Effective Field Theory of the Disordered Weyl Semimetal

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In disordered Weyl semimetals, mechanisms of topological origin lead to the protection against Anderson localization, and at the same time to different types of transverse electromagnetic response—the anomalous Hall and the chiral magnetic effect. We here apply field theory methods to discuss the manifestation of these phenomena at length scales that are beyond the scope of diagrammatic perturbation theory. Specifically, we show how an interplay of symmetry breaking and the chiral anomaly leads to a field theory containing two types of topological terms. Generating the unconventional response coefficients of the system, these terms remain largely unaffected by disorder, i.e., information on the chirality of the system remains visible even at large length scales.

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Weyl semimetals are paradigmatic examples of gapless topological condensed matter systems. A Weyl semimetal comprises an even number of linearly dispersive band touching points embedded in a three-dimensional Brillouin zone. The presence of these hot spots implies a response to perturbations that is intermediate between that of metals and insulators (see Ref. [1] for a review). This "semimetallicness" also signifies the physics of the disordered system [2–5]. On the one hand, the vanishing of the nodal density of states weakens disorder scattering cross sections; on the other hand, sufficiently strong disorder will generate a finite band center density of states to eventually overpower the above effect. It has been shown [4] that the above competition manifests itself in the presence of a critical disorder strength. Above it the system flows towards a clean fixed, while the regime of strong impurity scattering is realized in the opposite case. It is the purpose of the present Letter to derive and discuss the effective theory describing the latter phase at length scales exceeding the system's scattering mean free path.

At large length scales, impurity scattering will render the motion of individual excitations diffusive, driving the system in the universality class of the $3d$ Anderson metal (i.e., above the phase transition point separating a $3d$ metal from an insulator). This expectation is, in fact, a certainty given that a single Weyl node may be interpreted as an effective surface theory of a bulk $4d$ topological insulator; finite conduction is protected by topology. At the same time, topology implies a number of differences distinguishing the Weyl system from a generic metal. First, an individual Weyl node breaks parity symmetry, and it is known [6] that the breaking of discrete symmetries is generally remembered, even in the presence of strong disorder. Indeed, we will find that the low energy theory of an individual node system contains a parity breaking non-Abelian Chern-Simons (CS) term, which describes the survival of the so-called chiral magnetic effect (CME) [7,8] in the disordered environment. Second, it has been shown that a system comprising two Weyl nodes separated in momentum space shows an anomalous Hall effect (AHE) [9]. Within the field theoretic framework below, this effect will derive from a 3D extension of a two-dimensional topological θ term, familiar from the theory of the quantum Hall effect.

Field theory.—Our starting point is the binodal Hamiltonian (cf. Fig. 1)

$$\hat{H} = v\hat{k}\sigma_3^n + (vb + \mu) + V(\mathbf{x}), \tag{1}$$

where $\hat{k} \equiv \mathbf{k}\cdot\boldsymbol{\sigma}$, $\boldsymbol{\sigma}$, is a vector of Pauli matrices, $\hat{\mathbf{k}}$ the vector momentum operator, and v a characteristic velocity. The Pauli matrix σ_3^n acts in a two-component space discriminating between two nodes split by a vector $2\mathbf{b} \equiv 2b\mathbf{e}_3$ in momentum space and an increment 2μ in energy. The model is coupled to disorder by a Gaussian distributed potential $V(\mathbf{x})$ with variance γ_0. We discriminate between disorder correlated over length scales $\gtrsim b^{-1}$, which is soft in the sense that the two Weyl nodes are not coupled by impurity scattering, and the opposite case of short range correlated disorder mixing the nodes. A high momentum cutoff $|\mathbf{k}| < \Lambda$ limits the range of linearizability of an underlying lattice model.

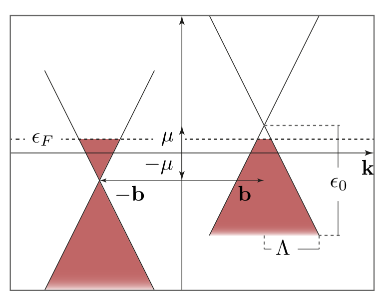

FIG. 1 (color online). Schematic of two Weyl nodes split in energy and momentum by 2μ and $2\mathbf{b}$, respectively.

