PRL **114,** 251603 (2015) PHYSICAL REVIEW LETTERS week ending 26 JUNE 2015

Absence of Quantum Time Crystals

Haruki Watanabe[1,*] and Masaki Oshikawa[2,†]
[1]Department of Physics, University of California, Berkeley, California 94720, USA
[2]Institute for Solid State Physics, University of Tokyo, Kashiwa 277-8581, Japan

(Received 28 March 2015; published 24 June 2015)

In analogy with crystalline solids around us, Wilczek recently proposed the idea of "time crystals" as phases that spontaneously break the continuous time translation into a discrete subgroup. The proposal stimulated further studies and vigorous debates whether it can be realized in a physical system. However, a precise definition of the time crystal is needed to resolve the issue. Here we first present a definition of time crystals based on the time-dependent correlation functions of the order parameter. We then prove a no-go theorem that rules out the possibility of time crystals defined as such, in the ground state or in the canonical ensemble of a general Hamiltonian, which consists of not-too-long-range interactions.

DOI: 10.1103/PhysRevLett.114.251603 PACS numbers: 11.30.-j, 05.70.Ln, 73.22.Gk

Recently, Wilczek proposed a fascinating new concept of time crystals, which spontaneously break the continuous time translation symmetry, in analogy with ordinary crystals that break the continuous spatial translation symmetry [1–3]. Li *et al.* soon followed with a concrete proposal for an experimental realization and observation of a (space-) time crystal, using trapped ions in a ring threaded by an Aharonov-Bohm flux [4–6]. While the proposal of time crystals was quite bold, it is, on the other hand, rather natural from the viewpoint of relativity: since we live in the Lorentz invariant space-time, why don't we have time crystals if there are ordinary crystals with a long-range order in spatial directions?

However, the very existence, even as a matter of principle, of time crystals is rather controversial. For example, Bruno [7] and Nozières [8] discussed some difficulties in realizing time crystals. However, since their arguments were not fully general, several new realizations of time crystals, which avoid these no-go arguments, were proposed [9,10].

In fact, a part of the confusion can be attributed to the lack of a precise mathematical definition of time crystals. Here, we first propose a definition of time crystals in the equilibrium, which is a natural generalization of that of ordinary crystals and can be formulated precisely also for time crystals. We then prove generally the absence of time crystals defined as such, in the equilibrium with respect to an arbitrary Hamiltonian which consists of not-too-long-range interactions. We present two theorems: one applies only to the ground state, and the other applies to the equilibrium with an arbitrary temperature.

Naively, time crystals would be defined in terms of the expectation value $\langle\hat{O}(t)\rangle$ of an observable $\hat{O}(t)$. If $\langle\hat{O}(t)\rangle$ exhibits a periodic time dependence, the system may be regarded as a time crystal. However, the very definition of eigenstates $\hat{H}|n\rangle = E_n|n\rangle$ immediately implies that the expectation value of any Heisenberg operator $\hat{O}(t) \equiv e^{i\hat{H}t}\hat{O}(0)e^{-i\hat{H}t}$ in the Gibbs equilibrium ensemble is time independent. To see this, recall that the expectation value $\langle\hat{X}\rangle$ is defined as $\langle\hat{X}\rangle \equiv \langle 0|\hat{X}|0\rangle$ at zero temperature $T = 0$ and $\langle\hat{X}\rangle \equiv \mathrm{tr}(\hat{X}e^{-\beta\hat{H}})/Z = \sum_n\langle n|\hat{X}|n\rangle e^{-\beta E_n}/Z$ at a finite temperature $T = \beta^{-1} > 0$, where $|0\rangle$ is the ground state and $Z \equiv \mathrm{tr}(e^{-\beta\hat{H}})$ is the partition function. Clearly, $\langle n|\hat{O}(t)|n\rangle$ is time independent since two factors of $e^{\pm iE_n t}$ cancel against each other and hence $\langle\hat{O}(t)\rangle$ is time independent.

