Time-Delayed Quantum Feedback Control

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A theory of time-delayed coherent quantum feedback is developed. More specifically, we consider a quantum system coupled to a bosonic reservoir creating a unidirectional feedback loop. It is shown that the dynamics can be mapped onto a *fictitious* series of cascaded quantum systems, where the system is driven by past versions of itself. The derivation of this model relies on a tensor network representation of the system-reservoir time propagator. For concreteness, this general theory is applied to a driven two-level atom scattering into a coherent feedback loop. We demonstrate how delay effects can qualitatively change the dynamics of the atom and how quantum control can be implemented in the presence of time delays.

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Introduction.—Delayed autonomous feedback, where a signal is directly fed back to a system after a controllable time delay, is an important control tool for classical systems [1–3]. It is highly attractive as a tool for stabilizing non-equilibrium states of fast dynamical systems, where avoiding any time-costly signal processing is crucial. Such stabilization is of great experimental and technological relevance [4–6]. In particular, delayed autonomous feedback has been used to stabilize the high frequency dynamics of optical systems and high speed electrical circuits [7,8].

Autonomous feedback is also receiving substantial and growing interest for controlling quantum systems [9–16]. Because of the relatively short coherence time and the fast dynamics of quantum systems, the very fast feedback control possible with autonomous feedback is highly desirable. In addition, any measurement of the feedback signal will necessarily destroy its quantum character, making a fully quantum mechanical feedback loop that preserves coherence attractive from a fundamental point of view. Compelling evidence that this type of coherent feedback can outperform any measurement-based counterpart for important quantum information processing tasks has been given [17,18].

A natural way of implementing coherent feedback control loops is by coupling remote quantum systems via waveguides [19–22]. Time delays are unavoidable in practice in such setups and are likely to become important if current experiments are scaled up to larger and more complex networks [23–25]. Despite this, relatively little theoretical research has been done on delay effects for coherent quantum feedback. A major obstacle is the lack of tractable and general theoretical models for treating the highly non-Markovian dynamics induced by this type of feedback. The theoretical difficulty lies in the quantum correlations between the control target system and the in-loop quantum field: The field cannot simply be traced out, and one has to deal with a highly entangled quantum state over a continuum of degrees of freedom.

Previous investigations have typically been limited to negligible delays [26,27], linear systems [18,28], or systems with special symmetries such as conservation of excitation number [29,30]. For linear systems, some very promising theoretical demonstrations of the usefulness of delayed autonomous feedback to stabilize quantum systems have been given recently. In Ref. [29] it was shown how it can be used to stabilize Rabi oscillations of an atom-cavity system in the single-excitation limit, and in Ref. [31] how it can enhance entanglement generated in a biexciton cascade in a quantum dot. Another study demonstrated that delayed coherent feedback might be used as a way of controlling the rate of convergence towards a nonequilibrium steady state in many-atom cavity quantum electrodynamics [32].

In this Letter we go beyond linear systems and develop a general and tractable theoretical model for time-delayed coherent quantum feedback. This opens up research in a largely unexplored regime of quantum feedback control. In particular, it allows for treating the important case of driven, nonlinear systems, something which should be of immediate experimental relevance. We consider a generic setup where an arbitrary quantum system is coupled to a bosonic field forming a feedback loop. We show that the system's density matrix can be found by evolving a time propagator in an extended system space, followed by a generalized partial trace operation. The evolution in this larger space is given by a differential equation for a time propagator in Lindblad form. Interestingly, we can interpret this evolution as an unconventional quantum cascade [33,34], where the system is driven by past versions of itself.

The derivation of our model uses so-called tensor network representations of quantum mechanical states and operators [35]. These tools have their origin at the intersection of condensed matter and quantum information, where they are used to efficiently handle entangled many-body quantum systems. Recently, an intimate connection was made between continuum limits of certain tensor networks and output fields of open quantum systems [36–38]. We develop these ideas further and find a novel application of tensor networks in handling the dynamics of a highly non-Markovian open quantum system. These developments could be of interest in themselves as a new approach to non-Markovian open systems theory.

