Entropy Inequality Violations from Ultraspinning Black Holes

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We construct a new class of rotating anti–de Sitter (AdS) black hole solutions with noncompact event horizons of finite area in any dimension and study their thermodynamics. In four dimensions these black holes are solutions to gauged supergravity. We find that their entropy exceeds the maximum implied from the conjectured reverse isoperimetric inequality, which states that for a given thermodynamic volume, the black hole entropy is maximized for Schwarzschild-AdS space. We use this result to suggest more stringent conditions under which this conjecture may hold.

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The study of the thermodynamics of black holes in anti–de Sitter space (AdS) has received much attention since the seminal paper of Hawking and Page [1]. The AdS case is of particular interest because thermodynamic equilibrium is straightforwardly defined and physical phenomena in the bulk admit a gauge duality description via a dual thermal field theory.

A novel topic of active study in recent years is the proposal that the mass of an AdS black hole should be interpreted as the enthalpy of spacetime [2]. This idea is a consequence of considering the cosmological constant Λ to be a thermodynamic variable [3] analogous to pressure in the first law [2,4–14], which has a number of implications. First, as a thermodynamic quantity, Λ must have a conjugate variable, the natural interpretation of which is a thermodynamic volume associated with the black hole. (For a discussion on extending thermodynamic volume beyond black hole spacetimes, see Refs. [12,13].) Second, in the presence of a nonzero Λ, the standard Smarr formula no longer holds. This problem is remedied when Λ is permitted to vary in the first law and the corresponding term is added to the Smarr relation [2]. Third, the extended phase space allows one to rewrite black hole thermodynamic equations as equations of state analogous to those of everyday simple substances, obtaining, for example, a gravitational analogue of the van der Waals fluid, triple points, and reentrant phase transitions (for a review see Ref. [10]) and the notion of a holographic heat engine [11].

The proposed relationship between the cosmological constant and the pressure is

$$P = -\frac{1}{8\pi}\Lambda = \frac{(d-1)(d-2)}{16\pi l^2}, \tag{1}$$

where d is the number of spacetime dimensions. The thermodynamic volume V for the asymptotically AdS black hole spacetimes is then defined so that the following extended first law of black hole thermodynamics holds:

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi\delta Q + V\delta P, \tag{2}$$

a result supported by geometric arguments [2]. Here, M, J, T, and S stand for the mass, angular momentum, temperature, and the entropy of the black hole, while the Ω_i are the angular velocities and Φ is the electric potential, all measured with respect to infinity. The corresponding Smarr relation

$$\frac{d-3}{d-2}M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2}\Phi Q - \frac{2}{d-2}VP \tag{3}$$

can be derived from a scaling (dimensional) argument [2].

An interesting property of the thermodynamic volume is that, in all the cases studied so far, it satisfies what is known as the reverse isoperimetric inequality [8,10]. Indeed, it was conjectured in Ref. [8] (see Ref. [14] for the de Sitter version) that the isoperimetric ratio

$$\mathcal{R} = \left[\frac{(d-1)V}{\omega_{d-2}}\right]^{[1/(d-1)]}\left(\frac{\omega_{d-2}}{A}\right)^{[1/(d-2)]} \tag{4}$$

always satisfies $\mathcal{R} \geq 1$. Here, V is the thermodynamic volume, A is the horizon area, and ω_d stands for the area of the space orthogonal to constant (t, r) surfaces; for a d-dimensional unit sphere $\omega_d = 2\pi^{[(d+1)/2]}/\Gamma[(d+1)/2]$. This result can be interpreted as implying that Schwarzschild-AdS black holes are "maximally entropic": for a black hole of a given "volume" V its entropy is maximized for Schwarzschild-AdS space.

