Mass Gap for Black-Hole Formation in Higher-Derivative and Ghost-Free Gravity

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We study a spherical gravitational collapse of a small mass in higher-derivative and ghost-free theories of gravity. By boosting a solution of linearized equations for a static point mass in such theories we obtain in the Penrose limit the gravitational field of an ultrarelativistic particle. Taking a superposition of such solutions we construct a metric of a collapsing null shell in the linearized higher-derivative and ghost-free gravity. The latter allows one to find the gravitational field of a thick null shell. By analyzing these solutions we demonstrate that in a wide class of the higher dimensional theories of gravity as well as for the ghost-free gravity there exists a mass gap for mini-black-hole production. We also found conditions when the curvature invariants remain finite at $r = 0$ for the collapse of the thick null shell.

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It is generally believed that the theory of general relativity (GR) should be modified to improve its ultraviolet (UV) behavior and remove singularities. One of the options is to allow terms in the gravitational action that contain more than two derivations. The UV properties of the higher-derivative theory of gravity are usually better than in GR. In particular, fourth order gravity can be made renormalizable [1]. At the same time, the gravitational potential of a point mass in the Newtonian limit of such theories is usually finite (see, e.g., Refs. [2,3] and references therein). However, higher-derivative gravity possesses new unphysical degrees of freedom (ghosts) [1,2]. The problem of ghosts can be solved if one allows an infinite number of derivatives in the gravity action, that makes it nonlocal. Ghost-free theories of gravity are discussed in Refs. [4–7]. Their application to the problem of singularities in cosmology and black holes can be found in Ref. [8].

In this Letter we study gravitational collapse of a small mass in higher-derivative (HD) and ghost-free (GF) theories of gravity. We obtain solutions of the linearized equations for such theories for a spherical collapse of null fluid. We demonstrate that if a static gravitational field of a point mass in the HD and GF gravity is regular at $r = 0$ [3], then the metric for the collapsing object has the same property. This means, that the perturbation of the metric, which is proportional to the collapsing mass M, is smooth and uniformly bounded, so that the higher in M corrections can be neglected in the leading order. This implies that for the collapse of a small mass an apparent horizon is not formed. In other words, for this wide class of HD and GF theories of gravity there exists a mass gap for mini-black-hole production. This property is a consequence of the existence of the UV length scale, where such theories become different from GR. For the Weyl modified gravity, this was shown a long time ago in Ref. [9].

We study the linearized gravity equations on the flat Minkowski background $\eta_{\mu\nu}$ and write the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The most general action for the higher-derivative theory of gravity which contains not higher than the second power of $h_{\mu\nu}$ is [5,6]

$$S = -\int d^4x \left[\frac{1}{2} h_{\mu\nu} a(\Box)\Box h^{\mu\nu} + h^{\sigma}_{\mu} b(\Box)\partial_\sigma \partial_\nu h^{\mu\nu} + hc(\Box)\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2} hd(\Box)\Box h + h^{\lambda\sigma}\frac{f(\Box)}{\Box}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}\right], \quad (1)$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$. In general, five nonlinear functions of the box operator obey the following three relations:

$$a + b = 0, \qquad c + d = 0, \qquad b + c + f = 0. \quad (2)$$

Thus the action S contains in fact only two independent arbitrary functions of the box operator. In order to recover GR in the infrared domain these functions must satisfy the following conditions: $a(0) = c(0) = -b(0) = -d(0) = 1$.

Let us list Lagrangians $\mathcal{L}$ for some special interesting examples [6]. (1) General relativity, $\mathcal{L} = R$: $a = c = 1$; (2) $L(R)$ gravity, $\mathcal{L}(R) = \mathcal{L}(0) + \mathcal{L}'(0)R + 1/2\mathcal{L}''(0)R^2 + \ldots$: $a = 1$, $c = 1 - \mathcal{L}''(\Box)$; (3) Weyl gravity, $\mathcal{L} = R - \mu^{-2} C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$: $a = 1 - \mu^{-2}\Box$, $c = 1 - \frac{1}{3}\mu^{-2}\Box$; (4) higher-derivative gravity: $a = \prod_{i=1}^{n}(1 - \mu_i^{-2}\Box)$, $c = \prod_{k=1}^{n_c}(1 - \nu_k^{-2}\Box)$ (for simplicity, in what follows, we assume that masses μ_i are different); and (5) ghost-free gravity: $a = c = \exp(-\Box/\mu^2)$. In the linearized approximation the Weyl and $L(R)$ theories of gravity are nothing but special cases of the general HD gravity.

