Phase Diffusion in Unequally Noisy Coupled Oscillators

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We consider the dynamics of two directionally coupled unequally noisy oscillators, the first oscillator being noisier than the second oscillator. We derive analytically the phase diffusion coefficient of both oscillators in a heterogeneous setup (different frequencies, coupling coefficients, and intrinsic noise intensities) and show that the phase coherence of the second oscillator depends in a nonmonotonic fashion on the noise intensity of the first oscillator: as the first oscillator becomes less coherent, i.e., worse, the second one becomes more coherent, i.e., better. This surprising effect is related to the statistics of the first oscillator which provides a source of noise for the second oscillator, that is non-Gaussian, bounded, and possesses a finite bandwidth. We verify that the effect is robust by numerical simulations of two coupled FitzHugh-Nagumo models.

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The dynamics of two coupled stochastic oscillators is a well-studied textbook problem [1–3]. In a self-sustained oscillator, noise results in diffusion of the oscillator's phase, breaking the oscillation coherence [1,4]. The coherence or quality of oscillations is characterized by the quality factor, which is reciprocal to the effective phase diffusion coefficient [1,4]. Hence the names "good" and "bad" used in Ref. [1] and in this Letter: the good oscillator is the one with a high quality of oscillations and with slow phase diffusion; the bad oscillator is characterized by a smaller quality factor and faster diffusion of its phase. When two such oscillators are coupled, the diffusion of their phases can be suppressed due to synchronization [2,3]. In a situation when unequally noisy oscillators are bidirectionally (nonsymmetrically) coupled, the coherence of the bad oscillator can be improved via synchronization by the less noisy, the good, oscillator [1,2]: the increase of coupling strength from the good oscillator to the bad oscillator results in suppression of the effective diffusion coefficient of the bad oscillator. Conversely, the coherence of the good oscillator can be degraded by the influence of the bad oscillator; i.e., the effective diffusion coefficient of the good oscillator increases with the increase of the coupling strength from the bad to the good oscillators.

Unequally noisy directionally coupled oscillators serve as a model of many natural systems, in particular, in neuroscience: synaptically coupled neurons are an example of directional coupling and neurons possess various kinds of noise sources and for no reason should be assumed equally noisy [5,6]. One particular example is the oscillatory sensory hair cell in amphibians, exhibiting stochastic mechanical oscillations of the hair bundle [7,8] and less noisy electrical oscillations of the membrane potential [9], which are bidirectionally coupled [10].

Intuitively, it is expected that because of coupling, the increase of noise intensity in one oscillator should degrade the coherence of both oscillators. Here we show how and when this intuition breaks down. We report a surprising result that an increase of noise in the bad oscillator can worsen the good oscillator only up to some extent. Further increase of noise in the bad oscillator actually suppresses the phase diffusion of the good oscillator, so that its effective diffusion coefficient passes through a maximum. First, we use a generic model of two coupled stochastic phase oscillators to derive exact results for the effective diffusion coefficients of individual oscillators' phases. Second, we demonstrate the generality of the observed effect of noise-induced diffusion suppression with numerical simulation of bidirectionally coupled noisy relaxation oscillators.

Theory for coupled phase oscillators.—We consider two noisy limit cycle oscillators with the natural frequencies $\omega_{1,2}$ governed by the equations for their phases $\phi_{1,2}$:

$$\dot{\phi}_j = \omega_j + G_{k\to j}\sin(\phi_k - \phi_j) + \sqrt{2D_j}\xi_j(t), \qquad (1)$$

where random processes $\xi_{1,2}(t)$ are uncorrelated Gaussian white noises with $\langle \xi_j(t)\xi_i(t')\rangle = \delta_{i,j}\delta(t-t')$ and $i, j = 1, 2$. For two coupled Stuart-Landau oscillators, driven by complex white Gaussian noise sources $\eta_j(t)$,

$$\dot{z}_j = (1 + i\omega_j)z_j - |z_j|^2 z_j + G_{k\to j} z_k + \sqrt{2D_j}\eta_j(t), \qquad (2)$$

Eq. (1) is obtained when amplitude fluctuations, $z_j(t) = \rho_j(t)e^{i\phi_j(t)}$, are neglected, i.e., $\rho_j \equiv 1$.