Below, we introduce the model putting emphasis on developing an intuitive picture of the dynamics. Technical details are left to the Supplemental Material [39]. As a concrete example we consider a two-level atom coupled to a coherent feedback loop. We demonstrate two simple yet remarkable possibilities for delayed feedback control for this example: (1) spontaneous decay acting only for a controllable time, τ, and (2) stabilizing Rabi oscillations far beyond the atom's coherence time in the absence of feedback. We discuss how these effects can be observed in a circuit quantum electrodynamics architecture [47].

Physical setup.—We consider a quantum system coupled to a single unidirectional bosonic field at two different spatial positions, $x = 0$ and $x = l$, as depicted in Fig. 1. The field mediates a feedback loop for the system [48]. We further assume that an arbitrary phase shift, ϕ, can be applied to the field between these two positions, such that the time delay and the phase are independent parameters. The system-field Hamiltonian is $H = H_S + H_B + V$, where H_S is the system Hamiltonian, $H_B = \int_0^\infty d\omega \omega b^\dagger(\omega) b(\omega)$ the free field Hamiltonian, and V the interaction Hamiltonian,

$$V = i \int_{-\infty}^{\infty} d\omega \sqrt{\frac{\kappa_1}{2\pi}} [a_1 b^\dagger(\omega) - \text{H.c.}] + i \int_{-\infty}^{\infty} d\omega \sqrt{\frac{\kappa_2}{2\pi}} [a_2 b^\dagger(\omega) e^{-i\omega\tau + i\phi} - \text{H.c.}], \quad (1)$$

where $\tau = l/c$ is the time delay (c being the speed of light), $\sqrt{\kappa_{1,2}}$ is the coupling strength at the two positions, $x = 0, l$, respectively, a_1 and a_2 are two system operators, and H.c. stands for Hermitian conjugate. The field modes, $b(\omega)$, satisfy $[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$. For generality, we allow the two system operators, a_1 and a_2, to be different, but they could very well refer to the same operator—for example, the dipole operator of a two-level atom or a cavity mode annihilation operator. The assumptions behind Eq. (1) are standard for open quantum systems, typically valid when the system is described by some frequency $\omega_S \gg \kappa_{1,2}$; see, e.g., Ref. [49].

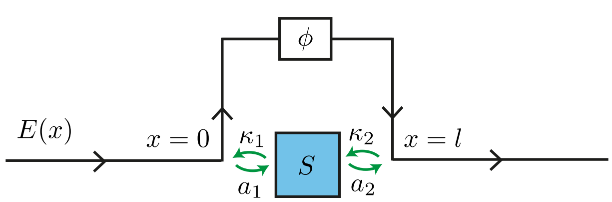

FIG. 1 (color online). Schematic of the setup. A unidirectional bosonic field, $E(x)$, interacts with the system, S, at positions $x = 0$ and $x = l$. The interaction at $x = 0$ ($x = l$) is with a system operator a_1 (a_2) and a rate κ_1 (κ_2). We assume that an arbitrary phase shift, ϕ, can be applied to the field between $x = 0$ and $x = l$, such that the time delay, $\tau = l/c$, and the phase are independently controllable.

To make the discussion more concrete, let us pause to consider a relevant example. A possible implementation is an optical cavity consisting of two mirrors, where the reflected field of one mirror is guided to be used as an input field on the other mirror (the inputs and outputs could be separated by circulators). In this case, one has the interaction in Eq. (1) with $a_1 = a_2 = a$, for a system annihilation operator a, satisfying $[a, a^\dagger] = 1$. $\kappa_{1,2}$ are in this example the linewidths of the two respective mirrors. H_S describes the internal dynamics of the cavity, which could be nonlinear due to the presence of other quantum degrees of freedom interacting with the cavity field. The equation of motion for the annihilation operator in the Heisenberg picture can be found to be (see, e.g., Ref. [49])

$$\dot{a}(t) = i[H_S, a(t)] - \frac{1}{2}(\kappa_1 + \kappa_2) a(t) - \sqrt{\kappa_1} b_{\text{in}}(t) - \sqrt{\kappa_2} e^{i\phi} [b_{\text{in}}(t - \tau) + \sqrt{\kappa_1} a(t - \tau)]. \quad (2)$$