Let us consider first static solutions of the linearized gravity equations. In the Newtonian limit the stress-energy tensor is $\tau_{\mu\nu} = \rho(\vec{r})\delta^0_\mu\delta^0_\nu$, and the metric is of the form

$$ds^2 = -(1 + 2\varphi)dt^2 + (1 - 2\psi + 2\varphi)d\ell^2. \quad (3)$$

The functions φ and ψ obey the following equations:

$$a(\triangle)\triangle\psi = 8\pi G\rho, \tag{4}$$

$$[a(\triangle) - 3c(\triangle)](\triangle\varphi - 2\triangle\psi) = 8\pi G\rho. \tag{5}$$

Here $\triangle$ is a usual flat Laplace operator in a flat 3D space with metric $d\ell^2$, and G is the gravitational coupling constant. After solving Eq. (4) and finding the potential ψ, one can find the second potential φ by solving Eq. (5).

For a point mass $\rho = m\delta(\vec{r})$ the solution is spherically symmetric. We call it finite if $\varphi(r)$ and $\psi(r)$ near $r = 0$ have the form

$$\psi(r) \sim \psi_0 + \psi_1 r + \frac{1}{2}\psi_2 r^2 + O(r^3), \tag{6}$$

$$\varphi(r) \sim \varphi_0 + \varphi_1 r + \frac{1}{2}\varphi_2 r^2 + O(r^3). \tag{7}$$

A finite solution is not necessary a regular one. Really, the Kretschmann invariant $\mathcal{R}^2 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ for the metric, Eqs. (3), (6), and (7), is the form

$$\begin{aligned}\mathcal{R}^2 &= \frac{A_2}{r^2} + \frac{A_1}{r} + O(1),\\ A_2 &= 8(4\psi_1^2 - 5\psi_1\varphi_1 + 3\varphi_1^2),\\ A_1 &= 16[\psi_1(5\psi_2 - 4\varphi_2) - 4\varphi_1(\psi_2 - \varphi_2)].\end{aligned} \tag{8}$$

The quantity A_2 is a positive definite quadratic form of variables ψ_1 and φ_1, and it vanishes only when $\psi_1 = \varphi_1 = 0$. In such a case the quantity A_1 vanishes as well, so that $\mathcal{R}^2$ is finite at $r = 0$. We call such a solution regular. We also call a solution ψ regular, if $\psi_1 = 0$. For a special class of theories, where $a = c$, one has $\psi = 2\varphi$ and a solution which is ψ regular is at the same time a regular one.

We denote $\hat{O} = a(\triangle)\triangle$,

$$Q(\xi) = \hat{O}^{-1}(\triangle = -\xi) = -[\xi a(-\xi)]^{-1} \tag{9}$$

and assume that $Q(\xi)$ can be written as the Laplace transform of some function $f(s)$

$$Q(\xi) = \int_0^\infty ds f(s) e^{-s\xi}, \tag{10}$$

$$f(s) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} d\xi Q(\xi) e^{s\xi}. \tag{11}$$

The second relation is nothing but the inverse Laplace transform. A parameter α must be chosen so that the integration path in Eq. (11) lies in the domain of the analyticity of $Q(\xi)$.

A formal solution of the operator equation $\hat{O}\,\hat{G} = -\hat{I}$ can be written by using the Laplace transform [Eq. (10)]. It contains the exponent $\exp(s\triangle)$, which in the x representation is nothing but the heat kernel

$$\langle x'|e^{s\triangle}|x\rangle = K(|x - x'|; s) = \frac{e^{-|x-x'|^2/(4s)}}{(4\pi s)^{3/2}}. \tag{12}$$

Thus the potential $\psi(r)$ for a point mass is

$$\psi(r) = 8\pi G m \int_0^\infty ds f(s) K(r; s) \tag{13}$$

$$= \frac{Gm}{\pi i r}\int_{\alpha - i\infty}^{\alpha + i\infty} d\xi Q(\xi) e^{-\sqrt{-\xi} r}. \tag{14}$$

We consider at first a case of HD gravity. We assume that the function $Q(\xi)$ has simple poles and write it in the form

$$Q(\xi) = -\left[\xi \prod_{i=1}^{n}(1 + \xi/\mu_i^2)\right]^{-1}. \tag{15}$$

This covers the gravitational theories (1)–(4) listed above, except for some degenerate cases.