Without loss of generality, we assume that $D_1 \geq D_2$; i.e., oscillator 1 is bad and oscillator 2 is good. The parameters $G_{2\to1}, G_{1\to2} \geq 0$ are the coupling strengths from the good to the bad oscillator and vice versa, respectively. In the following we derive the effective diffusion coefficients of the oscillators' phases, $\mathcal{D}_{\mathrm{eff},1,2} = \lim_{t\to\infty}(d/dt)\mathrm{var}[\phi_{1,2}(t)]$. Following Malakhov [1,2], we introduce the phase difference, $\psi(t) = \phi_1(t) - \phi_2(t)$, and a linear combination of the phases, $\theta(t) = G_{1\to2}\phi_1(t) + G_{2\to1}\phi_2(t)$. The phase diffusion of the first oscillator, for instance, is related to those of ψ and θ by

$$\mathcal{D}_{\mathrm{eff},1} = \frac{1}{G^2}\left(G_{2\to1}^2\mathcal{D}_{\mathrm{eff},\psi} + \mathcal{D}_{\mathrm{eff},\theta} + G_{2\to1}\frac{d}{dt}\langle\Delta\psi\Delta\theta\rangle\right), \tag{3}$$

where $\Delta\theta = \theta - \langle\theta\rangle$, $\Delta\psi = \psi - \langle\psi\rangle$, and $G = G_{2\to1} + G_{1\to2}$. Importantly, $\mathcal{D}_{\mathrm{eff},1}$ is affected not only by $\mathcal{D}_{\mathrm{eff},\psi}$ and $\mathcal{D}_{\mathrm{eff},\theta}$ but also by the correlation between ψ and θ. The latter variables follow the dynamics

$$\dot{\psi} = \nu - G\sin\psi + \sqrt{2D}\xi_\psi(t), \tag{4}$$

$$\dot{\theta} = G_{2\to1}\omega_2 + G_{1\to2}\omega_1 + \sqrt{2Q}\xi_\theta(t), \tag{5}$$

where $\nu = \omega_1 - \omega_2$ is the frequency mismatch, $\xi_{\psi,\theta}(t)$ are *correlated* Gaussian white noises, $\langle\xi_\psi(t)\xi_\theta(t')\rangle = 2R\delta(t-t')$, with intensities $R = G_{1\to2}D_1 - G_{2\to1}D_2$, $D = D_1 + D_2$, $Q = G_{1\to2}^2D_1 + G_{2\to1}^2D_2$. The dynamics of θ, Eq. (5), is just a biased diffusion with mean frequency (velocity) $\langle\dot{\theta}\rangle = G_{2\to1}\omega_2 + G_{1\to2}\omega_1$ and diffusion coefficient $\mathcal{D}_{\mathrm{eff},\theta} = Q$. The mean velocity for the phase difference $\psi(t)$ governed by Adler's equation (4) has been given by Stratonovich [11]:

$$\langle\dot{\psi}\rangle = 2\pi(1 - e^{-2\pi\nu/D})/\mathcal{I}. \tag{6}$$

The diffusion coefficient for ψ has been derived independently in Refs. [12] and [13] and can be written as follows [13]:

$$\mathcal{D}_{\mathrm{eff},\psi} = 4\pi^2D\mathcal{I}^{-3}\int_0^{2\pi}dxI_+^2(x)I_-(x). \tag{7}$$

In Eqs. (6) and (7) we have used $\mathcal{I} = \int_0^{2\pi}dxI_+(x)$, $I_\pm(x) = D^{-1}e^{\mp V(x)/D}\int_x^{x\pm2\pi}dye^{\pm V(y)/D}$, $V(\psi) = -\nu\psi - G\cos\psi$. Our contribution here is to calculate an exact expression for the cross-correlation term $\langle\Delta\psi\Delta\theta\rangle$ in Eq. (3). Malakhov [1,2] provided only heuristic arguments limited to zero detuning, $\nu = 0$. The noise ξ_θ that is driving θ can be split into a part $\bar{\xi}_\psi$ that is independent of ξ_ψ [and of $\psi(t)$] and one that is proportional to ξ_ψ: $\sqrt{2Q}\xi_\theta(t) = A\xi_\psi(t) + B\bar{\xi}_\psi(t)$ with $A = R/D$. Using that $\Delta\theta = \int^t dt'[A\xi_\psi(t') + B\bar{\xi}_\psi(t')]$, we can write the cross-correlation term as follows:

$$\begin{aligned}\frac{d}{dt}\langle\Delta\psi\Delta\theta\rangle &= A\left\langle\int_{-\infty}^{t}dt'\Delta\dot{\psi}(t)\xi_\psi(t') + \Delta\dot{\psi}(t')\xi_\psi(t)\right\rangle\\ &= A\int_{-\infty}^{\infty}d\tau\langle C_{\xi,\Delta\dot{\psi}}(\tau)\rangle \overset{\mathrm{WKT}}{=} AS_{\xi,\Delta\dot{\psi}}(0) =\\ &\overset{\mathrm{NT}}{=} A\chi_{\Delta\dot{\psi}}(0)S_{\xi_\psi,\xi_\psi}(0) = 2AD\frac{d\langle\dot{\psi}(\nu)\rangle}{d\nu} \overset{\mathrm{def}}{=} 2R\mu_\psi(\nu),\end{aligned}$$

where we used the Wiener-Khinchin theorem (WKT) [14] and the Novikov theorem (NT) [15] as indicated. In the equation above, $C_{\xi,\Delta\dot{\psi}}(\tau)$ and $S_{\xi,\Delta\dot{\psi}}(\omega)$ are the cross-correlation function and cross-spectrum between the noise and $\Delta\dot{\psi}$, respectively; $\chi_{\xi,\Delta\dot{\psi}}(\omega)$ is the weak-signal susceptibility of Adler's equation (4). Additionally, we defined the *differential* mobility $\mu_\psi(\nu)$ as the local derivative of the mean frequency, Eq. (6), with respect to the detuning ν. In a special case of zero detuning, $\nu = 0$, we recover Malakhov's result: $(d/dt)\langle\psi\Delta\theta\rangle = 2R\mu(0) = 2(R/D)\mathcal{D}_{\mathrm{eff},\psi}$. Using the equations above for the correlation term, we arrive at the main analytical result, the diffusion coefficients of individual oscillators

$$\mathcal{D}_{\mathrm{eff},1} = \frac{G_{2\to1}^2\mathcal{D}_{\mathrm{eff},\psi} + 2G_{2\to1}R\mu_\psi + Q}{G^2}, \tag{8}$$

$$\mathcal{D}_{\mathrm{eff},2} = \frac{G_{1\to2}^2\mathcal{D}_{\mathrm{eff},\psi} - 2G_{1\to2}R\mu_\psi + Q}{G^2}. \tag{9}$$

It can be proven that both coefficients are even functions of the detuning ν; hence, it suffices to study their dependence on positive values of ν.

Main effect for coupled phase oscillators.—The effective diffusion coefficient of the first oscillator $\mathcal{D}_{\mathrm{eff},1}$ increases with growing noise intensity D_1 [Figs. 1(a) and 1(c)]. That