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To access the transport properties of the system at energies ϵ, we introduce a replicated generating functional Z defined by the action

$$S[\bar{\psi},\psi] = -i\int d^3x\bar{\psi}(\epsilon + i\delta\tau_3 - \hat{H})\psi. \tag{2}$$

Here $\psi = \{\psi^r_{s,i,n}(\mathbf{x})\}$ is an $8R$-component vector of Grassmann variables where $n = 1, 2$ labels the two nodes, $i = 1, 2$ labels the components of a Weyl spinor, $s = \pm$ distinguishes between advanced and retarded (AR) Green functions, i.e., $(\tau_3)_{ss'} = s\delta_{ss'}$, and $r = 1, \ldots, R$ is a replica index. Transport observables may be computed by introducing suitably defined source variables, followed by an analytic continuation $R \to 0$. However, to keep the notation simple, we focus on the source-free functional for the moment.

To explore the influence of disorder on the system, we integrate over V to generate the quartic contribution $(\gamma/2)\int dx(\bar{\psi}\psi)^2$. The fate of this nonlinearity under changes of the cutoff Λ has been studied [5,10] by evaluating the results of renormalized perturbation theory in $2+\epsilon$ dimensions [11] at $\epsilon = 1$. It has been found that for bare amplitudes larger than a critical value $\gamma^* = \pi^2 v^2/\Lambda$ the effective disorder strength increases under renormalization. In this Letter, the focus will be on the perturbatively inaccessible regime beyond the scattering mean free path, $\Lambda^{-1} \equiv l \sim \gamma/v^2$. We start by decoupling [12] the nonlinear scattering vertex by a matrix field $B = \{B^{rr'}_{ss',ii',n}\}$, whose role is to describe the phase coherent propagation of pair amplitudes $\psi^r_{\sin}\bar{\psi}^{r'}_{s'i'n}$ in the system, see Fig. 2. The difference between the cases of hard and soft disorder, respectively, is that in the former [latter] case the two nodes couple to the same ($B = B \otimes \mathbb{I}^n$) [independent ($B = \mathrm{bdiag}(B_1, B_2)^n$)] matrix fields. We first consider the soft case, in which the two nodes can be discussed separately; the effect of impurity mixing can be described by a locking $B_1 = B_2$ at any later stage. Writing $B = B_1$ for notational simplicity, we integrate over the Grassmann variables to obtain the effective action

$$S[B] = -\frac{1}{2\gamma}\int d^3x\mathrm{tr}B^2 - \mathrm{tr}\ln(\hat{G}[B]), \tag{3}$$

where $\hat{G}[B] = (\epsilon + i\delta\tau_3 - \hat{H}_0 - B)^{-1}$, and $\hat{H}_0$ is the clean Hamiltonian.

A variation of the action with respect to B yields the mean field equation $\bar{B} \overset{!}{=} \gamma\mathrm{tr}\hat{G}(x, x; [\bar{B}])$, which is solved [11] by the diagonal ansatz $\bar{B} = -i\kappa\tau_3$. Physically, the solution $\bar{B}$ plays the role of an impurity self-energy, evaluated within the self-consistent Born approximation, cf. Fig. 2 (top right). Specifically, at $\epsilon = 0$ (semimetal) one finds $\kappa = (2/\pi)v\Lambda(1 - \gamma^*/\gamma)$ [13], while far away from the