The Heaviside expansion theorem [10] gives the following expression for $f(s)$:

$$f(s) = -\left(1 - \sum_{i=1}^{n} P_i^{-1} e^{-\mu_i s}\right), \tag{16}$$

where $P_i = \prod_{j=1, j\neq i}^{n}(1 - \mu_j^2/\mu_i^2)$. Taking the integral in Eq. (13), one obtains

$$\psi(r) = -2Gmr^{-1}\left(1 - \sum_{i=1}^{n} P_i^{-1} e^{-\mu_i r}\right), \tag{17}$$

For GR $f(s) = 1$ and one has

$$\psi(r) = 2\varphi(r) = -2Gm/r. \tag{18}$$

For a theory with higher derivatives, where $n \geq 1$, the potential $\psi(r)$ near $r = 0$ has a form like Eq. (6) with

$$\psi_0 = -2GmS_1, \qquad \psi_1 = GmS_2, \qquad S_k = \sum_{i=1}^{n} \mu_i^k P_i^{-1}. \tag{19}$$

We used here a relation $S_0 = 1$. Thus a theory with higher derivatives is ψ regular, when the condition $S_2 = 0$ is satisfied.

For the GF gravity $f(s) = -\vartheta(s - \mu^{-2})$ and one reproduces the result of Ref. [5],

$$\psi(r) = 2\varphi(r) = -2Gm\,\mathrm{erf}(\mu r/2)/r. \qquad (20)$$

This solution is regular at $r = 0$.

We demonstrate now, how using a solution of Eq. (4) for a static point mass one can obtain a solution for an ultrarelativistic particle. Let us write the flat space metric in the form $d\ell^2 = dy^2 + d\zeta_\perp^2$, and suppose that the source, generating the gravitational field, moves along the y axis with a constant velocity β. To find the gravitational field of the moving source we make the following boost transformation:

$$t = \lambda_- v + \lambda_+ u, y = \lambda_- v - \lambda_+ u. \qquad (21)$$

Here $\lambda_\pm = (1 \pm \beta)\gamma/2$ and $\gamma = (1 - \beta^2)^{-1/2}$. In the limit $\gamma \to \infty$ one gets $y \sim -\gamma u$, $t \sim \gamma u$, $\ell^2 \sim \gamma^2 u^2 + \zeta_\perp^2$, and

$$ds^2 = -dudv + d\zeta_\perp^2 + dh^2,$$
$$dh^2 = \Phi du^2, \qquad \Phi = -2\lim_{\gamma\to\infty}(\gamma^2\psi). \qquad (22)$$

We assume that the energy of the particle, $M = \gamma m$, remains constant in this (Penrose) limit. We use also the following relation:

$$\lim_{\gamma\to\infty}\gamma e^{-\frac{\gamma^2 u^2}{4s}} = \sqrt{4\pi s}\,\delta(u). \qquad (23)$$

Using these relations and Eq. (13) one gets

$$\Phi = -4GMF(\zeta_\perp^2)\delta(u), \qquad (24)$$

$$F(z) = \int_0^\infty \frac{ds}{s} f(s) e^{-z/(4s)}. \qquad (25)$$

For GR, one has $f(s) = 1$ and $F(z) = \ln(z/\eta^2)$, where η is an infrared cutoff parameter. The relations Eqs. (22) and (24) correctly reproduce the well-known Aichelburg-Sexl solution for the gravitational field of an ultrarelativistic particle ("photon") in GR.

Using Eq. (16) for $f(s)$ for the HD gravity and taking integral in Eq. (25) one finds

$$F(z) = \ln(z/\eta^2) + 2\sum_{i=1}^{n} P_i^{-1} K_0(\mu_i\sqrt{z}). \qquad (26)$$

In the limit $\mu_i \to \infty$ the second term in the right-hand side vanishes and one obtains the correct expression for GR. In the presence of the higher derivatives the leading term of the function $F(z)$ at small z is

$$F(z) \sim C - \frac{1}{4}S_2 z(\ln z - 2c) - \frac{1}{4}Sz + O(z^2), \qquad (27)$$

where $c = 1 + \ln 2 - \gamma$, $\gamma = 0.5772$ is the Euler constant, and $S = \sum_{i=1}^{n}\mu_i^2\ln(\mu_i^2)P_i^{-1}$. For the ghost-free gravity one has [11]

$$F(z) = \ln z + \gamma + \mathrm{Ei}(1, z) \sim z - \frac{1}{4}z^2 + O(z^3). \qquad (28)$$