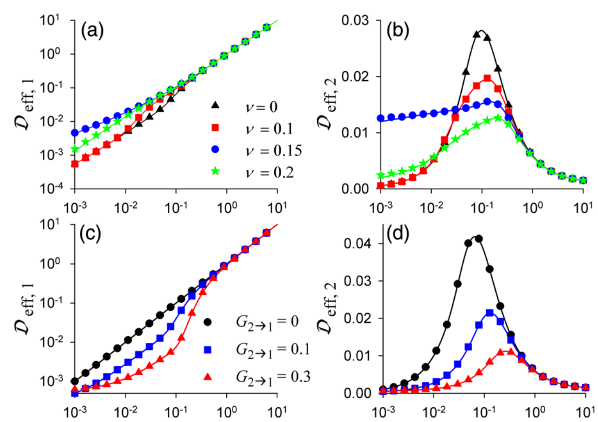

FIG. 1 (color online). Effective diffusion coefficients of two oscillators $\mathcal{D}_{\mathrm{eff},1,2}$ versus noise intensity in the first oscillator D_1. Symbols show the result of direct numerical simulation of an ensemble of 2^{20} pairs of stochastic oscillators, Eq. (1). Solid lines correspond to the theory. (a),(b) Effect of the frequency detuning ν, with $G_{2\to1} = 0.05$, $G_{1\to2} = 0.1$, $D_2 = 10^{-3}$, $\omega_{1,2} = 1 \pm \nu/2$. (c),(d) Effect of the coupling strength $G_{2\to1}$, with $\omega_{1,2} = 1$, $G_{1\to2} = 0.1$, $D_2 = 10^{-3}$.

is, as expected, the first oscillator becomes more noisy as more noise is pumped into it. However, the effective diffusion coefficient of the second oscillator $\mathcal{D}_{\mathrm{eff},2}$ first increases with D_1, reaches a maximum, and then decreases [Figs. 1(b) and 1(d)]. In other words, driving the bad oscillator with stronger noise results in an improved coherence of the good oscillator. The effect is observed for various values of the detuning ν and the coupling strength $G_{2\to1}$. The increase of $G_{2\to1}$ (coupling from the good to the bad) decreases the phase diffusion coefficients in both oscillators, as the good oscillator synchronizes the bad one, which suppresses the effective diffusion of the phase difference $\mathcal{D}_{\mathrm{eff},\psi}$.

Figures 1(a) and 1(b) also show that for weak noise the phase diffusion coefficients of both oscillators are at maximum at the critical value of $\nu_c = G$ [blue circles in Figs. 1(a) and 1(b)], which corresponds to the border of the synchronization region, i.e., the saddle-node bifurcation in the noiseless Adler equation (4). Such an enhancement of the diffusion occurs because of the maximum in $\mathcal{D}_{\mathrm{eff},\psi}(\nu)$ at the critical detuning ν_c, and was studied in detail in Refs. [13,16,17].

To explain the phenomenon of suppression of the phase diffusion in the good oscillator, we consider the case of tuned unidirectionally coupled oscillators ($G_{2\to1} = \nu = 0$), for which the effect is most pronounced [black solid line in Fig. 1(d)]. In this case the equations for the phase diffusion coefficients are simplified to $\mathcal{D}_{\mathrm{eff},1} = D_1$, $\mathcal{D}_{\mathrm{eff},2} = D_1 + \mathcal{D}_{\mathrm{eff},\psi}(1 - 2D_1/D)$, $\mathcal{D}_{\mathrm{eff},\psi} = D[I_0(G_{1\to2}/D)]^{-2}$, where $I_0(x)$ is a modified Bessel function [with $I_0(0) = 1$ and $I_0(x) \sim \exp(x)/\sqrt{2\pi x}$ for $x \to \infty$]. For weak noise in the first oscillator, $D_1 \approx D_2 \ll G_{1\to2}$, the effective diffusion of the phase difference is small, $\mathcal{D}_{\mathrm{eff},\psi} \ll D$. Thus, $\mathcal{D}_{\mathrm{eff},2} \approx D_1$ and increases with D_1. On the contrary, for $D_1 \to \infty$, $I_0(G_{1\to2}/D_1) \approx 1$ and $\mathcal{D}_{\mathrm{eff},2} \approx D_2$. That is, for large noise in the first oscillator, the phase diffusion coefficient of the second oscillator tends to its uncoupled value given by the noise intensity in the second oscillator D_2. We note that in the case of unidirectional coupling, the first oscillator can be considered as a source of noise, $\sin\phi_1$ and $\cos\phi_1$, to the second oscillator. Unlike the uncorrelated Gaussian noise sources $\xi_{1,2}(t)$ in Eq. (1), the noise due to the first oscillator is non-Gaussian, bounded, and correlated with the autocorrelation function $C_1(\tau) = \frac{1}{2}e^{-D_1|\tau|}\cos\omega_1\tau$. While the variance of this bounded noise is invariant with respect to variations of D_1, its correlation time is $\tau_{\mathrm{cor},1} = 1/D_1$, and the corresponding power spectral density (PSD) is a Lorenzian centered at ω_1. In the limit $D_1 \to \infty$, this bounded noise approaches its white limit, but because its variance is fixed, the power within any frequency band, and in particular within the frequency band of the second oscillator, vanishes. Hence, the maximal value of the effective diffusion coefficient of the second oscillator in Figs. 1(b) and 1(d) can be understood as the consequence of two competing tendencies. First, a small amount of noise in the bad oscillator breaks synchronization: an increase of noise enhances the effective phase diffusion of the good oscillator. Second, a further increase in D_1 eventually suppresses the power that is transferred to the good oscillator and, consequently, the second oscillator assumes its coherent uncoupled dynamics.