Here, we construct a new ultraspinning limit to the singly spinning Kerr-AdS metric in d spacetime dimensions that yields a new class of black hole solutions whose entropies exceed this maximum bound. The ultraspinning transformation we employ begins with a Kerr-AdS black hole written in a coordinate system that rotates at infinity. We then boost this rotation to the speed of light and compactify the corresponding azimuthal direction. In so doing we qualitatively change the structure of the spacetime since it is no longer possible to return to a frame that does not rotate at infinity. When $d = 4$ the obtained metrics are equivalent (upon inclusion of charge) to a class of black hole solutions of gauged supergravity in four dimensions, recently derived in Ref. [15] and later elaborated upon in Ref. [16]. Insofar as the solutions we construct are understood as string theory ground states, through the AdS/CFT correspondence, topics such as microscopic degeneracy can be studied [17]. The class of black holes we construct all have horizons that are noncompact yet have finite area (and therefore entropy). We find that this particular feature is sufficient to ensure that their entropy exceeds the maximal bound implied by the reverse isoperimetric inequality [8]; as such they provide the first counterexample to this conjecture. Such black holes are therefore “superentropic”.

Let us start with the four-dimensional Kerr–Newman-AdS solution [18], written in the “standard Boyer–Lindquist form” [19]

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_a}{\Sigma_a}\left[dt - \frac{a\sin^2\theta}{\Xi}d\phi\right]^2 + \frac{\Sigma_a}{\Delta_a}dr^2 + \frac{\Sigma_a}{S}d\theta^2 \\
&\quad + \frac{S\sin^2\theta}{\Sigma_a}\left[a dt - \frac{r^2+a^2}{\Xi}d\phi\right]^2, \\
\mathcal{A} &= -\frac{qr}{\Sigma_a}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right),
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\Sigma_a &= r^2 + a^2\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}, \\
S &= 1 - \frac{a^2}{l^2}\cos^2\theta, \\
\Delta_a &= (r^2+a^2)\left(1 + \frac{r^2}{l^2}\right) - 2mr + q^2
\end{aligned} \tag{6}
$$

with the horizon r_h defined by $\Delta_a(r_h) = 0$. The thermodynamic quantities obeying Eq. (2) were calculated in Refs. [4,6,8]; in particular, the thermodynamic volume V and the horizon area A read

$$
\begin{aligned}
V &= \frac{2\pi}{3}\frac{(r_h^2+a^2)(2r_h^2l^2 + a^2l^2 - r_h^2a^2) + l^2q^2a^2}{l^2\Xi^2 r_h}, \\
A &= \frac{4\pi(r_h^2+a^2)}{\Xi},
\end{aligned} \tag{7}
$$

and satisfy the isoperimetric inequality $\mathcal{R} \geq 1$.

Let us now consider a new ultraspinning limit as follows. We first replace everywhere $\psi = \phi/\Xi$, and then take the $a \to l$ limit. In this way we obtain a new solution of the Einstein–Maxwell equations, the superentropic black hole, given by

$$
\begin{aligned}
ds^2 &= -\frac{\Delta}{\Sigma}[dt - l\sin^2\theta d\psi]^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\Sigma}{\sin^2\theta}d\theta^2 \\
&\quad + \frac{\sin^4\theta}{\Sigma}[l dt - (r^2+l^2)d\psi]^2, \\
\mathcal{A} &= -\frac{qr}{\Sigma}(dt - l\sin^2\theta d\psi),
\end{aligned} \tag{8}
$$

where

$$
\Sigma = r^2 + l^2\cos^2\theta, \qquad \Delta = \left(l + \frac{r^2}{l}\right)^2 - 2mr + q^2. \tag{9}
$$

In this form, ψ is a noncompact coordinate, which we now compactify

$$
\psi \sim \psi + \mu, \tag{10}
$$

where the parameter μ is dimensionless. The metric (8) can be shown to be the same as that obtained in Refs. [15,16] via the following change of coordinates:

$$
\tau = t, \qquad p = l\cos\theta, \qquad \sigma = -\psi/l, \qquad L = \mu/l. \tag{11}
$$

The location of the horizon r_+ is determined by the largest root of $\Delta(r)$. In order for horizons to exist, the mass parameter must satisfy

$$
\begin{aligned}
m &\geq 2r_0\left(\frac{r_0^2}{l^2} + 1\right), \\
r_0^2 &\equiv \frac{l^2}{3}\left[-1 + \left(4 + \frac{3}{l^2}q^2\right)^{1/2}\right].
\end{aligned} \tag{12}
$$

The case where equality holds corresponds to an extremal black hole.

