Multiple Observers Can Share the Nonlocality of Half of an Entangled Pair by Using Optimal Weak Measurements

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We investigate the trade-off between information gain and disturbance for von Neumann measurements on spin-$\frac{1}{2}$ particles, and derive the measurement pointer state that saturates this trade-off, which turns out to be highly unusual. We apply this result to the question of whether the nonlocality of a single particle from an entangled pair can be shared among multiple observers that act sequentially and independently of each other, and show that an arbitrarily long sequence of such observers can all violate the Clauser-Horne-Shimony-Holt–Bell inequality.

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Introduction.—A central paradigm in quantum theory is that measurements are necessarily disturbing—in order to probe the properties of a system one must perturb it [1]. The measurement postulate [2] states that performing what is referred to as a "strong" measurement collapses the system into one of the eigenstates of the measured observable; this type of measurement offers the maximum information about the system.

On the other hand, there exist measurement schemes that disturb the system infinitesimally, offering only a small amount of information about the state. Such "weak" measurements are often considered in conjunction with postselection [3], a formalism that has precipitated the study of weak values [4]. Of course, these measurements are important by themselves, even without postselection. Indeed, all macroscopic measurements are weak measurements [5,6].

Here we consider measurements of all intermediate strengths focusing on the trade-off between the degree of disturbance and the amount of information we gain about the system. This trade-off has been explored extensively, both in the context of specific measuring devices [7,8] and abstract measurement representations [9–12].

The subject of our investigation is the von Neumann–type measurement, which is characterized by the pointer of the measuring device being displaced proportionally to the value of the measured observable. This offers arguably the most direct connection between the measured physical quantity and the reading of the measuring device. We are interested in deriving the optimal measurements, i.e., those that maximize the information gain for a given disturbance to the system.

In this Letter, we consider the case of dichotomic measurements on spin-$\frac{1}{2}$ particles.

The information gain and disturbance can be modified by changing the initial state of the pointer as well as the strength of the coupling between the system and the measuring device. However, the optimal information gain vs disturbance trade-off cannot be achieved by only tuning the coupling strength (which is equivalent to rescaling the state of the pointer). Rather the initial state of the pointer must be appropriately chosen. We determine the optimal pointer state, and find it to be very counterintuitive. In particular, it is nothing like the Gaussian wave packet that is almost universally considered and considerably outperforms it.

We then use a simple bipartite scenario involving successive measurements to find a constraint on the trade-off, in a similar vein to those derived in [10,12]. The trade-off attained by the optimal pointer saturates this constraint.

Since von Neumann measurements are, on the one hand rich enough to allow us to tune this trade-off, and on the other hand simple enough to allow manageable calculations, they enable us to raise and answer a new fundamental question in nonlocality: can the nonlocality of an entangled pair of particles be distributed among multiple observers, that act sequentially and independently of each other? We consider the scenario that a single observer has access to one of the particles of an entangled pair, and a group of observers have access to the second particle. Each observer in the second group acts independently, performing a measurement on the particle before passing it on to the next member of the group. We address the question of whether the single observer with the first particle can see nonlocal correlations with *all* of the members in the second group.

Crucially, we find that each member in the second group cannot perform a very weak measurement, since this is unable to extract enough information to observe nonlocal correlations. Hence the state is disturbed significantly, and it is not clear that subsequent observers can still observe nonlocal correlations. Nevertheless, we show that an arbitrary number of independent observers can indeed see consecutive violations of the CHSH (Clauser-Horne-Shimony-Holt)-Bell inequality. As well as teaching us about the nature of nonlocality, this problem illuminates the nature of the information gain vs disturbance trade-off.

von Neumann measurement pointers for spin-$\frac{1}{2}$ particles.—In a von Neumann–type measurement, the pointer is shifted proportional to the eigenvalues of the measured observable

$$|\Psi\rangle \otimes |\varphi(q)\rangle \longrightarrow \sum_a \langle a|\Psi\rangle \cdot |a\rangle \otimes |\varphi(q - g_0 a)\rangle, \quad (1)$$

where Ψ and $\varphi(q)$ are the initial states of the system and pointer, respectively, the index a refers to the eigenbasis of the observable, q is the position of the pointer, and g_0 is a coupling constant. The outcome of the measurement is then provided by reading the position of the pointer.

The evolution (1) is generated by the interaction Hamiltonian $H(t) = g(t)A \otimes p$, where A is the measured observable, p is the momentum operator of the pointer conjugate to q, and $g(t)$ is nonzero only during a short time interval and normalized so that $\int g(t)dt = g_0$. Here, we take $g_0 = 1$, which can be done without loss of generality by simply rescaling the pointer state [13].