Here we have defined an input field $b_{\text{in}}(t) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)} b_0(w)$, where $b_0(\omega)$ are the initial values for $b(\omega)$ in the Heisenberg picture. Equation (2) has the form of a delay differential equation [50], and it makes the effect of the feedback quite clear. However, since the Heisenberg equations involve coupling between system and field operators, they are typically not efficiently solvable in practice. Also, no corresponding master equation for the reduced system density matrix exists, in general, due to the finite time delay. In the following, we present a practical scheme to integrate this type of dynamics by embedding the system in a larger space.

A cascade of information from the past.—Our main result is a tractable model for the system dynamics after eliminating the field degrees of freedom. The model suggests an intuitive picture where the system is driven by past versions of itself in a cascaded fashion. We present the model here and develop this picture, while leaving the technical details of the derivation to the Supplemental Material [39].

To find the system state, $\rho_S(t)$, at an arbitrary time $(k-1)\tau \leq t < k\tau$, for $k = 1, 2, \ldots$, we evolve a time propagator for a fictitious *cascade* of k identical copies of the system. The time propagator, which we label $\mathcal{E}_s(t)$, is a superoperator on an extended system, $S^{\otimes k}$, and obeys a differential equation in the form of a cascaded master equation, as introduced by Carmichael and Gardiner [33,34]. Note that we are considering here the master equation as a differential equation for the *propagator*, and not for a density matrix. For simplicity, we consider an incoming vacuum field, and we assume that the system and the field are in a product state at time $t = 0$ (the in-loop field is also initially in the vacuum state). As shown in the Supplemental Material [39], the differential equation for the propagator then takes the form

$$\frac{d}{ds}\mathcal{E}_s(t)=\sum_{l=0}^{k}\left\{-\frac{i}{2}\mathcal{H}[H_{l,l+1}(s)]+\mathcal{D}[L_{l,l+1}(s)]\right\}\mathcal{E}_s(t). \quad (3)$$

The integration variable, s, is an auxilliary time variable, and the equation is to be integrated up to $s=\tau$, with the initial condition $\mathcal{E}_0(t)\equiv\mathcal{I}^{\otimes k}$, where $\mathcal{I}$ is the system identity superoperator. We have labeled k identical system copies by S_l, $l=1,\ldots,k$. The superoperators $\mathcal{H}$ and $\mathcal{D}$ are defined by

$$\mathcal{H}[X]\bullet=[X,\bullet], \quad (4)$$

$$\mathcal{D}[X]\bullet=X\bullet X^\dagger-\frac{1}{2}X^\dagger X\bullet-\frac{1}{2}\bullet X^\dagger X, \quad (5)$$

and the operators $H_{l,l+1}$ and $L_{l,l+1}$ are given by

$$H_{l,l+1}=H_S^{(l)}+H_S^{(l+1)}+i\sqrt{\kappa_1\kappa_2}(e^{i\phi}a_1^{(l)\dagger}a_2^{(l+1)}-\text{H.c.}), \quad (6)$$

$$L_{l,l+1}=\sqrt{\kappa_1}a_1^{(l)}+\sqrt{\kappa_2}e^{i\phi}a_2^{(l+1)}, \quad (7)$$

except for $H_{0,1}=H_S^{(1)}$, $H_{k,k+1}=H_S^{(k)}$, $L_{0,1}=\sqrt{\kappa_2}e^{i\phi}a_2^{(1)}$, and $L_{k,k+1}=\sqrt{\kappa_1}a_1^{(k)}$. The superscript denotes the system on which an operator acts. Finally, we have defined $A^{(l)}(s)=A^{(l)}$ for all $l<k$, and $A^{(k)}(s)=\theta(t-(k-1)\tau-s)A^{(k)}$, where $\theta(s)$ is the Heaviside step function for any system operator A.