The compactification of ψ introduces no conical singularities. The induced metric on the horizon approaches a metric of constant negative curvature on a quotient of the space $\mathbb{H}^2$ near $\theta = 0, \pi$ [16]. It is straightforward to show that the Ricci scalar for the induced metric on the horizon is everywhere finite. The full $4d$ metric is everywhere asymptotically anti–de Sitter, with the coordinate ψ becoming null on the conformal boundary [16].

The fundamental thermodynamic parameters of the superentropic black hole are

$$M = \frac{\mu m}{2\pi}, \qquad J = Ml, \qquad \Omega = \frac{l}{r_+^2 + l^2},$$
$$T = \frac{1}{4\pi r_+}\left(3\frac{r_+^2}{l^2} - 1 - \frac{q^2}{l^2 + r_+^2}\right),$$
$$S = \frac{\mu}{2}(l^2 + r_+^2) = \frac{A}{4}, \qquad \Phi = \frac{q r_+}{r_+^2 + l^2}, \qquad Q = \frac{\mu q}{2\pi}. \tag{13}$$

The angular velocity Ω is that of the horizon and the mass and angular momentum were computed using the method of conformal completion [20–22] using the associated Killing vectors ∂_τ and ∂_ψ. respectively. Note also the "chirality condition" $J = Ml$.

The thermodynamic characteristics displayed here appear, at first glance, to differ from those presented in Ref. [16]. However, the quantities are easily shown to be the same when one takes note of two points. First, the quantity μ in Eq. (13) is related to L from Ref. [16] by $\mu = lL$. Second, the angular momentum computed in Ref. [16] is computed with respect to the coordinate σ, which has dimension $[L]^{-1}$, rendering it to be an angular momentum per unit length. The quantities in Eq. (13) all have scaling dimensions consistent with their Kerr-Newman-AdS counterparts.

We now consider the thermodynamics in extended phase space. In addition to considering pressure and volume terms, we also consider μ as a thermodynamic parameter. In the context of asymptotic Schrödinger geometries the compactified null length can be interpreted as a chemical potential [23]. As discussed in Ref. [16], ψ becomes identified as a compact null coordinate on the conformal boundary. Since μ is associated with the compactification of ψ, we therefore interpret μ as being related to a chemical potential and denote its thermodynamic conjugate as K.

To determine K and the thermodynamic volume V, we demand consistency of the extended first law:

$$dM = TdS + VdP + \Omega dJ + \Phi dQ + Kd\mu. \tag{14}$$

Doing so yields

$$V = \frac{r_+ A}{3} = \frac{2}{3}\mu r_+(r_+^2 + l^2), \tag{15}$$

$$K = \frac{(l^2 - r_+^2)[(r_+^2 + l^2)^2 + q^2 l^2]}{8\pi l^2 r_+ (r_+^2 + l^2)}. \tag{16}$$

The above thermodynamic quantities obey the Smarr relation (3); note that there is no contribution from a $K\mu$ term as μ is a dimensionless quantity. It is interesting to note that the thermodynamic volume V found here is reminiscent of the naive geometric volume (the integral of $\sqrt{-g}$ "inside" the event horizon) of the Kerr-AdS black hole (studied in detail in Ref. [8]), in strong contrast to the traditional ultraspinning black holes, for which the naive geometric volume negligibly contributes to V [10]. Note that V does not explicitly depend on the black hole charge q, as in the case of nonrotating charged AdS black holes [9].