Yet it is too early to reject the idea of time crystals just from this observation, since a similar argument would preclude ordinary (spatial) crystals. One might naively define crystals from a spatially modulating expectation value of the density operator $\hat{\rho}(\vec{x}) = e^{-i\hat{\vec{P}}\cdot\vec{x}}\hat{\rho}(\vec{0})e^{i\hat{\vec{P}}\cdot\vec{x}}$. The unique ground state of the Hamiltonian in a finite box is nevertheless symmetric and hence $\hat{\vec{P}}|0\rangle = 0$, implying that $\langle\hat{\rho}(\vec{x})\rangle$ is constant over space at $T = 0$. Likewise, at a finite temperature, $\langle\hat{\rho}(\vec{x})\rangle \equiv \mathrm{tr}[e^{-i\hat{\vec{P}}\cdot\vec{x}}\hat{\rho}(\vec{0})e^{i\hat{\vec{P}}\cdot\vec{x}}e^{-\beta\hat{H}}]/Z$ cannot depend on position since $\hat{\vec{P}}$ and $\hat{H}$ commute. More generally, the equilibrium expectation value of any order parameter vanishes in a finite-size system, since the Gibbs ensemble is always symmetric. This, of course, does not rule out the possibility of spontaneous symmetry breaking.

A convenient and frequently used prescription to detect a spontaneous symmetry breaking is to apply a symmetry-breaking field. For example, in the case of antiferromagnets on a cubic lattice, we apply a staggered magnetic field $h_s(\vec{R}) = h\cos(\vec{Q}\cdot\vec{R})$ $[\vec{Q} \equiv (\pi/a)(1,\ldots,1)]$ by adding a term $-\sum_{\vec{R}}h_s(\vec{R})\hat{s}^z_{\vec{R}}$ to the Hamiltonian, where $\vec{R}$'s are lattice sites and $\hat{s}^z_{\vec{R}}$ is the spin on the site $\vec{R}$. One computes the expectation value of the macroscopic order parameter, which is the staggered magnetization in the case of an antiferromagnet, under the symmetry breaking field and then take the limit $V \to \infty$ and $h \to 0$ in this order. The nonvanishing expectation value of the macroscopic order parameter, in this order of limits, is often regarded as a definition of spontaneous symmetry breaking (SSB). In the case of crystals, we apply a potential $v(\vec{x}) = h\sum_{\vec{G}} v_{\vec{G}} \cos(\vec{G}\cdot\vec{x})$ with a periodic *position* dependence. Here, $\vec{G}$'s are the reciprocal lattice of the postulated crystalline order.

This prescription is quite useful but unfortunately is not straightforwardly applicable to time crystals. The symmetry-breaking field for time crystals has to have a periodic *time* dependence. In the presence of such a field, the "energy" becomes ambiguous and is defined only modulo the frequency of the periodic field, making it difficult to select states or to take statistical ensembles based on energy eigenvalues. Therefore, an alternative definition of time crystals is called for, and indeed we will propose a definition of time crystals which is applicable to very general Hamiltonians.

Time-dependent long-range order.—In order to circumvent the problem in defining time crystals using a time-dependent symmetry-breaking field, here we define time crystals based on the long-range behavior of correlation functions. In fact, all conventional symmetry breakings can be defined in terms of correlation functions, without introducing any symmetry-breaking field. That is, we say the system has a *long-range order* (LRO) if the equal-time correlation function of the local order parameter $\hat{\phi}(\vec{x},t)$ satisfies

$$\lim_{V\to\infty}\langle\hat{\phi}(\vec{x},0)\hat{\phi}(\vec{x}',0)\rangle \to \sigma^2 \neq 0 \qquad (1)$$

