state and, as proven in Proposition 1 below, has the greatest variation of loss over states. The measurement was set to $0.87|\phi\rangle\langle\phi| + 0.95|\phi^\perp\rangle\langle\phi^\perp|$, where $\{|\phi\rangle, |\phi^\perp\rangle\}$ is a randomly chosen orthonormal basis, to model an imperfect detector.

Analysis.—To derive the decay curve in Eq. (4), note that averaging over all sequences corresponds to independently averaging over all g_{k_j}'s, so that

$$\mathbb{E}_{\mathbf{k}} Q_{\mathbf{k}} = \mathrm{Tr}[Q \bar{\mathcal{G}} \circ \mathcal{E} \circ \ldots \circ \bar{\mathcal{G}} \circ \mathcal{E}(\rho)], \tag{6}$$

where $\bar{\mathcal{G}}(\rho) = |\mathcal{G}|^{-1} \sum_{g \in \mathcal{G}} g \rho g^\dagger$ (noting that a unitary channel corresponds to unitary conjugation). Since $\mathcal{G}$ is a unitary 1-design (and a linear map), $\bar{\mathcal{G}}(A) = \mathrm{Tr}(A)\mathbb{I}/d$ for all $d \times d$ matrices A [15,19]. Therefore, $\bar{\mathcal{G}} \circ \mathcal{E}(\rho) = S(\rho|\mathcal{E})\mathbb{I}/d$ and $\bar{\mathcal{G}} \circ \mathcal{E}(\mathbb{I}/d) = S(\mathcal{E})\mathbb{I}/d$, so Eq. (6) simplifies to Eq. (4).

The average survival rate obtained via our protocol is one possible figure of merit that could be used to characterize loss, with an important alternative being the worst-case loss. However, as we now prove, the average loss provides a practical bound for the worst-case loss:

Proposition 1.—For any quantum channel $\mathcal{E}$ and state ρ for a d-dimensional system,

$$L(\rho|\mathcal{E}) \leq L(\mathcal{E}) d.$$

Moreover, for all d there exist channels $\mathcal{E}$ and states ρ that saturate this bound.

Proof of Proposition 1.—Let ρ and $\mathcal{E}$ be arbitrary states of and channels for a d-dimensional system. Since the trace is linear and any valid state can be written as $\rho = \tau \mathrm{Tr}\rho$, where τ is a unit-trace density matrix, the survival rate is independent of $\mathrm{Tr}\rho$, so we assume $\mathrm{Tr}\rho = 1$ without loss of generality.

Let $\rho' = (\mathbb{1} - \rho)/(d-1)$, which is a valid quantum state since it is Hermitian and positive semidefinite by construction and has unit trace. Since ρ' is a valid quantum state, the probability of detecting a system in the state ρ' after applying $\mathcal{E}$ is a true probability and thus

$$\frac{dS(\mathcal{E}) - S(\rho|\mathcal{E})}{d-1} = \frac{\mathrm{Tr}[\mathcal{E}(\mathbb{1}) - \mathcal{E}(\rho)]}{d-1} = \mathrm{Tr}\mathcal{E}(\rho') \leq 1, \quad (7)$$

where we have used the fact that quantum channels and the trace are linear. Rearranging and substituting $L = 1 - S$ gives the desired bound.

To see that the bound is saturated, fix d and consider the channel

$$\mathcal{E}(\rho) = [\mathbb{1} + (\alpha - 1)|0\rangle\langle 0|]\rho[\mathbb{1} + (\alpha - 1)|0\rangle\langle 0|] \quad (8)$$

for $\alpha \in [0, 1]$. For this channel,

$$\mathcal{E}(|j\rangle\langle j|) = \begin{cases} \alpha^2 |0\rangle\langle 0| & j = 0 \\ |j\rangle\langle j| & j \neq 0, \end{cases} \quad (9)$$

so $L(|j\rangle\langle j||\mathcal{E}) = \delta_j(1 - \alpha^2)$ and

$$L(\mathcal{E}) = \frac{1}{d}\sum_j L(|j\rangle\langle j||\mathcal{E}) = \frac{1-\alpha^2}{d}. \quad (10)$$

Therefore, there exists a channel $\mathcal{E}$ and a state ρ such that $L(\rho|\mathcal{E}) = L(\mathcal{E})d$. ∎

For average survival rates close to 1, the estimate of $S(\rho|\mathcal{E})D(Q)$ can be used to directly estimate $D(Q)$ since

$$S(\rho|\mathcal{E})D(Q) \in (\{1 - d[1 - S(\mathcal{E})]\}D(Q), D(Q)) \quad (11)$$

by Proposition 1. Consequently, $S(\rho|\mathcal{E})D(Q)/S(\mathcal{E})$ will give an estimate of $D(Q)$ that is accurate to within a factor of $(d-1)L(\mathcal{E})$. Estimating $D(Q)$ can be used to estimate the efficiency of the detector as

$$\eta = \frac{D(Q)}{D(Q_{\mathrm{ideal}})}, \quad (12)$$

where Q_{ideal} and Q are the ideal and actual measurement operators. That is, η is the ratio of observed to expected detector "clicks," averaged (independently) over all states.