The effect is robust against amplitude fluctuations. Numerical simulations of coupled Stuart-Landau oscillators, Eq. (2), yielded the same nonmonotonic dependence of the phase diffusion coefficient of the less noisy oscillator on the noise intensity of the more noisy oscillator (not shown). The main effect of the amplitude fluctuations is that for $D_1 \to \infty$, the phase diffusion coefficient of the second oscillator $\mathcal{D}_{\mathrm{eff},2}$ is somewhat larger than that of the corresponding pure phase oscillators.

Simulation results for coupled nonlinear oscillators.— The theory developed above suggests that the effect is generic and should be observed in a wide range of directionally coupled oscillators. In particular, nonsymmetric directional coupling underlies the interaction of neurons via chemical synapses. Here we demonstrate the effect using numerical simulation of two bidirectionally coupled FitzHugh-Nagumo (FHN) neuronal oscillators [18]. The model's equations read

$$\dot{v}_j = v_j - \frac{v_j^3}{3} - w_j + I_{\mathrm{ext}} + G_{k\to j}v_k + \sqrt{2D_j}\xi_j(t),$$
$$\dot{w}_j = 0.01(v_j + 0.7 - 0.8w_j), \quad (10)$$

where $j,k = 1,2$; the fast variables $v_{1,2}$ are the membrane potential of a cell and $w_{1,2}$ are slow recovery variables. The term $G_{k\to j}v_k$ mimics a synaptic current from the kth to the jth neuronal oscillator. As before, the Gaussian white noise terms in Eq. (10), $\xi_{1,2}$, are uncorrelated, and $D_1 > D_2$, so the first is the bad oscillator, while the second is the good one. In the following, we set the value of the "external current" parameter I_{ext} to 0.4, so that in the deterministic uncoupled case each oscillator is in a stable limit cycle regime.

For weak noise in the first oscillator, neurons are synchronized [Fig. 2(a1)] and show coherent periodic firing, as indicated by a narrow peak at their natural frequency in the PSD and by slowly decaying autocorrelation functions [Figs. 2(b1) and 2(c1)]. For an intermediate value of D_1 the first neuron fires faster than the second [Figs. 2(a2) and 2(b2)]; synchronization breaks down as indicated by two distinct peaks at the mean firing rates of neurons in the PSD of the second neuron [Fig. 2(b2)]. This deteriorates the coherence of the second oscillator, as indicated by rapidly decaying autocorrelation in Fig. 2(c2). Large noise in the first oscillator results in fast irregular firing [Fig. 2(a3)] with a flat and low PSD and peaked autocorrelation [Figs. 2(b3) and 2(c3), dotted lines]. In marked contrast, the second neuron regains its oscillation coherence expressed by a sharp peak in its PSD and