for $|\vec{x}-\vec{x}'|$ much greater than any microscopic scales. One can equivalently use the integrated order parameter $\hat{\Phi} \equiv \int_V d^d x \hat{\phi}(\vec{x},0)$, for which the long-range order is defined as $\lim_{V\to\infty}\langle\hat{\Phi}^2\rangle/V^2 = \sigma^2 \neq 0$. For example, in the case of the quantum transverse Ising model $\hat{H} = -\sum_{\langle\vec{r},\vec{r}'\rangle}\sigma^z_{\vec{r}}\sigma^z_{\vec{r}'} - \Gamma\sum_{\vec{r}}\sigma^x_{\vec{r}}$, the local order parameter $\hat{\phi}(\vec{r},t)$ is identified with $\sigma^z_{\vec{r}}$ or its coarse graining. It has been proven quite generally that the LRO $\sigma \neq 0$ guarantees the corresponding SSB, namely, a nonvanishing expectation value of the order parameter in the limit of the zero symmetry-breaking field taken after the limit $V\to\infty$ [11,12]. While the reverse is not proved in general, it is expected to hold in many systems of interest.

A crystalline order can also be defined by the correlation function. Namely, if the long-range correlation approaches to a periodic function

$$\lim_{V\to\infty}\langle\hat{\phi}(\vec{x},0)\hat{\phi}(\vec{x}',0)\rangle \to f(\vec{x}-\vec{x}') \qquad (2)$$

for sufficiently large $|\vec{x}-\vec{x}'|$, the system exhibits a spontaneous crystalline order [13]. Equivalently, $\lim_{V\to\infty}\langle\hat{\Phi}_{\vec{G}}\hat{\Phi}_{-\vec{G}}\rangle/V^2 = f_{\vec{G}} \neq 0$ signals a density wave order with wave vector $\vec{G}$, where $\hat{\Phi}_{\vec{G}} = \int_V \mathrm{d}^d x \hat{\phi}(\vec{x},0)e^{-i\vec{G}\cdot\vec{x}}$. Note again that $\langle\hat{\phi}(\vec{x},0)\rangle$ itself is a constant over space in the Gibbs ensemble, which is symmetric. For instance, we set $\hat{\phi} = \hat{\rho}$ for ordinary crystals and $\hat{\phi} = \hat{s}_\alpha$ for spin-density waves. In terms of the LRO, one can therefore characterize crystals using only the symmetric ground state or ensemble, which itself does not have a finite expectation value of the order parameter [14].

Let us now define time crystals, in an analogous manner to the characterization of ordinary crystals in terms of the spatial LRO. Generalizing Eqs. (1) and (2), we could say the system is a time crystal if the correlation function $\lim_{V\to\infty}\langle\hat{\phi}(\vec{x},t)\hat{\phi}(0,0)\rangle \to f(t)$ is nonvanishing for large enough $|\vec{x}|$ and exhibits a nontrivial periodic oscillation in time t (i.e., is not just a constant over time). In terms of the integrated order parameter defined above, the condition reads

$$\lim_{V\to\infty}\langle e^{i\hat{H}t}\hat{\Phi}e^{-i\hat{H}t}\hat{\Phi}\rangle/V^2 = f(t). \qquad (3)$$

When f is a periodic function of both space and time, we call it a space-time crystal, in which case we have

$$\lim_{V\to\infty}\langle e^{i\hat{H}t}\hat{\Phi}_{\vec{G}}e^{-i\hat{H}t}\hat{\Phi}_{-\vec{G}}\rangle/V^2 = f_{\vec{G}}(t), \qquad (4)$$

where $f_{\vec{G}}(t)$ is the Fourier component of $f(t,\vec{x})$. For example, Li *et al.* [4] investigated a Wigner crystal in a ring threaded by a Aharonov-Bohm flux and predicted its spontaneous rotation, which would be a realization of a space-time crystal. If this were indeed the case, the density at (x_1,t_1) and (x_2,t_2) would be correlated as illustrated in Fig. 1.