The generator in Eq. (3) is exactly the generator for a cascaded chain of k identical quantum systems, as introduced by Carmichael and Gardiner [33,34]. An illustration is given in Fig. 2. The evolution would describe a cascade in the usual sense if the time propagator, $\mathcal{E}_s(t)$, is applied to an initial state on the k-fold system space, $S^{\otimes k}$. However, the feedback problem is different, and the solution, $\rho_S(t)$, is found by imposing a peculiar type of "boundary conditions" on the propagator, as we will now explain.

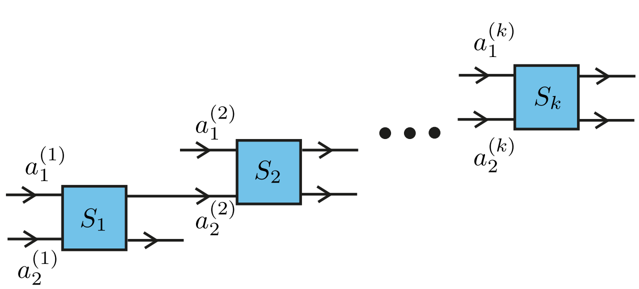

FIG. 2 (color online). The time propagator, $\mathcal{E}_s(t)$, in Eq. (3) can be recognized as the propagator for a cascade of k identical systems, S_l. We can think of the copies as representing past versions of the system, i.e., the system is being driven by itself from the past.

First of all, the integration variable, s, in Eq. (3) is to be understood as a fictitious time variable, and the equation is to be integrated up to $s=\tau$, as already stated. $\rho_S(t)$ is then found by acting with $\mathcal{E}_\tau(t)$ on an initial state $\rho_{S_1}(0)$ for the first system, S_1, and essentially mapping the output of system S_l to the input of system S_{l+1}, for $l=1,\ldots,k-1$. The desired solution will be given as the output of system k, $\rho_S(t)=\rho_{S_k}(t)$. To explain this in more detail, we first have to introduce a generalized trace operation on the level of superoperators. For a superoperator, $\mathcal{A}$, that acts on a tensor product of *identical* systems, $S_1\otimes\cdots\otimes S_k$, we define the following generalized trace:

$$\text{Tr}_{(S_{l'},S_l)}\mathcal{A}\bullet=\sum_{ij}\langle i_l|\mathcal{A}(\bullet\otimes|i_{l'}\rangle\langle j_{l'}|)|j_l\rangle, \quad (8)$$

where $|i_l\rangle$ and $|i_{l'}\rangle$ are bases for the two respective systems, S_l and $S_{l'}$. Note that with $l=l'$, this is a partial trace, in the usual sense, but on the level of superoperators. More generally, this operation can be understood as mapping the output of system S_l to the input of system $S_{l'}$.

We are now ready to write down an expression for $\rho_S(t)$, given the $\mathcal{E}_\tau(t)$ found in Eq. (3):

$$\rho_S(t)=\text{Tr}_{(S_k,S_{k-1})}\cdots\text{Tr}_{(S_2,S_1)}\mathcal{E}_\tau(t)\rho_{S_1}(0). \quad (9)$$

This equation, together with Eq. (3), constitutes our main result, as the two equations provide a practical scheme to find $\rho_S(t)$ for an arbitrary time t. In practice, the solution is thus found by first integrating Eq. (3) up to time $s=\tau$, and subsequently computing the reduced state, $\rho_S(t)$, by acting on the initial state and taking the generalized partial trace in Eq. (9). To help build an understanding of Eq. (9), we illustrate the trace operation diagrammatically for the case $k=3$ in Fig. 3. In the Supplemental Material [39], we give a derivation of Eqs. (3) and (9) using a tensor network representation of the time propagator.