In a strong measurement the pointer's initial state is narrower than the distance between the eigenvalues, i.e., $\langle\varphi(q-a)|\varphi(q-a')\rangle = \delta_{aa'}$; hence, reading the pointer's position provides full information of the measured physical quantity and collapses the system into the corresponding eigenstate of the observable.

Conversely, if the pointer spread is very large, covering the entire spectrum of eigenvalues, reading the pointer position provides essentially no information since $\langle\varphi(q-a)|\varphi(q-a')\rangle \approx 1$ and the system is not perturbed,

$$\begin{aligned} |\Psi'\rangle_{|q_0} &= \sum_a \langle a|\Psi\rangle\langle q_0|\varphi(q-a)\rangle|a\rangle \\ &\approx \langle q_0|\varphi(q)\rangle \sum_a \langle a|\Psi\rangle|a\rangle = \langle q_0|\varphi(q)\rangle|\Psi\rangle. \quad (2) \end{aligned}$$

This is the limit of a weak measurement.

We now consider measurements in between the two extremes. Focusing on spin-$\frac{1}{2}$ particles, the initial state of the spin in the eigenbasis of the measured observable is $|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, hence

$$|\Psi\rangle \otimes |\varphi(q)\rangle \longrightarrow \alpha|\uparrow\rangle \otimes |\varphi(q-1)\rangle + \beta|\downarrow\rangle \otimes |\varphi(q+1)\rangle. \quad (3)$$

For simplicity, we consider pointer states with symmetric modulus, i.e., $|\varphi(q)| = |\varphi(-q)|$. We also take $\varphi(q)$ to be real valued, without loss of generality, since complex pointers are shown not to outperform real ones (see the Supplemental Material [14], Part B).

To determine the disturbance produced by the measurement, we compute the system postmeasurement state by tracing out the pointer (see [14], Part A)

$$\rho' = F|\Psi\rangle\langle\Psi| + (1-F)(\pi^+|\Psi\rangle\langle\Psi|\pi^+ + \pi^-|\Psi\rangle\langle\Psi|\pi^-), \quad (4)$$

where $\pi^+ = |\uparrow\rangle\langle\uparrow|$ and $\pi^- = |\downarrow\rangle\langle\downarrow|$. The quantity F is independent of the state of the spin, and is the scalar product of the displaced pointer states,

$$F = \int_{-\infty}^{+\infty} \varphi(q+1)\varphi(q-1)dq. \quad (5)$$

We call F the "quality factor" of the measurement since it is the proportion of the postmeasurement state that corresponds to the original state. The remainder corresponds to the state decohered in the measurement eigenbasis, as it would have been if measured strongly.

The other quantity of interest is the information gain. Since we are measuring a dichotomic observable, we digitize the reading of the pointer, associating positive positions to the outcome $+1$ and negative positions to -1 (see discussion in the Supplemental Material [14], Part C). The probability of the outcomes ±1 is then (see [14], Part A)

$$P(\pm1) = G\langle\Psi|\pi^\pm|\Psi\rangle + (1-G)\frac{1}{2}. \quad (6)$$

G is also independent of the state of the spin, and depends on the width of the pointer compared to the distance between the eigenvalues,

$$G = \int_{-1}^{+1} \varphi^2(q)dq. \quad (7)$$

The first term in (6) represents the contribution of the probability as if there was a strong measurement, so we call G the precision of the measurement. The other term, $(1-G)\frac{1}{2}$, corresponds to a random outcome.

Consider, for example, the simple case of a square pointer state: $\varphi(q) = 1/(\sqrt{2\Delta})$ for $-\Delta < q < +\Delta$ and zero elsewhere. If the spread Δ is smaller than 1, then reading the pointer's position provides full information of the measured spin; i.e., $\Delta \le 1$ corresponds to a strong measurement: $F = 0$, $G = 1$. When $\Delta > 1$, we find that $G = 1 - F$. Hence square pointers correspond to measuring strongly with probability G and producing a random result, without measuring, with probability $1 - G$.

However, square pointers are far from optimal: Gaussian wave packets achieve a better trade-off between F and G (see Fig. 1), but are still not optimal.

Optimal pointers.—Since F and G are solely functionals of the pointer state it is natural to look for the one that achieves the best trade-off by using variational calculus (see the Supplemental Material [14], Part B).