One might think that we could define time crystals based on the time dependence of equal-*position* correlation functions. Should we adopt this definition, however, rather trivial systems would qualify as time crystals. For example, consider a two-level system $\hat{H} = -\Omega_0\sigma_z/2$ at $T=0$ and

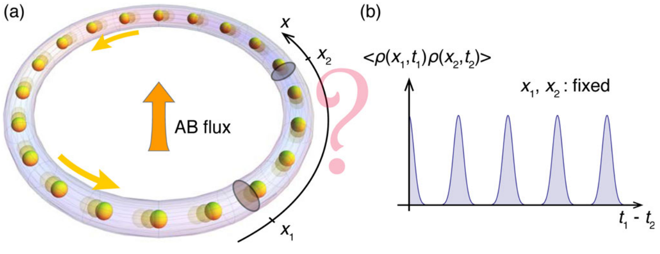

FIG. 1 (color online). Time-dependent correlation. (a) Wigner crystal of ions in a ring threaded by an Aharonov-Bohm flux, as proposed in Ref. [4] as a possible realization of a time crystal. (b) Illustration of the time dependent correlation function, should the time crystal be indeed realized as a spontaneous rotation of the density wave (crystal) in the ground state, as proposed. The density-density correlation function between (x_1,t_1) and (x_2,t_2) must exhibit an oscillatory behavior as a function of $t_1 - t_2$ for fixed x_1 and x_2.

set $\hat{\phi}(t) \equiv \sigma_x(t) = e^{i\hat{H}t}\sigma_x e^{-i\hat{H}t} = \sigma_x \cos\Omega_0 t + \sigma_y \sin\Omega_0 t$. The correlation function $\langle 0|\hat{\phi}(t)\hat{\phi}(0)|0\rangle$ of the ground state $|0\rangle = (1,0)^T$ exhibits a periodic time dependence $e^{-i\Omega_0 t}$. The same applies to the equal-position correlation function in independent two-level systems spread over the space. Clearly we do not want to classify such a trivial, uncorrelated system as a time crystal. "Crystal" should be reserved for systems exhibit correlated, coherent behaviors, which are captured by long-distance correlation functions, be it an ordinary crystal or a time crystal.

Absence of long-range time order at $T = 0$.—We now prove quite generally that time crystals defined above are not possible in ground states. More precisely, we show

$$\frac{1}{V^2}|\langle 0|\hat{A}e^{-i(\hat{H}-E_0)t}\hat{B}|0\rangle - \langle 0|\hat{A}\,\hat{B}\,|0\rangle| \leq C\frac{t}{V}, \tag{5}$$

where E_0 is the ground-state energy. Equation (5) holds for any Hermitian operators $\hat{A} = \int_V d^d x \hat{a}(\vec{x})$ and $\hat{B} = \int_V d^d x \hat{b}(\vec{x})$, where $\hat{a}(\vec{x})$ and $\hat{b}(\vec{x})$ are local operators that act only near $\vec{x}$. The constant C may depend on $\hat{A}$, $\hat{B}$, and $\hat{H}$ but not on t or V. Once we prove Eq. (5), we can immediately see that $f(t)$ in Eq. (3) is time independent, by setting $\hat{A} = \hat{B} = \hat{\Phi}(0)$ and taking the limit $V \to \infty$ for $t = o(V)$. We can also apply Eq. (5) to space-time crystals characterized by $f(t, \vec{x})$. Although $\hat{\Phi}_{\vec{G}}$ may not be Hermitian, one can always decompose it to the sum of two Hermitian operators. Applying Eq. (5) for each of them, one can see all $f_{\vec{G}}(t)$'s, and, hence, $f(t, \vec{x})$, are time independent.

To show Eq. (5), we use the trick to represent the change in time by an integral

$$\begin{aligned} &|\langle 0|\hat{A}e^{-i(\hat{H}-E_0)t}\hat{B}|0\rangle - \langle 0|\hat{A}\,\hat{B}\,|0\rangle| \\ &\quad = \left|\int_0^t \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s}\langle 0|\hat{A}e^{-i(\hat{H}-E_0)s}\hat{B}|0\rangle\right| \\ &\quad \leq \int_0^t \mathrm{d}s|\langle 0|\hat{A}(\hat{H}-E_0)e^{-i(\hat{H}-E_0)s}\hat{B}|0\rangle|. \end{aligned} \tag{6}$$