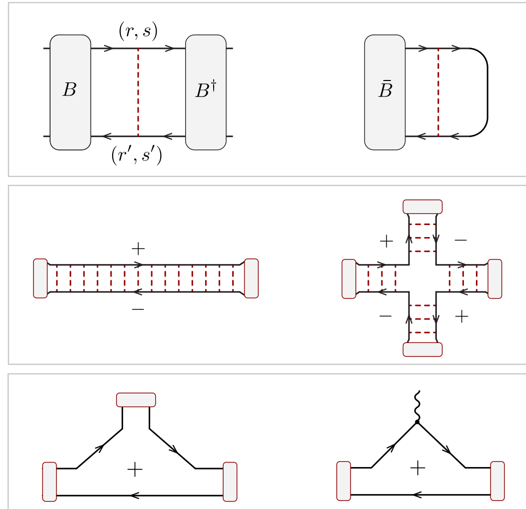

FIG. 2 (color online). Building blocks of the field theory. Upper panel: decoupling the impurity scattering by a Hubbard-Stratonovich field which describes the particle-hole interference; the mean field value of the field is determined by the self-consistent Born approximation impurity self-energy (right). Center panel: soft modes (generated by the fields A of the text) describe diffusive propagation (left) and nonlinear "interaction" due to higher order scattering vertices (right). Bottom panel: two "triangle diagrams" generating an A^3 (left) and $A\partial A$ (right) vertex, with the wavy line representing a derivative vertex.

Weyl node (metal) $\kappa = \gamma\pi\nu$, where $\nu = \epsilon^2/2\pi^2 v^3$ is the clean density of states.

To understand the meaning of the mean field symmetry breaking for the physics at large scales, notice that before disorder averaging the action possessed "replica rotation symmetry" under global unitary transformations $\psi \to U\psi$, where $U \in G \equiv \mathrm{U}(2R)$ acts in replica and advanced retarded space. Disorder averaging leads to spontaneous symmetry breaking $\delta \to \kappa$, much like in a ferromagnet where rotational symmetry gets broken by mean field magnetization. This analogy suggests organizing the "soft fluctuations" in the system as $B \equiv i\kappa Q \equiv i\kappa T\tau_3 T^{-1}$, where fluctuations $T \in G/H$, $H \equiv \mathrm{U}(R) \times \mathrm{U}(R)$, noncommutative with τ_3 will turn out to be diffusively propagating Goldstone modes, conceptually analogous to magnons. We also observe that our system possesses two distinct types of symmetries: a global symmetry $Q \to T_0 Q T_0^{-1}$ under uniform transformations $T_0 \in G$ (corresponding to a uniform change of a magnetization axis) and a trivial local gauge symmetry $T \to Tk(\mathbf{x}), k(\mathbf{x}) \in H$ that does not affect the "order parameter field" Q.

Our goal is to derive an action for the generators $T^{-1}\partial_i T \equiv A_i$ of soft Goldstone mode fluctuations $T(\mathbf{x})$ that will describe the propagation of diffusion modes and their "interaction" due to quantum interference [cf. Fig. 2 (center panel)]. We expect this action to contain a backbone describing the universality class of the diffusive $3d$-Anderson metal, plus contributions of topological origin accounting for the anomalous response properties of the Weyl system. As we show below this action emerges from an interplay of Goldstone mode fluctuations with the chiral anomalies of the Dirac Weyl nodes. Readers primarily interested in results may skip this discussion and proceed directly to the discussion of the main result, Eqs. (5) and (7).

Effective field theory.—Since $\mathrm{tr}(Q^2) = \mathrm{const}$, the action we need to consider reads $S_0[Q] \equiv \mathrm{tr}\ln(\epsilon - v\not{k} + i\kappa Q)$. One might be tempted to start the expansion program with a similarity transformation

$$S_0[Q] \overset{?}{=} \mathrm{tr}\ln(T^{-1}(\epsilon - v\not{k} + i\kappa Q)T) = \mathrm{tr}\ln(\epsilon - v\not{k} + iv\not{A} + i\kappa\tau_3) \equiv S[A], \tag{4}$$