Choosing states and measurements.—Our protocol is robust to SPAM errors in that the choice of the state ρ and the measurement operator Q only effect the value of the constant $S(\rho|\mathcal{E})D(Q)$. However, there are two choices of Q and ρ that have particular benefits.

(i) The most useful scenario corresponds to choosing $\mathcal{G}$ to be a unitary 2-design [15] and choosing $\rho, Q \approx |0\rangle\langle 0|$ as in randomized benchmarking [11]. There are two major advantages to this choice. First, with this choice the same data can also be used to estimate the unitarity of $\mathcal{E}$, a quantitative measure of how the noise $\mathcal{E}$ affects the purity of input states [12]. Second, estimating the constant prefactor in Eq. (4) with this choice is particularly useful because it allows an additional and vital constraint to be imposed when fitting randomized benchmarking data to the fidelity decay curve. In Ref. [11], it was shown that the fidelity decay curve is

$$A(\mathcal{E}')p^m + B(\mathcal{E}'), \quad (13)$$

where p is related to the average gate fidelity, $\mathcal{E}'$ is the average error under the convention that the experimental implementation of g is written as $\mathcal{E}_g = \mathcal{E}' \circ g$ (in contrast to our choice of $\mathcal{E}_g = g \circ \mathcal{E}$), and

$$A(\mathcal{E}') = \mathrm{Tr}\left[Q\mathcal{E}'\left(\rho - \frac{\mathbb{1}}{d}\right)\right],$$
$$B(\mathcal{E}') = \mathrm{Tr}\left[Q\mathcal{E}'\left(\frac{\mathbb{1}}{d}\right)\right]. \quad (14)$$

If the alternative convention of writing errors as $\mathcal{E}' \circ g$ is applied to Eq. (6), then the constant prefactor $S(\rho|\mathcal{E})D(Q)$ in Eq. (4) becomes $B(\mathcal{E}')$. Since the fidelity decay curve is in terms of observable properties, it is independent of the choice of convention, so $B(\mathcal{E}') = S(\rho|\mathcal{E})D(Q)$. Obtaining a precise estimate of the constant term for randomized benchmarking is important for two reasons. First, underestimating the constant term $B(\mathcal{E}')$ [and hence overestimating the coefficient $A(\mathcal{E}')$] will lead to an overestimate of the decay parameter p or, equivalently, an underestimate of the average gate infidelity. That is, underestimating the constant term will falsely indicate that the gates are performing better than they actually are. Second, the values of the constants A and B in randomized benchmarking are not completely independent: they must satisfy particular constraints in order to correspond to physical Markovian noise processes. In particular, for qubits, note that

$$B(\mathcal{E}) - A(\mathcal{E}) = \mathrm{Tr}[Q\mathcal{E}(\rho^{\perp})], \quad (15)$$

where $\rho^{\perp}$ is the state whose Bloch vector is antiparallel to that of ρ. Therefore, $B(\mathcal{E}) - A(\mathcal{E})$ is a probability and so must be nonnegative if the noise is truly Markovian. Consequently, if $B(\mathcal{E}) - A(\mathcal{E})$ is (strongly) negative, then either the noise is non-Markovian or strongly gate dependent, so the estimate of the average gate infidelity in randomized benchmarking is not known to be accurate. Moreover, if the prefactor $S(\rho|\mathcal{E})D(Q)$ in Eq. (4) is estimated by setting $m = 1$, then the resulting estimate is

unaffected by the presence of non-Markovian effects between sequential operations (since there is only one operation applied). Therefore, if the estimate obtained by setting $m = 1$ differs from the estimate obtained from fitting the randomized benchmarking data under the protocol of Ref. [11], then this disagreement indicates that non-Markovian effects are present in the data for the latter.