The integrand can be bounded by the Schwarz inequality as

$$\begin{aligned} &|\langle 0|\hat{A}(\hat{H}-E_0)^{1/2}e^{-i(\hat{H}-E_0)s}(\hat{H}-E_0)^{1/2}\hat{B}|0\rangle| \\ &\quad \leq \sqrt{\langle 0|\hat{A}(\hat{H}-E_0)\hat{A}|0\rangle\langle 0|\hat{B}(\hat{H}-E_0)\hat{B}|0\rangle} \\ &\quad = \frac{1}{2}\sqrt{\langle 0|[\hat{A},[\hat{H},\hat{A}]]|0\rangle\langle 0|[\hat{B},[\hat{H},\hat{B}]]|0\rangle}. \end{aligned} \tag{7}$$

Each of $\hat{H}$, $\hat{A}$, and $\hat{B}$ involves a spatial integration and introduces a factor of V, while each commutation relation reduces a factor of V, assuming that the equal-time commutation relation of any two operators $\hat{\phi}_1(\vec{x}, t)$ and $\hat{\phi}_2(\vec{x}', t)$ can be nonzero only near $\vec{x} = \vec{x}'$. Hence, $\|[\hat{A},[\hat{H},\hat{A}]]\|$ is at most of the order of $V^{3-2} = V$ [15,16]. The same is true for $\|[\hat{B},[\hat{H},\hat{B}]]\|$. Therefore, combining Eqs. (6) and (7), we get the desired Eq. (5).

In this estimate of $\|[\hat{A},[\hat{H},\hat{A}]]\|$, we assumed the locality of the Hamiltonian; i.e., $\hat{H}$ is an integral of the Hamiltonian density $\hat{h}(\vec{x})$, which contain only local terms. It is easy to see that the same conclusion holds even when there are interactions among distant points, provided that the interaction decays exponentially as a function of the distance. One can further relax this assumption to power-law decaying interactions $r^{-\alpha}$ ($\alpha > 0$). When $0 < \alpha < d$, $\|[\hat{A},[\hat{H},\hat{A}]]\|$ can be order of $V^{2-(\alpha/d)}$ and the right-hand side of Eq. (5) should be accordingly modified to $CtV^{-(\alpha/d)}$, where d is the spatial dimension of the system. When $\alpha \geq d$, Eq. (5) holds without any change. In both cases, as long as $\alpha > 0$, $f(t)$ remains time independent in the limit $V \to \infty$ for a fixed finite t.

Absence of long-range time order at a finite T.—The argument presented above cannot directly be extended to excited eigenstates $|n\rangle$, because $(\hat{H} - E_0)^{1/2}$ in Eq. (6) would then be replaced by $(\hat{H} - E_n)^{1/2}$ but the latter is not well defined. Instead, here we employ the Lieb-Robinson bound [17] to discuss finite temperatures.

The result of Lieb and Robinson is that [17]

$$\|[e^{i\hat{H}t}a(\vec{x})e^{-i\hat{H}t}, b(\vec{y})]\| \leq \min\{C_1, C_2 e^{-\mu(|\vec{x}-\vec{y}|-vt)}\}, \tag{8}$$

where constants $C_{1,2}$, μ, and v may depend on $\hat{a}$, $\hat{b}$, and $\hat{H}$. This bound is valid only for a local Hamiltonian. The physical meaning of Eq. (8) is that there exists an upper bound on the velocity at which information can propagate in quantum systems.

To prove the time independence of $f(t)$ in Eq. (3) and $f_{\vec{G}}(t)$ in Eq. (4), let us introduce a new correlation function defined by the commutation relation

$$\begin{aligned} g_{AB}(t) &\equiv \langle [e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}, \hat{B}]\rangle / V^2 \\ &= \int_V \mathrm{d}^d x \langle [e^{i\hat{H}t}\hat{a}(\vec{x})e^{-i\hat{H}t}, \hat{b}(\vec{0})]\rangle / V. \end{aligned} \tag{9}$$

The Lieb-Robinson bound (8) tells us that $|g_{AB}(t)| \leq [C_3 + C_4(vt)^d]/V$ for some constants $C_{3,4}$ that do not depend on t or V. Hence, as long as $t = o(V^{1/d})$, $|g_{AB}(t)| \to 0$ as $V \to \infty$.