How can we now understand the dynamics induced by the feedback field? Equation (3) suggests that the dynamics is given by a cascade of instances of the system, where each instance is driven by a past version of itself, from a time τ earlier. What is highly nontrivial is that the feedback field that returns after a time τ is already quantum correlated with the system it is driving. This leads us to Eq. (9): It is this equation that correctly accounts for the quantum correlations *in time* in the cascade picture.

Delayed coherent feedback for a two-level atom.—We illustrate the theory with a simple example: a two-level atom coupled to a coherent feedback loop. Both spontaneous decay and resonance fluorescence through the feedback loop are considered. This setup can, e.g., describe an atom placed a (large) distance from a mirror, a problem with a long history in quantum optics (see Ref. [51] and the references therein). In the absence of a drive, the problem can be solved analytically due to there being only a single

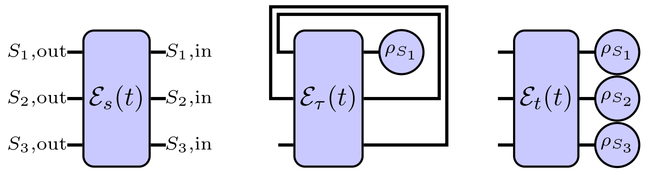

FIG. 3 (color online). A diagrammatic illustration of Eq. (9). (Left panel) The propagator, $\mathcal{E}_s(t)$, for the case $k = 3$. The map is represented by a shape with lines attached to represent the input and output spaces. The labels indicate the systems associated to the lines. (Middle panel) Equation (9) takes a state as input to system S_1, while the output of system S_1 is mapped to the input of S_2, and similarly the output of S_2 to the input of S_3. The final output is a state for system S_3. (Right panel) For comparison, we show the application of the propagator to a state, $\rho_{S_1} \otimes \rho_{S_2} \otimes \rho_{S_3}$, on the threefold system space. This case corresponds to a conventional quantum cascade of three identical systems [33,34] (the choice of a product initial state is not essential). This diagrammatic notation is developed further in the Supplemental Material [39].

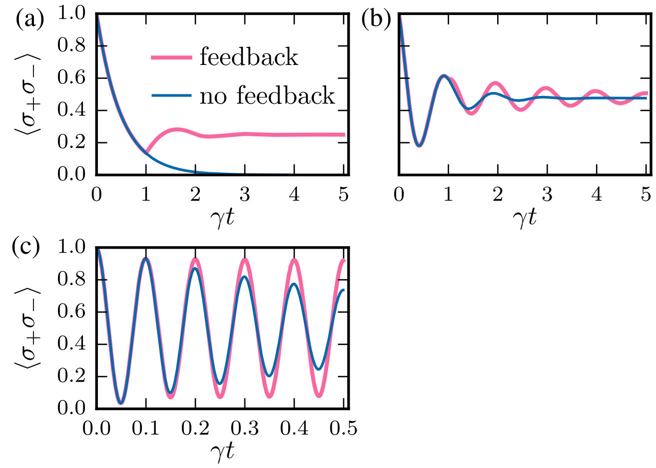

FIG. 4 (color online). Time-delayed coherent feedback control of a two-level atom for three different parameter sets, as indicated in the figure. The pink (light gray) lines show the numerical results with feedback, while for comparison the blue (dark gray) lines show analogous simulations without feedback (i.e., $\kappa_1 = 2\gamma$, $\kappa_2 = 0$). (a) $\mathcal{E}/\gamma = 0$, $\gamma\tau = 1.0$, (b) $\mathcal{E}/\gamma = \pi$, $\gamma\tau = 1.0$, and (c) $\mathcal{E}/\gamma = 10\pi$, $\gamma\tau = 0.1$.

conserved excitation between the system and the reservoir [52]. In the driven case, the problem has, to the best of our knowledge, previously only been considered in an approximate sense, employing perturbation theory in various limits [51].