(ii) Alternatively, given any allowed choice of $\mathcal{G}$, choosing $Q \approx \mathbb{I}$ and ρ to be any unit-trace density matrix will maximize the value of the constant prefactor in Eq. (4), reducing the number of experiments required to obtain a desired precision (since $\mathbb{E}_{\mathbf{k}} Q_{\mathbf{k}}$ is close to 1 for sufficiently small m). Note that these data can be collected under the same experimental configuration as case (i), where $Q = |0\rangle\langle 0|$ and $\mathcal{G}$ is a unitary 2-design, by simply reincorporating the outcomes associated with $\mathbb{I} - Q$ that are discarded in case (i). These data give independent information because, by assumption, the probabilities of these two outcomes are not constrained to add to 1 due to the presence of loss.

Coherent leakage.—A distinct, but closely related error to loss is (coherent) leakage, wherein the system is "leaked" from the qubit subspace to other energy levels. Leakage errors are non-Markovian errors on the qubit subspace since the system can return to the qubit subspace. Coherent leakage is a known consequence of control imperfections in certain implementations of the coupling gate in ion traps [20] and the controlled-phase gate in superconducting qubits [21,22]. Figure 2 shows the results of our protocol given a model of coherent leakage, in particular, an error model for a random (fixed) unitary acting on a qutrit with a random relative phase between the leakage level and the qubit levels. The results initially appear to fit a single exponential decay, but then quickly converge to a constant, similar to the behavior observed in Ref. [23]. Consequently,

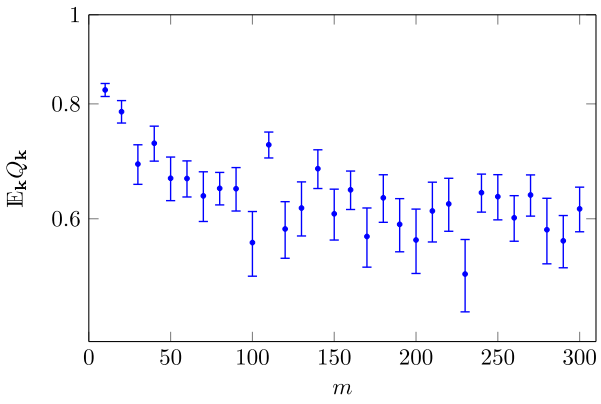

FIG. 2 (color online). Semilog plot demonstrating the signature of non-Markovian leakage under our protocol. Numerical results obtained for a model of coherent leakage from a qubit subspace to a third level under a small random unitary on the full qutrit space. The data points are the estimates of $\mathbb{E}_{\mathbf{k}} Q_{\mathbf{k}}$ for $m = 10, 20, \ldots, 300$ obtained by sampling 30 random sequences of single-qubit Pauli operators and the error bars are the standard errors of the mean.

if experimental data for our protocol does not neatly fit a single exponential, one explanation would be that there is a leakage level that has not been accounted for. A simple protocol for estimating rates of coherent leakage was provided previously in Ref. [24].

Conclusion.—In this Letter, we have presented a platform-independent and robust protocol for characterizing the average loss rate due to noisy implementations of operations. Our protocol can also be used to estimate the detector efficiency, provided the loss rate due to noisy operations is sufficiently small. Since our protocol is easy to implement, it is also a promising technique for experimentally optimizing quantum control, as done, e.g., in Ref. [25] using randomized benchmarking experiments.

Experimentally implementing our protocol yields a single exponential decay curve which can be fitted to our analytical expression to obtain the average loss rate. If the experimental data deviates significantly from a single decay curve, the experimental noise is either strongly gate dependent or non-Markovian. We have illustrated that the decay can be observed and fitted in practice through numerical simulations of loss for a specific error model and also that non-Markovian leakage to an ancillary level results in a deviation from a single exponential. However, fully characterizing how the present protocol (and other randomization-based protocols) behave in the presence of non-Markovian noise remains an open problem.

Our protocol is scalable and robust against state-preparation and measurement errors. However, particular choices of preparations and measurements give extra information. If the set of gates is chosen to be a unitary 2-design and the preparation and measurement are the same as those used in standard randomized benchmarking, then our current protocol can be applied to directly estimate one of the parameters in randomized benchmarking and thus provides a test to indicate non-Markovian noise. Furthermore, with this choice of preparation and measurement, the same data obtained via our protocol can be used to estimate the unitarity presented in Ref. [12] and thus to estimate how close the noise is to depolarizing noise.

As with standard randomized benchmarking, obtaining rigorous confidence intervals on the parameters obtained from our protocol is still an open problem, though techniques bounding the number of sequences to be sampled [17] and using Bayesian methods to refine prior information [26] should also be applicable to our protocol.

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