On the other hand, we have $g_{AB}(t) = f_{AB}(t) - f_{BA}(-t)$, where $f_{AB}(t) \equiv \langle e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}\hat{B}\rangle / V^2$. By inserting the complete set $1 = \sum_n |n\rangle\langle n|$, it can be readily shown that

$$f_{AB}(t) = \int_{-\infty}^{\infty} \mathrm{d}\omega \rho_{AB}(\omega) e^{-i\omega t}, \tag{10}$$

$$g_{AB}(t) = \int_{-\infty}^{\infty} \mathrm{d}\omega (1 - e^{-\beta\omega})\rho_{AB}(\omega) e^{-i\omega t}, \tag{11}$$

where $\rho_{AB}(\omega)$ is defined by

$$\sum_{n,m}\frac{\langle m|\hat{A}|n\rangle\langle n|\hat{B}|m\rangle e^{-\beta E_m}}{ZV^2}\delta(\omega-E_n+E_m). \tag{12}$$

Since $\lim_{V\to\infty}g_{AB}(t)=0$ for any given real value of t, $(1-e^{-\beta\omega})\lim_{V\to\infty}\rho_{AB}(\omega)=0$. Combined by the sum rule $\int d\omega\rho_{AB}(\omega)=f_{AB}(0)=\langle\hat{A}\,\hat{B}\rangle/V^2$, we find

$$\lim_{V\to\infty}\rho_{AB}(\omega)=\delta(\omega)\lim_{V\to\infty}\langle\hat{A}\,\hat{B}\rangle/V^2. \tag{13}$$

Therefore, $f_{AB}(t)$, as a function of finite t in the thermodynamic limit $V\to\infty$, can be at most a finite constant that does not depend on time. Thus (space-)time crystals do not exist at a finite temperature either.

Grand-canonical ensemble.—Let us discuss systems with variable number of particles. The equilibrium of those systems can be described by a grand-canonical ensemble. It is given by the Boltzmann-Gibbs distribution with respect to $\hat{\mathcal{H}}=\hat{H}-\mu\hat{N}$, where μ is the chemical potential determined by the property of the particle reservoir. Namely, the expectation value of an observable $\hat{X}$ is given by $\langle\hat{X}\rangle_\mu\equiv\mathrm{tr}(\hat{X}e^{-\beta\hat{\mathcal{H}}})/Z_\mu$, where $Z_\mu\equiv\mathrm{tr}e^{-\beta\hat{\mathcal{H}}}$. Although the statistical weight is given in terms of $\hat{\mathcal{H}}$, the time evolution of the Heisenberg operator $\hat{\Psi}(t)$ is still defined by $\hat{H}$, i.e., $\hat{\Psi}(t)\equiv e^{i\hat{H}t}\hat{\Psi}(0)e^{-i\hat{H}t}$. This mismatch can produce some trivial time dependence as we shall see now. If we define $\hat{\Psi}_\mu(t)\equiv e^{i(\hat{H}-\mu\hat{N})t}\hat{\Psi}(0)e^{-i(\hat{H}-\mu\hat{N})t}$ and assume $[\hat{N},\hat{\Psi}(0)]=-q\hat{\Psi}(0)$ with q a real number that represents the U(1) charge of Ψ, then $\hat{\Psi}(t)=\hat{\Psi}_\mu(t)e^{-iq\mu t}$. Therefore, even if $f_\mu=\lim_{V\to\infty}[\langle\hat{\Psi}_\mu(t)\hat{\Psi}_\mu^\dagger(0)\rangle_\mu/V^2]$ is time independent as we proved above, $f(t)=\lim_{V\to\infty}[\langle\hat{\Psi}(t)\hat{\Psi}^\dagger(0)\rangle_\mu/V^2]$ has a trivial time dependence $f(t)=f_\mu e^{-iq\mu t}$. This is consistent with the well-known fact that the order parameter of a Bose-Einstein condensate has the trivial time dependence [18] as $\langle\hat{\psi}(\vec{x},t)\rangle=\psi_0e^{-i\mu t}$. This type of time dependence has been discussed [19] also in the context of time crystals [20,21]. However, Volovik pointed out that this kind of time dependence cannot be measured as long as the particle number is exactly conserved [22]. Indeed, the overall phase of condensate cannot be measured unless one couples the condensate to another one. We will discuss this phenomenon in the following section.