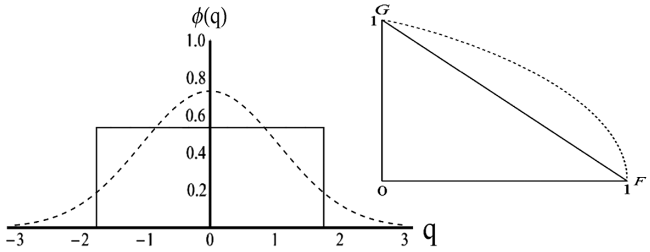

FIG. 1. Square (solid line) and Gaussian (dashed line) pointers of equal width $\Delta = 1.5$ with (inset) the corresponding trade-off between the precision G and quality factor F.

Interestingly, we find that for any quality factor F, there is an entire family of optimal pointer states that achieve the maximum precision G. Each element of this family is defined by the choice of an arbitrary function $f(q)$ in the interval $-1 < q < +1$ such that the norm of the pointer state within this interval is the precision G (7). The function is then copied to all other regions between adjacent odd points $q = 2n - 1$ and $q = 2n + 1$ with the relative height of the function in each region falling under an exponential envelope that depends on G,

$$\varphi(q) = f(q - 2n)\left(\sqrt{\frac{1-G}{1+G}}\right)^{|n|} \quad \forall q \in (2n-1, 2n+1], n \in \mathbb{Z}. \tag{8}$$

Two such optimal pointer states are plotted in Fig. 2, along with the trade-off compared to that of Gaussian pointers. For an optimal pointer state, the trade-off is given by

$$F^2 + G^2 = 1. \tag{9}$$

A bound on the disturbance-precision trade-off.— Interestingly, the above trade-off (9) can also be deduced from a simple Bell-inequality–type scenario (Fig. 3). Alice and Bob each possess one-half of a singlet state of spin-$\frac{1}{2}$ particles. Alice receives a binary input $x \in \{0, 1\}$, and performs a strong projective measurement of her spin along a corresponding direction $\bar{u}_x$; we label her outcome $a = \pm 1$. Bob receives two consecutive binary inputs $y_1, y_2 \in \{0, 1\}$, and performs two consecutive spin measurements along corresponding directions $\bar{w}_{y_1}$ and $\bar{v}_{y_2}$; his outputs are labeled b_1 and b_2 (± 1). Bob's first measurement has intermediate strength, while his second is a strong measurement.

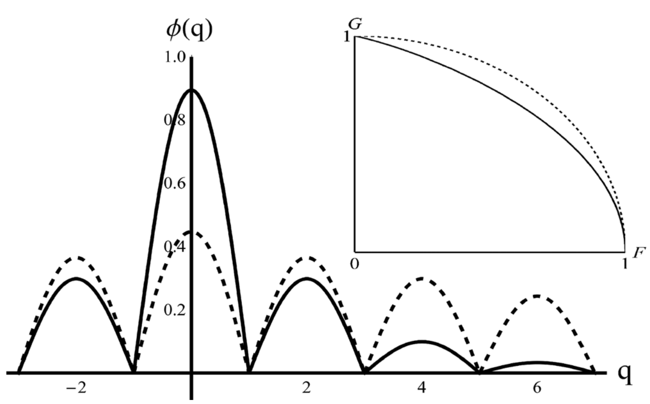

FIG. 2. Plot of two optimal pointer distributions, $\{G = 0.8, F = 0.6\}$ (solid) and $\{G = 0.2, F = 0.98\}$ (dashed). Inset: Comparison of the optimal trade-off (dashed) to that attained by the Gaussian pointer (solid).

Such a scenario is characterized by the conditional probabilities of the outcomes, $P(ab_1b_2|xy_1y_2)$. To calculate these, we require the state of Bob's spin after his first measurement of intermediate strength. This is different from the state in Eq. (4), since here we require the postmeasurement state *given the specific outcome* b_1, and thus trace only over either positive or negative pointer positions, respectively.