then to be followed by an expansion in the "non-Abelian gauge field" A. However, due to the notorious chiral anomaly of relativistic fermion systems, this operation is invalid; the action needs to be regularized first. Following a strategy previously applied to the 2D d-wave superconductor, we regularize by subtraction of a term $S[Q] \equiv S_0[Q] - S_\eta[Q]$, where S_η differs from S_0 by a replacement $\kappa \to \eta \searrow 0$, and setting $\epsilon = 0$. In the limit $\eta \to 0$ the Q dependence of $S_\eta[Q]$ drops out so that $S_{\eta\to 0}[Q] = 0$. On the other hand, for large momenta $v|k| \gg q$ and fixed η, the two action contributions cancel against each other, i.e., $S[Q]$ is UV regularized. The similarity transformation may now safely be applied to both $S_{0,\eta}$ to obtain an effective action $S[Q] = (S_0[A] - S_\eta[A])_{\mathrm{reg}}$, where the subscript "reg" means that only UV finite contributions to the subsequent expansion in A are to be kept. Notice that in the language of the A fields, the formerly trivial invariance under local transformations k is no longer manifest. Rather, the A's transform as non-Abelian gauge fields, $A_i \to k^{-1}(A + k\partial_i k^{-1})k$, and ensuring gauge invariance of the theory becomes a nontrivial consistency check.

In the expansion of the action, we keep terms of order two $[O(\partial A, A^2)]$ and three $[O(A^3, \partial A)]$ derivatives. To second order we obtain the result

$$S_d[A] = \frac{\sigma_{xx}^1}{8}\sum_i \int d^3x \mathrm{tr}([A_i, \tau_3]^2),$$
$$S_{\mathrm{top}}[A] = -\frac{\sigma_{xy}^1}{2}\epsilon^{3ij}\int d^3x \mathrm{tr}(\tau_3 \partial_i A_j), \tag{5}$$

where the longitudinal and Hall conductivity of node 1, σ_{xx}^1 and σ_{xy}^1, are determined by the microscopic model parameters as discussed below. We note that the action (5) affords the manifestly gauge-invariant reformulation

$$S_d[Q] = \frac{\sigma_{xx}}{8}\int d^3x \mathrm{tr}(\partial Q^2),$$
$$S_{\mathrm{top}}[Q] = -\frac{\sigma_{xy}}{8}\epsilon^{3ij}\int dx \mathrm{tr}(Q\partial_i Q \partial_j Q). \tag{6}$$

The first contribution S_d has been constructed in Ref. [11] on phenomenological grounds within a nonregularized framework [14]. Note that the representation of S_{top} as a full derivative in the second line of Eq. (5) shows that this term becomes a boundary action describing circulating boundary currents if $\sigma_{xy} \neq 0$. However, before discussing the physics of these expressions further, we complete the derivation of the action and consider terms of cubic order in A.

The terms of $O(A^3)$ are the "triangle graphs" pervasive in the theory of $(2+1)$- or $(3+0)$-dimensional relativistic gauge theories. On general grounds [15] we expect the appearance of a Chern-Simons action at this order. A straightforward if lengthy calculation indeed yields the result

$$S_{\mathrm{CS}}[A] = S_{\mathrm{CS}}^{\mathrm{I}}[A] + S_{\mathrm{CS}}^{\mathrm{II}}[A],$$
$$S_{\mathrm{CS}}^{\mathrm{I}}[A] = -\frac{i\epsilon^{ijk}}{8\pi}\sum_{s=\pm} s \int d^3x \mathrm{tr}(A_i P^s \partial_j A_k P^s),$$
$$S_{\mathrm{CS}}^{\mathrm{II}}[A] = -\frac{i\epsilon^{ijk}}{12\pi}\sum_{s=\pm} s \int d^3x \mathrm{tr}(A_i P^s A_j P^s A_k P^s), \tag{7}$$

where $P^{\pm}$ is a projector on advanced or retarded indices, and the two contributions originate in the diagrams shown in the bottom panel of Fig. 2. Apart from the presence of the projector matrices, this has the characteristic structure of a non-Abelian CS term. (However, inasmuch as A does not describe a genuine external gauge field, Eq. (7) does not define a "real" CS action. The situation rather bears similarity to that considered in Refs. [16,17] in somewhat different physical contexts.)