Spontaneous oscillation of nonequilibrium states.—In order to extract the time dependence of the condensate order parameter, the system has to be attached to another system to allow change of the number of particles. As a simplest setup, we may prepare two condensates with different chemical potentials μ_1,μ_2 and measure their time-dependent interference pattern $\propto e^{-i(\mu_1-\mu_2)t}$ in terms of the current between the condensates, or, equivalently, the change of the number of particles in each condensate. This is nothing but the ac Josephson effect. In fact, in Ref. [9], a proposal of time crystal based on this effect was made.

However, in order to observe the ac Josephson effect, the initial state simply must not be in the equilibrium. In order to see this, it is helpful to use the mapping of the ac Josephson effect in two coupled condensates to a quantum spin in a magnetic field. For simplicity, let us consider condensates of bosons without any internal degree of freedom, and suppose there is only one single-particle state in each condensate. Then the system can be described by the two set of bosonic annihilation-creation operators, $a,a^\dagger$ and $b,b^\dagger$. The effective Hamiltonian of the system, in the limit of zero coupling between the two condensates, is given as $H=\mu_1a^\dagger a+\mu_2b^\dagger b=(\mu_1-\mu_2)\frac{a^\dagger a-b^\dagger b}{2}+\frac{\mu_1+\mu_2}{2}N$, where $N=a^\dagger a+b^\dagger b$ is the total number of particles in the coupled system. Let us assume that the coupling to the outside environment is negligible in the time scale we are interested in. N is then exactly conserved and can be regarded as a constant. As a consequence, the second term in the Hamiltonian proportional to N can be ignored.

With N being exactly conserved, this system of coupled condensates can be mapped to a quantum spin model by identifying the bosons as Schwinger bosons. The Hamiltonian now reads $H=BS^z+\mathrm{const}$, where $B=\mu_1-\mu_2$ and S^z is the z component of the quantum spin with the spin quantum number $S=(N-1)/2$. Similarly, the current operator between the two condensates is given by $J\propto-i(a^\dagger b-b^\dagger a)=2S^y$. The ac Josephson effect, in the quantum spin language, is just a Larmor precession about the magnetic field. The oscillatory behavior of the current in the ac Josephson effect just corresponds to the oscillation of the excitation value of S^y in the Larmor precession.

In order to observe the Larmor precession, the initial state must have a nonvanishing expectation value of the transverse component (S^x or S^y). This excludes the ground state, in which the spin is fully polarized along the magnetic field in the z direction, as well as thermodynamic equilibrium at arbitrary temperature. In Ref. [9] it was argued that, by taking the limit of weak coupling, the dissipation can be made arbitrarily small. While this is certainly true, the lack of dissipation does not mean that the system is in an equilibrium, as it is clear by considering the spin Larmor precession in a magnetic field. Our result, which is valid for equilibrium, of course does not exclude such spontaneous oscillations of *nonequilibrium* quantum states. The latter, however, are well known and should not be called time crystals without a further justification.

Discussion.—In this Letter, we proposed a definition of time crystals and proved their absence in the equilibrium. The present result brings back the question: why there is no time crystal, even though there surely exist crystals with a spatial long-range order? We should recall that Lorentz invariance does not mean the complete equivalence between space and time: the time direction is still distinguished by the different sign of the metric. This leads to a fundamental difference in the spectrum: while the eigenvalues of the Hamiltonian (the generator of translation in time direction) is bounded from below, the eigenvalues of the momentum (the generator of translation in a spatial direction) is unbounded. Moreover, the equilibrium is determined by the Hamiltonian and the system is generally not Lorentz invariant in the thermodynamic equilibrium. Therefore, as far as the equilibrium as defined in standard statistical mechanics is concerned, it is not surprising to find a fundamental difference between space and time.