The obtained metric, Eqs. (22) and (25), can be used to find a solution for the linearized HD and GF gravity equations for a collapsing spherical thin null shell. For this purpose one considers a set of "photons," passing through a fixed point P of the Minkowski spacetime. In the continuous limit this set fills the surface of the null cone, with the vertex at P. We additionally assume that the density of this spherical distribution of the "photons" is uniform and the corresponding mass per a unit solid angle is $M/4\pi$. Since we are working in the linear approximation the resulting gravitational field for such a distribution is $ds^2 = ds_0^2 + \langle dh^2\rangle$, where $\langle dh^2\rangle$ is obtained by averaging of a single photon metric over their spherical distribution. The calculations give [11]

$$ds^2 = -dt^2 + dr^2 + r^2 d\omega^2 + \langle dh^2\rangle, \qquad z = r^2 - t^2,$$
$$\langle dh^2\rangle = -2GMr^{-1}F(z)[(dt - tdr/r)^2 + zd\omega^2/2]. \qquad (29)$$

Let us denote

$$g = (\nabla\rho)^2, \qquad \rho^2 \equiv g_{\theta\theta} = r^2 - \frac{GM}{r}zF(z), \qquad (30)$$

then the equation $g = 0$ determines a position of the apparent horizon, if the latter exists. In the linear in M approximation this function is

$$g = 1 - 2GMr^{-1}q(z), \qquad q(z) = zF'(z), \qquad (31)$$

where $(\cdots)' = d(\cdots)/dz$. For GR (as well as for $L(R)$ gravity) $q(z) = 1$.

Using Eq. (26) one finds that for the HD gravity

$$q(z) = 1 - \sqrt{z}\sum_{i=1}^{n}\mu_i P_i^{-1} K_1(\mu_i\sqrt{z}). \qquad (32)$$

For small z one has

$$q(z) = -\frac{1}{4}S_2 z(\ln z - 2c + 1) - \frac{1}{4}Sz + O(z^2). \qquad (33)$$

Let us demonstrate now that the function g is positive for small enough M, and, hence, the apparent horizon does not exist. Let us notice that outside the null shell $|t|/r < 1$. We denote $t = \pm\sqrt{1 - \beta^2}r$, $0 \le \beta \le 1$, then one has ($y_i = \beta\mu_i r$)

$$q(z)/r = \beta\sum_{i=1}^{n}\mu_i P_i^{-1} Z(y_i), \qquad Z(y) = \frac{1}{y} - K_1(y). \qquad (34)$$

The function $Z(y)$ is positive and takes maximal value 0.399 at $y = 1.114$. Thus

$$|q(z)|/r < 0.4 \sum_{i=1}^{n} \mu_i |P_i|^{-1}. \tag{35}$$

This implies that for small enough value of the mass M the invariant g is positive everywhere outside the shell. In other words, for such mass M the collapse of the null shell does not produce a mini-black-hole. This means that for the class of the higher-derivative theory of gravity [Eq. (15)] with $n \geq 1$ there is a mass gap for the mini-black-hole production. The value of this map is determined by the characteristic length scale μ^{-1} of the theory. The apparent horizon does not exist if $GM\mu \lesssim 1$. The same conclusion is valid for the GF theory of gravity [11].

It is possible to calculate the curvature invariants for the metric in Eq. (29). In particular, the Kretschmann curvature invariant $\mathcal{R}^2$ in the lowest order in M is

$$\mathcal{R}^2 = \frac{48G^2M^2}{r^6}\mathcal{F}, \qquad \mathcal{F} = 2z^2q'^2 - 2zqq' + q^2. \tag{36}$$

Using Eq. (33) for small z one finds

$$\mathcal{F} \sim \frac{1}{16} z^2[(w^2+4w+5)S_2^2 + 2(w+2)SS_2 + S^2], \tag{37}$$

where $w = \ln z - 2c$. This means that Kretschmann curvature vanishes on the null shells. However, in a general case it is divergent at $r = 0$.

In the model of a thin shell, an assumption is made that the energy of the incoming pulse has a deltalike profile. This assumption is not realistic for the theories under consideration. If μ^{-1} is the characteristic time scale of the HD and GF theories, one can expect that the minimal duration of the energy flux cannot be smaller than μ^{-1}. We demonstrate now that for the collapse of the shell with a finite thickness for the class of ψ-regular theories the curvature is finite.