Note that the CS action does not afford a representation in terms of Q fields, which reflects the lack of complete gauge invariance of this action piece [15]; however, one verifies that under a gauge transformation by $k \equiv \mathrm{bdiag}(k_+, k_-)^{\mathrm{AR}} \in H$, the CS action transforms as $S_{\mathrm{CS}}[A] \to S_{\mathrm{CS}}[A] + S_{\mathrm{top}}[k]$, where

$$S_{\mathrm{top}}[k] = \frac{i}{24\pi}\sum_{s=\pm} s \int d^3x \mathrm{tr}(k_s^{-1}\partial k_s)^{\wedge 3}. \tag{8}$$

The integral yields a quantized value, viz., $24\pi^2 \times n_s$, where n_s is the is winding number of a configuration $k_s^{\mathrm{ff}} \in \mathrm{SU}(2)$ in three-dimensional space. For "large" gauge transformations with nonvanishing winding numbers, the CS action changes by a factor $i\pi(n_+ + n_-)$. The origin of this phenomenon was explained in Ref. [15], where it was shown that the regulator action $S_\eta \to S_\eta + i\pi(n_+ + n_-)$ too changes under a large gauge transformation due to zero crossings of the regularizing Dirac operator. However, the sum of the two contributions $S_{\rm CS} + S_\eta$ remains invariant.

Discussion.—The action $S_d + S_{\rm top}$ of the diffusion modes A appeared for the first time in connection with the multilayer quantum Hall effect [18], a system conceptually similar to the present one if the 3-direction of Weyl node splitting is interpreted as the stacking direction of a layered system of 2D quantum anomalous Hall insulators [9]. The action S_d controls the fluctuations of diffusion modes in terms of the dimensionless coupling constant $g_{xx} \equiv \sigma^1_{xx}\Lambda^{-1}$. Within the framework of our gradient expansion we find $\sigma_{xx} = (\epsilon^2 + 3\kappa^2)/6\pi\kappa v$, which simplifies to $\sigma^1_{xx} = \kappa/2\pi v$ at the Weyl node and asymptotes to the Drude conductivity $\sigma^1_{xx} = v^2/3\gamma$ at higher energies $\epsilon \gg \kappa$. At the nodes, $\epsilon = 0$, and at bare length scales $\Lambda \sim l^{-1}$ characteristic of the ballistic-diffusive crossover, the conductance g_{xx} takes values of $O(1)$, close to but larger [19] than the critical value g^* marking the $3d$ Anderson transition. The bare coefficient σ^1_{xy} in the action $S_{\rm top}$ is the contribution of node 1 to the Hall conductivity of the system at crossover length scales to the diffusive regime. Following the theory of the quantum Hall effect [20] we obtain σ_{xy} as a thermodynamic coefficient $\sigma_{xy} = \partial_B n$ that probes the electron concentration at fixed chemical potential $\mu = \epsilon$. Our final result $\sigma^{1/2}_{xy} = b/2\pi$ holds for both nodes and depends neither on the energy ϵ nor on the disorder strength κ. Both for soft and hard disorder, the two contributions to the Hall conductivity add, and we obtain $\sigma_{xy} = b/\pi$ in agreement with Ref. [9], a result known as the AHE.

The CS contribution to the action accounts for the thermodynamic response of the system to imbalances $(\mathbf{b}, \mu)$ between the nodes. For that we couple the system to an external field $a = \{a_i\}$, where $a_i = (B/2)\epsilon_{3ij}x^j$, $i = 1, 2$, represents an external magnetic field $B\mathbf{e}_3$, as well as to the source field $a_3 = a(x)\tau_3$. The latter is defined in such a way that differentiation of the partition function

$$\frac{i}{4\pi}\lim_{R\to 0}\frac{1}{R}\delta_{a(\mathbf{x})}Z[a] = -\frac{1}{\pi}\mathrm{Im}\langle \mathrm{tr}[G^+(\mathbf{x},\mathbf{x})j_3(\mathbf{x})]\rangle \equiv j_{3,\epsilon}$$