H. W. wishes to thank Ashvin Vishwanath for his useful and important suggestions. The authors are very grateful to Hal Tasaki and Tohru Koma for helping us improve the presentation. In particular, the proof for $T = 0$ presented in this Letter is an improved version due to Tohru Koma, after discussion with Hal Tasaki. The work of M. O. was supported by the "Topological Quantum Phenomena" (No. 25103706) Grant-in Aid for Scientific Research on Innovative Areas from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. H. W. appreciates financial support from the Honjo International Scholarship Foundation.

---

*hwatanabe@berkeley.edu
†oshikawa@issp.u-tokyo.ac.jp

[1] F. Wilczek, Quantum Time Crystals, Phys. Rev. Lett. **109**, 160401 (2012).

[2] P. Bruno, Comment on Quantum Time Crystals, Phys. Rev. Lett. **110**, 118901 (2013).

[3] F. Wilczek, Wilczek Reply, Phys. Rev. Lett. **110**, 118902 (2013).

[4] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, Space-Time Crystals of Trapped Ions, Phys. Rev. Lett. **109**, 163001 (2012).

[5] P. Bruno, Comment on Space-Time Crystals of Trapped Ions, Phys. Rev. Lett. **111**, 029301 (2013).

[6] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, Reply to Comment on "Space-Time Crystals of Trapped Ions", arXiv:1212.6959.

[7] P. Bruno, Impossibility of Spontaneously Rotating Time Crystals: A No-Go Theorem, Phys. Rev. Lett. **111**, 070402 (2013).

[8] P. Nozières, Time crystals: Can diamagnetic currents drive a charge density wave into rotation?, Europhys. Lett. **103**, 57008 (2013).

[9] F. Wilczek, Superfluidity and Space-Time Translation Symmetry Breaking, Phys. Rev. Lett. **111**, 250402 (2013).

[10] R. Yoshii, S. Takada, S. Tsuchiya, G. Marmorini, H. Hayakawa, and M. Nitta, Time crystal phase in a superconducting ring, arXiv:1404.3519.

[11] T. Kaplan, P. Horsch, and W. von der Linden, Order Parameter in Quantum Antiferromagnets, J. Phys. Soc. Jpn. **58**, 3894 (1989).

[12] T. Koma and H. Tasaki, Symmetry Breaking in Heisenberg Antiferromagnets, Commun. Math. Phys. **158**, 191 (1993).

[13] X.-G. Wen, *Quantum Field Theory of Many-Body Systems*, 2nd ed. (Oxford University Press, New York, 2004).

[14] R. Griffiths, Spontaneous Magnetization in Idealized Ferromagnets, Phys. Rev. **152**, 240 (1966).

[15] T. Koma and H. Tasaki, Symmetry Breaking and Finite-Size Effects in Quantum Many-Body Systems, J. Stat. Phys. **76**, 745 (1994).

[16] P. Horsch and W. von der Linden, Spin correlations and low lying excited states of the spin-$1/2$ Heisenberg antiferromagnet on a square lattice, Z. Phys. B **72**, 181 (1988).

[17] E. H. Lieb and D. W. Robinson, The finite group velocity of quantum spin systems, Commun. Math. Phys. **28**, 251 (1972).

[18] C. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, England, 2008).

[19] A. Nicolis and F. Piazza, Spontaneous symmetry probing, J. High Energy Phys. 06 (2012) 025.

[20] E. Castillo, B. Koch, and G. Palma, On the dynamics of fluctuations in time crystals, arXiv:1410.2261.

[21] M. Thies, Semiclassical time crystal in the chiral Gross-Neveu model, arXiv:1411.4236.

[22] G. Volovik, On the Broken Time Translation Symmetry in Macroscopic Systems: Precessing States and Off-Diagonal Long-Range Order, JETP Lett. **98**, 491 (2013).