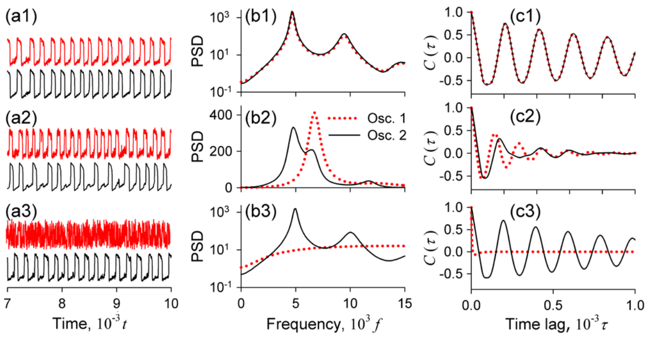

FIG. 2 (color online). Numerical simulations of coupled FitzHugh-Nagumo models. Rows 1, 2, and 3 correspond to the values of noise intensity in the first oscillator, $D_1 = 0.01$, 0.1, and 1.0, respectively. (a) Time traces of oscillators' voltage variables. Upper trace (red line) shows $v_1(t)$, lower trace (black line) shows $v_2(t)$. (b) Power spectral densities (PSDs), corresponding to time traces in (a). (c) Normalized autocorrelation functions $C(\tau)$ corresponding to PSDs in (b). In panels (b) and (c), dotted (red) and solid (black) lines correspond to the first and second oscillators, respectively. Other parameters are $G_{2\to1} = 0$, $G_{1\to2} = 0.1$, $I_{\mathrm{ext}} = 0.4$, $D_2 = 10^{-3}$.

by slowly decaying autocorrelation function [Figs. 2(b3) and 2(c3), solid line]. The instantaneous phases of FHN oscillators $\phi_j(t)$ were estimated from the sequences of their spike times $t_i^{(j)}$, $i = 1, \ldots, N_j$, where $N_j(t)$ is the number of spikes of the jth neuron during the time $(0, t]$ and $j = 1, 2$ [3], $\phi_j(t) = 2\pi[t - t_i^{(j)}]/[t_{i+1}^{(j)} - t_i^{(j)}] + 2\pi i$, $t_i^{(j)} < t < t_{i+1}^{(j)}$. As our theory for the coupled phase oscillators predicts, the phase diffusion coefficient of the second neuron shows nonmonotonic dependence on the noise intensity in the first oscillator (Fig. 3). The effect is most pronounced for unidirectional coupling $G_{2\to1} = 0$, and the peak of $D_{\mathrm{eff},2}$ shifts towards larger values of D_1 for increasing values of the coupling strength $G_{2\to1}$.

Conclusions.—To summarize, we have derived exact analytical expressions for the phase diffusion coefficients of coupled unequally noisy phase oscillators, a generic model serving various applications in natural sciences. Our theory predicted the counterintuitive effect of a

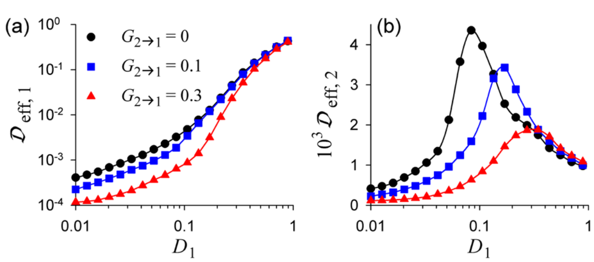

FIG. 3 (color online). Direct numerical simulation of an ensemble of 2^{15} pairs of stochastic FHN oscillators Eq. (10). Effective phase diffusion coefficient of the first and second oscillators $\mathcal{D}_{\mathrm{eff},1,2}$ versus noise intensity of the first oscillator D_1 for the indicated values of the coupling strength $G_{2\to1}$. Other parameters are $G_{1\to2} = 0.1$, $D_2 = 10^{-3}$.

nonmonotonic dependence of the diffusion coefficient of the less noisy oscillator on the noise intensity of the more noisy oscillator. This effect is distinct from other noise-induced phenomena, such as coherence [19], anticoherence [20], or incoherence resonance [21]. While the former are observed in a single oscillator and rely on specifics of excitable systems, the effect studied here is based upon generic properties of noisy limit cycle oscillators: (i) an oscillator is most sensitive to external perturbations within a band around its natural frequency (for an experimental verification, see, e.g., Ref. [7]); (ii) because of the phase diffusion, the oscillation's power spreads over a wide range around the oscillator's natural frequency with the increase of noise intensity. When two such oscillators are coupled and noise is increased in, say, the first one, the power transmitted to the second oscillator within its relevant frequency band decreases and so eventually the fluctuations of the second oscillator are mainly due to its internal noise source. Our result shows a strikingly simple example of how the non-Gaussian and temporally correlated nature of fluctuations affects oscillations in unexpected ways.

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