We find that the outcome dependent state of the spin-$\frac{1}{2}$ particle is dependent on both the quality factor F and precision G of the measurement

$$\rho'_{|b_1} = \frac{F}{2}\rho + \left(\frac{1 + b_1G - F}{2}\right)\pi^+\rho\pi^+ + \left(\frac{1 - b_1G - F}{2}\right)\pi^-\rho\pi^-, \tag{10}$$

where π^+ and π^- denote the projectors of the spin measurement, ρ is the premeasurement state, and $\rho'_{|b_1}$ is the unnormalized postmeasurement state of the system given the outcome b_1. From this state, one arrives at the conditional probability (see the Supplemental Material [14], part D),

$$P(ab_1b_2|xy_1y_2) = \frac{b_1G}{4}\left(\frac{a\bar{u}_x \cdot \bar{w}_{y_1} + b_2\bar{w}_{y_1} \cdot \bar{v}_{y_2}}{2}\right) + \frac{F}{4}\left(\frac{1 + ab_2\bar{u}_x \cdot \bar{v}_{y_2}}{2}\right) + \left(\frac{1-F}{4}\right)\left(\frac{1 + ab_2\bar{u}_x \cdot \bar{w}_{y_1}\bar{w}_{y_1} \cdot \bar{v}_{y_2}}{2}\right), \tag{11}$$

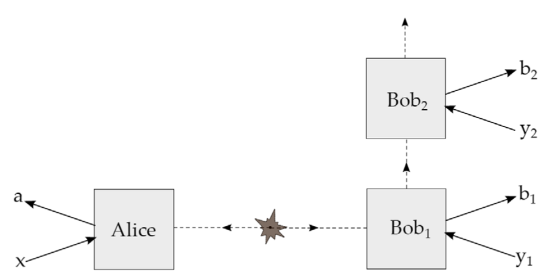

FIG. 3 (color online). Bell scenario involving a single Alice and multiple Bobs, where the dashed lines indicate a spin-$\frac{1}{2}$ particle being transmitted, and the solid lines the inputs and outputs.

which is nonsignaling between Alice and Bob, as expected.

Furthermore, being a probability, it must lie between 0 and 1. Choosing the measurement directions to be $\bar{u}_0 = \bar{Z}$, $\bar{w}_0 = -\bar{X}$, and $\bar{v}_0 = \bar{Z}\sin\theta - \bar{X}\cos\theta$ (where $\bar{Z}$ and $\bar{X}$ are two orthogonal directions in space), along with the outcomes $a = b_1 = b_2 = 1$, we obtain the inequality $P(111|000) = F\sin\theta + G\cos\theta \leq 1$. This is the expression of a tangent to the unit circle $F^2 + G^2 = 1$, at the point $\{\sin\theta, \cos\theta\}$. Varying over θ, we obtain all of the tangents to the unit circle as constraints on the pair $\{F, G\}$, and thus the pair must lie within the unit circle. The optimal pointer (described above) saturates this constraint.

Using such a Bell scenario to examine the trade-off is a natural method to study weak measurements in generalized probability theories, where one can expect the optimal trade-off to differ from the quantum trade-off.

Consecutive violations of the CHSH-Bell inequality.—Armed with an understanding of the trade-off between information gain and disturbance, we now raise a novel and fundamental question in nonlocality—can multiple observers share the nonlocality present in a single particle from an entangled pair? To answer this question, we consider the Bell scenario in Fig. 3, where Alice has one-half of an entangled pair of spin-$\frac{1}{2}$ particles, but instead of a single Bob performing two consecutive measurements, there are two Bobs that each perform a measurement one after the other on the second particle of the pair. The Bobs are independent; i.e., Bob_2 is ignorant of the direction that Bob_1 measures his spin in as well as the outcome of his measurement.

We investigate whether the statistics of the measurements of Bob_1 and Bob_2 can both be nonlocal with Alice by testing the conditional probabilities $P(ab_1|xy_1)$ and $P(ab_2|xy_2)$ against the CHSH inequality [15].

At first one may think it impossible to have simultaneous violations Alice-Bob_1 and Alice-Bob_2 because of the monogamy of entanglement [16] and of nonlocality [17,18]. However, these results assume no-signaling between all parties, while in our scenario Bob_1 implicitly signals to Bob_2 by his choice of measurement on the state before he passes it on. Hence, no monogamy argument holds, and one has to look more closely at the situation.

An unusual feature of this Bell scenario is that Bob_2's CHSH value depends on the *input bias* of Bob_1, i.e., the frequency with which Bob_1 received the input 0 versus the input 1. Even though the CHSH expression contains only conditional probabilities, the state that Bob_2 measures has been perturbed by Bob_1. Since Bob_2 is independent of Bob_1, his density matrix is the mixture of the states given each of Bob_1's two possible measurements, weighted by their relative frequencies. Hence, the input bias of Bob_1 affects the statistics of Bob_2's measurement.