The problem is defined by a system Hamiltonian $H_S = \mathcal{E}(\sigma_+ + \sigma_-)$, and coupling operators $a_1 = a_2 = \sigma_-$. Here $\sigma_- = |g\rangle\langle e|$ is the atomic lowering operator, and $\sigma_+ = (\sigma_-)^\dagger$. $\mathcal{E}$ is the drive amplitude, and we assume that the atom is driven on resonance for $\mathcal{E} > 0$. We take the rates to be identical, $\kappa_1 = \kappa_2 = \gamma$, and assume a phase shift of $\phi = \pi$ in the feedback loop.

Numerical results for the solution of Eqs. (3) and (9) are shown in Fig. 4. The panels show three different cases: (a) $\mathcal{E}/\gamma = 0$, (b) $\mathcal{E}/\gamma = \pi$, and (c) $\mathcal{E}/\gamma = 10\pi$. The delay is chosen to be $\gamma\tau = 1.0$ for the case $\mathcal{E}/\gamma = 0$, and otherwise equal to the Rabi oscillation period: $\tau = 2\pi/2\mathcal{E}$. The pink (light gray) lines show results with feedback, while the blue (dark gray) lines are analogous simulations without feedback, for comparison.

We note two remarkable features in Fig. 4: First, we consider the simplest case of spontaneous emission in panel a. In this case the atom decays exponentially to the ground state in the absence of feedback. In the presence of feedback, however, the feedback field starts driving the system after an initial transient period of time τ, after which the population grows and eventually stabilizes at a steady state value. In steady state, destructive interference between two contributions to the output field, one coming from direct scattering and one from scattering via the feedback loop, prohibits the system from decaying. Hence, we have the possibility of letting the atom decay only for a controllable time. In steady state the system is *dynamically* decoupled from the decay channel. This phenomenon of feedback-induced dynamical decoupling of an atom from a decay channel has been demonstrated previously [51,53,54].

Let us now look at the nonzero drive strengths shown in Figs. 4(b) and 4(c). Here, the feedback induces long-lived Rabi oscillations, far beyond the coherence time of the atom in the absence of feedback. We have chosen τ to coincide with the Rabi period, which is an optimal choice for stabilizing the Rabi oscillations. This means that τ should be considered as a control parameter in its own right. In the bottom panel with $\mathcal{E}/\gamma = 10\pi$ and $\tau = 0.1$, the decay is extremely slow after the initial transient period of τ. Numerical results have been verified with a brute force numerical integration of the full system plus reservoir dynamics for small values of $\gamma\tau$ [55]. This was done by representing the feedback reservoir by a finite number of modes, truncated to have a small total photon number. Such an approach, however, quickly becomes impractical for large $\gamma\tau$'s ($\gtrsim 0.1$).

The simple example we have considered here could be realized experimentally in a variety of different platforms. A particularly appealing implementation is a circuit quantum electrodynamics architecture with an artificial atom coupled to a one-dimensional waveguide [19,47,56,57]. A meandering waveguide can be made to couple to the artificial atom at two different locations, or the artificial atom can be placed in a semi-infinite waveguide where the end point serves as a mirror. Such a setup was recently demonstrated experimentally in Ref. [57]. A requirement to observe strong delay effects is $\gamma\tau \gtrsim 0.1$, which is readily achievable. In fact, significant delay effects are likely to be unavoidable even for moderate distances for strong coupling between the artificial atom and the waveguide.

Conclusions.—We have shown that the problem of an arbitrary quantum system coupled to a coherent, field-mediated feedback loop can be mapped onto a tractable problem in a larger system space. This theory also yields an intuitive picture that helps us to understand feedback mediated by a quantum field. For practical numerical integration, the approach presented here is superior to alternative approaches based on approximating the feedback reservoir by a lower-dimensional system when the time delay becomes comparable to the inverse linewidth of the emitting quantum system.

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