It is straightforward to see that these black holes are superentropic. Bearing in mind that our space is compactified according to Eq. (10), the orthogonal two-dimensional surface area takes the form $\omega_2 = 2\mu$. Consequently, the isoperimetric ratio (4) now reads

$$\mathcal{R} = \left(\frac{r_+ A}{2\mu}\right)^{1/3}\left(\frac{2\mu}{A}\right)^{1/2} = \left(\frac{r_+^2}{r_+^2 + l^2}\right)^{1/6} < 1. \tag{17}$$

Hence we have shown that our black holes always violate the reverse isoperimetric inequality.

This result stands in contrast to the "usual" ultraspinning limit of Kerr-AdS black holes [24] in which, as $a \to l$, the isoperimetric ratio approaches infinity, maximally satisfying the reverse isoperimetric inequality. The distinction arises because of the nature of the ultraspinning limit we are taking. Rather than keeping M fixed and letting the horizon area approach zero as $a \to l$ [10,24], here we require this limit be taken while demanding that the horizon area remain finite.

Unfortunately, this class of charged black holes does not have interesting phase behavior or critical phenomena. For example, in the charge-free case we obtain [using Eqs. (1) and (13) for the pressure and temperature, respectively]

$$P = \frac{T}{v} + \frac{1}{2\pi v^2} \tag{18}$$

for the equation of state, where the specific volume $v = 2r_+ = 6V/(A/l_P^2)$ [9,10]. Since all terms on the right-hand side are positive, it is not possible for this black hole to exhibit critical behavior in the charge-free case. Inclusion of electric charge leads to a more complicated equation of state, but likewise yields no interesting critical behavior. The same conclusion holds if alternate definitions of the specific volume are employed (see, e.g. those in Ref. [10]).

For completeness we connect our results with previous thermodynamic considerations [16] for these black holes. The quantities M and J are not independent; consequently, we should consider $M = M(S, P, Q, \mu)$ in deriving the Smarr formula. A more convenient choice of thermodynamic variables is $L_\pm = \frac{1}{2}(M \pm J/l)$, where L_- vanishes due to the chirality condition $J = Ml$ [16]. It is straightforward to show that the first law (14) becomes

$$TdS = \left(1 - \Omega\sqrt{\frac{3}{8\pi P}}\right)dL_+ - \left(V - \frac{\Omega L_+}{8P}\sqrt{\frac{6}{\pi P}}\right)dP - Kd\mu - \Phi dQ \tag{19}$$

and scaling arguments imply that the Smarr formula is

$$ZL_+ = 2(TS - V'P) + \Phi Q, \tag{20}$$

where Z and V' are the respective thermodynamic conjugates to L_+ and P, from Eq. (19). Note that, in the case where Λ and μ are not considered thermodynamic quantities, the terms proportional to dP and $d\mu$ are not present and the standard form of the first law [16] is recovered.

The preceding considerations suggest that perhaps

$$V' = \left(V - \frac{\Omega L_+}{8P}\sqrt{\frac{6}{\pi P}}\right) \tag{21}$$

could be regarded as the thermodynamic volume rather than V from Eq. (15). However, unlike V, V' suffers from the drawback of not being strictly positive for all values of the parameters, and the quantity L_+ in Eqs. (19) and (20) does not correspond to either a mass or an angular momentum. Furthermore, making this identification does not alter our basic result, as we find that the reverse isoperimetric inequality is still violated using V'. For these reasons we contend that V defined by Eq. (15) should be regarded as the thermodynamic volume for this class of black holes.