To begin with, we assume the measurements are unbiased; i.e., both Bobs receive the inputs 0 and 1 with equal probability. Clearly Bob_1 cannot perform a strong measurement, since he would destroy the entanglement, and prevent Bob_2 from being nonlocal with Alice. However, Bob_1 may not be able to observe nonlocality with a very weak measurement either. To see this precisely, consider that Alice and the Bobs initially share a singlet state, and that they perform the standard measurements that attain Tsirelson's bound for the CHSH inequality: i.e., Alice measures in the $\bar{Z}$ or $\bar{X}$ direction, corresponding to inputs 0 or 1, respectively, and the Bobs measure in the directions $-(\bar{Z}+\bar{X})/\sqrt{2}$ or $(-\bar{Z}+\bar{X})/\sqrt{2}$, for their respective inputs 0 or 1.

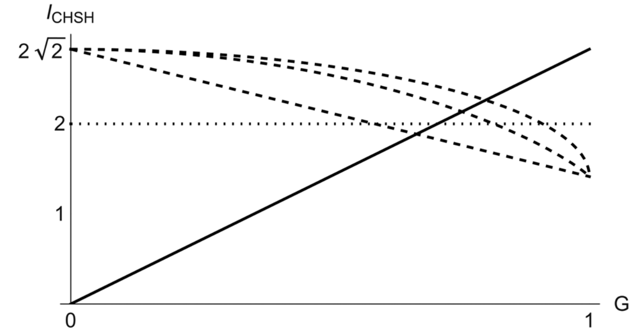

FIG. 4. Plot of $I^{(1)}_{\mathrm{CHSH}}$ (solid) as a function of the precision G of Bob_1, together with $I^{(2)}_{\mathrm{CHSH}}$ (dashed) for different pointer types, (from bottom) square, Gaussian and optimal.

Using the form of the CHSH expression [15] with the classical bound at 2 and the quantum bound at $2\sqrt{2}$, we find that the CHSH values of Alice with each Bob are given by $I^{(1)}_{\mathrm{CHSH}} = 2\sqrt{2}G$, and $I^{(2)}_{\mathrm{CHSH}} = \sqrt{2}(1+F)$, where G and F are the precision and quality factor of Bob_1's measurement. These are plotted in comparison to the classical bound in Fig. 4.

We see from the figure that Bob_1 must tune the precision of his measurement, as either a strong or weak measurement would prevent Bob_2 or himself, respectively, from seeing a CHSH violation. He must also use a pointer with a good trade-off—one cannot have a double violation using a square pointer, while it is possible with a Gaussian or optimal pointer.

Longer sequences of CHSH violations with biased inputs.—Since it is possible to have two Bobs simultaneously violate CHSH with Alice, the next natural question is whether there is a limit to the number of consecutive violations achievable.

We find that it is possible for more than two Bobs to violate CHSH with Alice, if the frequency of the inputs 0 and 1 to each Bob is not the same [19]. In the Supplemental Material [14], Part F, we provide an explicit measurement protocol that does so in the case that one of the inputs to the various Bobs occurs much more often than the other input. In this scenario, there is no limit to the number of Bobs that can violate CHSH with Alice—the larger the bias of the inputs, the longer the sequence of violations. However, in our protocol, the value of the CHSH violation in the sequence falls off superexponentially: if $V_n = I_{\mathrm{CHSH}}^{(n)} - 2$ is the maximum violation that can be achieved by Bob_n with Alice, we find that for large n (see [14], Part G),

$$V_{n+1} \approx \frac{V_n^3}{4}. \tag{12}$$

Discussion and open problems.—We have seen that the trade-off between information gain and disturbance for von Neumann measurements is strongly dependent on the initial state of the pointer, and the optimal pointer state differs considerably from the pointers considered usually, such as the Gaussian wave packet. An interesting question to ask is what form the optimal pointer takes for measurements on higher dimensional systems.

We also obtained a constraint on the trade-off by relating it to the probabilities in a simple Bell scenario. Such a method can be used to extend the concept of weak measurements to general nonlocal theories.

In the case of multiple observers violating a Bell inequality, we have numerical evidence that if the inputs to the various Bobs are unbiased, it is impossible to have more than a double violation of CHSH with Alice. Proving this analytically is an open problem. For general input bias, an open question is whether there exists a protocol that achieves a better CHSH violation than that found in this Letter. Also, one may generalize to the case when the Bobs have some information about each others' inputs and/or outcomes, this will presumably improve the CHSH violation. Finally, it would be interesting to include multiple Alices in the setup, and investigate if it is possible to have an arbitrarily long sequence of pairs of Alices and Bobs that violate a Bell inequality.

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