with the current operator $j_3(\mathbf{x}) = \sigma_3$ yields the contribution of states at energy ϵ to the equilibrium value of the three-current density of node 1. We compute this expression by adding the external field to the internal one, $A \to A + ia$, and substituting this configuration into the CS action. In the simplest approximation $A = 0$ (for the above "equilibrium" choice of source terms fluctuation corrections around $A = 0$ vanish in the replica limit regardless); we then obtain $S_{\rm CS}[a] = -(iB/x)\int dx a(\mathbf{x})$, and hence $j_{3,\epsilon} = (1/4\pi^2)B$. To obtain the full response of the system, we need to add the (opposite) contribution of the second node and integrate over filled energy states up to some Fermi energy ϵ_F. Taking into account that the existence of a bare linearization cutoff Λ implies a cutoff $|\epsilon| < |\epsilon_0 \pm \mu|$, $\epsilon_0 \equiv v\Lambda$, for the accessible energy states (cf. Fig. 1); this leads to

$$j_3 = \int_{-\epsilon_0+\mu}^{\epsilon_F} d\epsilon\, j_{3,\epsilon} - \int_{-\epsilon_0-\mu}^{\epsilon_F} d\epsilon\, j_{3,\epsilon} = \frac{\mu B}{2\pi^2}, \tag{9}$$

i.e., an equilibrium current proportional to an external magnetic field, the so-called CME [7,8] (for a discussion of how this result may be understood from the perspective of the Fermi-liquid theory, see Ref. [21]).

Renormalization.—What happens if short distance fluctuations in the field theory are integrated out to probe the physics at length scales beyond the ballistic-diffusive crossover regime? An answer to this question has been formulated in Ref. [18] within the framework of two loop renormalized perturbation theory for the dimensionless coupling constants $g_{\mu\nu} = \sigma_{\mu\nu}\Lambda^{-1}$ of the model. The result

$$\frac{dg_{xx}}{d\ln L} = g_{xx} - \frac{1}{3\pi^4 g_{xx}}, \qquad \frac{dg_{xy}}{d\ln L} = g_{xy}, \tag{10}$$

states that the longitudinal conductance scales with the system size L according to the predictions of one-parameter scaling theory (unaffected by the Hall conductance) towards Ohmic behavior $g_{xx} \overset{g_{xx}\gg 1}{\sim} L$. The Hall conductance shows linear scaling, $g_{xy} \propto L$, which means that the AHE remains unrenormalized by disorder; $\sigma_{xy} = \text{const}$ even at large length scales (in contrast to the Hall conductivity renormalized by instanton fluctuations [20] in a two-dimensional system). Finally, the coupling constant of the CS action is fixed by gauge invariance, and fluctuation corrections to $\langle j_3\rangle_B$ vanish in the replica limit. This means that within the framework of our theory the CME is fully protected against renormalization by disorder.

The disorder insensitivity of the topological response coefficients holds regardless of whether one probes the semimetallic Weyl nodes $\epsilon, \mu \sim \kappa$, or the metallic physics at $|\epsilon \pm \mu| \gg \kappa$. The essential difference between the two situations lies in the bare and renormalized values of the longitudinal conductance g_{xx}: in the former (latter) case, g_{xx} is initially small (large) to begin with. However, in either case, g_{xx} increases, and asymptotes to Ohmic behavior at large length scales. While the system then behaves similarly to a three-dimensional metal, the preserved nonvanishing of its two transverse transport coefficients betrays the underlying presence of two Dirac nodes.

Summarizing, we have microscopically derived a field theory description of disordered Weyl semimetals and metals at length scales exceeding the mean free path. The structure of the theory is essentially determined by an interplay of symmetry conditions and the chiral anomaly. This mechanism stabilizes metallic behavior at large length scales, along with various disorder-insensitive response coefficients of topological origin.

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Note added.—Recently, we became aware of the preprint [22] where an action similar to ours is motivated from a different perspective, viz., by dimensional reduction from a bulk $4d$-topological insulator.

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