Based on the limiting procedure introduced earlier, it is a straightforward matter to generalize the metric (8) to d dimensions. Starting with the singly spinning ($a_1 = a$ and other $a_i = 0$) Kerr-AdS solution, replacing everywhere $\phi = \psi\Xi$ and then taking the limit $a \to l$, we obtain

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}(dt - l\sin^2\theta d\psi)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\sin^2\theta}d\theta^2 \\ &+ \frac{\sin^4\theta}{\rho^2}[ldt - (r^2 + l^2)d\psi]^2 + r^2\cos^2\theta d\Omega_{d-4}^2, \end{aligned} \tag{22}$$

where

$$\Delta = \left(l + \frac{r^2}{l}\right)^2 - 2mr^{5-d}, \qquad \rho^2 = r^2 + l^2\cos^2\theta, \tag{23}$$

and $d\Omega_d^2$ denotes the metric element on a d-dimensional sphere. In this form, ψ is a noncompact coordinate, which we now compactify via $\psi \sim \psi + \mu$. It is straightforward to show that the metric (22) satisfies the Einstein-AdS equations. Setting $d = 4$ we recover the metric (8) with $q = 0$. Horizons exist in any dimension $d \geq 5$ provided $m > 0$.

The solution inherits a closed conformal Killing–Yano 2-form from the Kerr-AdS metric, given by $h = db$, where

$$b = (l^2\cos^2\theta - r^2)dt - l(l^2\cos^2\theta - r^2\sin^2\theta)d\psi. \tag{24}$$

This object together with the explicit symmetries of the metric guarantee complete integrability of geodesic motion as well as separability of the Hamilton-Jacobi, Klein-Gordon, and Dirac equations in the black hole background; see Ref. [25] for analogous results in the Kerr-AdS case.

Computing the thermodynamic quantities for this solution in extended phase space, we find

$$\begin{aligned} &M = \frac{\omega_{d-2}}{8\pi}(d-2)m, \qquad J = \frac{2}{d-2}Ml, \qquad \Omega = \frac{l}{r_+^2 + l^2}, \\ &T = \frac{1}{4\pi r_+ l^2}[(d-5)l^2 + r_+^2(d-1)], \\ &S = \frac{\omega_{d-2}}{4}(l^2 + r_+^2)r_+^{d-4} = \frac{A}{4}, \qquad V = \frac{r_+ A}{d-1}, \end{aligned} \tag{25}$$

where

$$\omega_d = \frac{\mu\pi^{(d-1/2)}}{\Gamma(\frac{d+1}{2})} \tag{26}$$

is the volume of the d-dimensional unit "sphere." Here, Ω is the angular velocity of the horizon and J and M have been computed via the method of conformal completion as the conserved quantities associated with the ∂_ψ and ∂_t Killing vectors, respectively.

By varying μ we get the following expression for its conjugate quantity K:

$$K = \frac{1}{\mu}(M - TS - \Omega J). \tag{27}$$

One can easily verify that these thermodynamic quantities satisfy the Smarr formula (3). The thermodynamic volume obtained here satisfies the isoperimetric inequality ($\mathcal{R} \leq 1$) in all dimensions, and so this class of black holes is also superentropic.

To summarize, we have constructed a class of black hole solutions to Einstein-AdS gravity that results from taking a new ultraspinning limit of Kerr-AdS black holes in d dimensions. These black holes are superentropic insofar as their entropy is larger than the maximum allowed by the reverse isoperimetric inequality [8] (shown to be obeyed by all previously known black hole solutions): they have a greater entropy than their thermodynamic volume would naively allow.

We attribute this behavior to be a consequence of their finite area but noncompact event horizons. We posit that the reverse isoperimetric inequality conjecture might hold under more stringent conditions: for a black hole with a thermodynamic volume $\mathcal{V}$ and with a compact horizon of area $\mathcal{A}$, the ratio (4) satisfies $\mathcal{R} \geq 1$. The proof of this conjecture and the implications of superentropic black holes for counting black hole microstates remain interesting open questions for further study.

Ultraspinning limits similar to the type we have taken here can be applied to multiply spinning Kerr-AdS black holes and charged Kerr-AdS black holes in higher dimensions, yielding further new classes of solutions. We shall elaborate further on this topic in the near future [26].

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