To obtain a solution of the HD and GF theories of gravity equations for such a thick shell we proceed as follows [11]. Consider a set of spherical null shells collapsing to the same spatial point $r = 0$, but passing it at different moments of time t. In the continuous limit, one obtains a distribution of the matter, that describes a spherical thick null shell which initially collapses and has a mass profile $M(t + r)$, and after passing through the center it recollapses with the mass profile $M(t - r)$. In the linear in M approximation the gravitational field of such a shell can be obtained by averaging the metrics given by Eq. (29). We denote by $\ll d\bar{h}^2 \gg$ the result of the averaging of the perturbation $\langle dh^2 \rangle$. For simplicity we present here the expression for $\ll d\bar{h}^2 \gg$ for the case when $\dot{M}$ is constant, and the time duration of the thick shell is b, so that the total mass M of the shell is $\dot{M}b$. In the domain of the intersection of the incoming and outgoing null fluid fluxes the metric is static. The calculations give (see Ref. [11] for more details)

$$\ll d\bar{h}^2 \gg = -\frac{2GM}{br}\left[c_0 dt^2 + c_2\frac{dr^2}{r^2} + \frac{1}{2}(c_0 r^2 - c_2)d\omega^2\right], \tag{38}$$

where $c_k = \int_{-r}^{r} dx x^k F(r^2 - x^2)$. It is easy to check that constant C, which enters Eq. (27), does not contribute to the curvature. For this reason we put $C = 0$. Using the expansion of $F(z)$ in Eq. (27), one obtains

$$c_0 = -\frac{r^3}{9}[(6u - 5)S_2 + 3S],$$
$$c_2 = -\frac{r^5}{225}[(30u - 31)S_2 + 15S], \tag{39}$$

where $u = \ln r - c - \ln 2$. For small M the function g remains positive, while the Kretschmann invariant is

$$\mathcal{R}^2 \sim \frac{32}{27}G^2\dot{M}^2[(36u^2 + 5)S_2^2 + 36uS_2S + 9S^2]. \tag{40}$$

Hence, a collapse of a thick null shell in the theory with higher derivatives results in the logarithmic singularity of the curvature. However, if such a theory is ψ regular, the curvature is finite. In particular, this property is valid for any regular theory with higher derivatives. For the ghost-free theory of gravity the Kretschmann invariant $\mathcal{R}^2 \sim \frac{32}{3}G^2\dot{M}^2\mu^4$ is always finite at $r = 0$ [11]. Let us denote by $\lambda = \mu^{-1}$ the fundamental length scale of the theory. If the time duration of the pulse T obeys the condition $T \geq \mu^{-1}$, then one can rewrite the obtained restriction on the value of the curvature in the form $|\mathcal{R}| \lesssim (GM\mu)\mathcal{R}_{\rm cr}$, where $\mathcal{R}_{\rm cr} = \lambda^{-2}$ is the critical curvature of the theory. Similar estimation with $\ln\mu$ corrections is valid for a regular HD gravity. This result means that for this class of theories the curvature remains uniformly limited, and for $M \ll \mu^{-1}$ it is much smaller than the critical curvature value. One can expect that in such a situation, the higher in curvature corrections, which are present in the full (not linearized) theory, can be neglected. This means that the above conclusions, concerning the absence of the apparent horizon for the collapse of small mass and regularity of the solutions, will remain valid in the full theory.

For the collapse of objects with large mass one can expect that the apparent horizon exists at least in the domain where the curvature is small and the Einstein gravity is valid. An interesting open question is under which conditions on the modified gravity theory a black hole does not contain singularities in its interior.

Let us summarize. We studied solutions of the linearized equations of the higher-derivative and ghost-free theories of gravity. At first we discussed the gravitational field of a point mass and obtained conditions when such a field is regular. Next, we derived the gravitational field of an ultrarelativistic particle for these theories, which generalizes the Aichelburg-Sexl solution of general relativity. And finally, we found a solution for the field of a relativistic collapsing object. The main result of the Letter is the demonstration that, for regular higher-derivative theories and for the ghost-free gravity, the gravitational field of the collapsing object of small mass remains regular, its curvature is finite, and the apparent horizon does not form. Besides addressing the singularity problem, the results presented in the Letter might be useful in general for studying a nonlocal gravity in the time-dependent domain.

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