Optimal State Estimation for Cavity Optomechanical Systems

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We demonstrate optimal state estimation for a cavity optomechanical system through Kalman filtering. By taking into account nontrivial experimental noise sources, such as colored laser noise and spurious mechanical modes, we implement a realistic state-space model. This allows us to obtain the conditional system state, i.e., conditioned on previous measurements, with a minimal least-squares estimation error. We apply this method to estimate the mechanical state, as well as optomechanical correlations both in the weak and strong coupling regime. The application of the Kalman filter is an important next step for achieving real-time optimal (classical and quantum) control of cavity optomechanical systems.

Introduction.—State estimation is a crucial task at the heart of control theory, both in the classical [1] and quantum domain [2]. For Gaussian systems, real-time state estimation can be achieved in an optimal manner using Kalman-Bucy filtering [3,4]. Since many physical systems are approximately Gaussian, Kalman filtering has been successfully implemented for a broad range of uses, for example, for navigation and tracking in aeronautics (including the Apollo project and the Global Positioning System) [5], as well as in the physical sciences, such as for suspension noise cancellation in gravitational wave detection [6], Heisenberg limited atomic magnetometry [7], or quantum-enhanced optical-phase tracking [8–11]. In this Letter, we introduce a new domain of applications by implementing Kalman filtering for cavity optomechanical systems. These systems represent a versatile light-matter interface in which optomechanical interactions inside optical or microwave cavities allow control over optical and mechanical degrees of freedom. While the first investigations go back to the late 1960s in the context of gravitational wave detectors [12,13], it is only the last few years that have seen the development of a completely new generation of micro- and nano-optomechanical solid-state devices with fast-growing application areas from classical sensing to quantum information processing [14].

State estimation of a cavity optomechanical system in real time is key for optimal state control and verification. The outstanding challenge is to obtain reliable information on the mechanical subsystem. In optomechanics, this is done through an optical cavity field, which imposes both additional noise and dynamical backaction effects that have to be taken into account. Until now, reconstructions of the mechanical dynamics have focused either on statistical

properties [15,16] or, for real-time reconstructions, on regimes of sufficiently weak coupling and negligible dynamical backaction effects [17–20]. The information obtained from real-time estimation about the mechanical quadratures can be used for active feedback control of the mechanical resonator [19–21]. However, the validity of these reconstruction schemes breaks down when either coupling strength, dynamical backaction effects, or noise become strong. Our Kalman-filtering approach overcomes this limitation and allows us to demonstrate real-time optimal state estimation for cavity optomechanical systems operating in arbitrary parameter regimes. From a quantum physics perspective, the Kalman filter solves the stochastic Schrödinger equation—a stochastic, nonlinear generalization of the Schrödinger equation—which is the canonical way to describe quantum systems subject to a continuous measurement via coupling to electromagnetic fields [2,22–24]. These concepts and their application to mechanical systems have been the subject of extensive theoretical research [25–29], but no experiments in the context of cavity optomechanics have been conducted so far.

Kalman filter.—In quantum theory, just as in classical theories, a continuously observed system can be described by a conditional state [2], i.e., a state that incorporates the total amount of knowledge that an observer has extracted from her set of measurements. Discarding this knowledge yields the unconditional state, which is an incoherent mixture of all possible conditional states. Our goal is to find the (multipartite) conditional states of the full cavity optomechanical system including mechanical and optical subsystems. We restrict ourselves to Gaussian dynamics and measurements, which is a valid assumption for the

existing realizations of optomechanical systems [14]. For this case, it has been shown [30] that the problem of finding the conditional state can be mapped to a classical estimation problem, which is solved by a Kalman filter. It produces a real-time state estimate from a continuous measurement trajectory, which is optimal in the sense of minimizing the mean-square estimation error. We describe the system by the following (linear) state-space model

$$\dot{\mathbf{x}}_t = A_t \mathbf{x}_t + \mathbf{w}_t, \tag{1a}$$

$$\mathbf{z}_t = C_t \mathbf{x}_t + \mathbf{v}_t, \tag{1b}$$

where $\mathbf{x}_t$ is a state vector in some appropriately chosen state space (e.g., the phase space of a harmonic oscillator), $\mathbf{z}_t$ is the outcome of a linear measurement on the system, and $\mathbf{w}_t$ and $\mathbf{v}_t$ describe process and measurement noise, respectively. Both $\mathbf{w}_t$ and $\mathbf{v}_t$ are assumed to be zero-mean Gaussian white-noise processes, which obey $\mathrm{Re}(\mathbb{E}[\mathbf{w}_t \mathbf{w}_s^T]) = W \delta(t - s)$ and $\mathrm{Re}(\mathbb{E}[\mathbf{v}_t \mathbf{v}_s^T]) = V \delta(t - s)$, where δ is the Dirac δ function, $\mathbb{E}[\cdot]$ denotes the expectation value with respect to the initial probability distribution describing system and noise, and $\mathrm{Re}(\cdot)$ the real part [31]. Process and measurement noise may be correlated, which is described by the cross-correlations $\mathrm{Re}(\mathbb{E}[\mathbf{w}_t \mathbf{v}_s^T]) = M \delta(t - s)$. A_t and C_t are real, matrix-valued functions that parametrize the state-space model and are fixed by the physical model of the system and measurement process. We aim to find the estimate $\hat{\mathbf{x}}_t$ of $\mathbf{x}_t$ that minimizes the mean-square estimation error $\mathbb{E}[\|\mathbf{x}_t - \hat{\mathbf{x}}_t\|^2]$ at a time t when taking into account the measurement results $\{\mathbf{z}_s : 0 \leq s \leq t\}$. This yields $\hat{\mathbf{x}}_t = \mathbb{E}[\mathbf{x}_t | \{\mathbf{z}_s : 0 \leq s \leq t\}]$, i.e., the conditional expectation value of $\mathbf{x}_t$ given the full measurement record. Evaluating this for system (1) yields the time-continuous version of the Kalman filter [4]

$$\dot{\hat{\mathbf{x}}}_t = A_t \hat{\mathbf{x}}_t + K_t(\mathbf{z}_t - C_t \hat{\mathbf{x}}_t), \tag{2a}$$

$$\dot{P}_t = A_t P_t + P_t A_t^T + W - K_t V K_t^T, \tag{2b}$$

where $K_t = (P_t C_t^T + M) V^{-1}$ is the so-called Kalman gain, and $P_t = \mathrm{Re}(\mathbb{E}[(\mathbf{x}_t - \hat{\mathbf{x}}_t)(\mathbf{x}_t - \hat{\mathbf{x}}_t)^T])$ is the estimation-error covariance.

We can ascribe a quantum theoretical meaning to $\hat{\mathbf{x}}_t$ and P_t by associating $\mathbf{x}$ with the Schrödinger operators that describe the quantum system, $\mathbf{x}_t$ with the corresponding Heisenberg operators that evolve under Eq. (1a) (their Heisenberg-Langevin equations), and $\mathbf{z}_t$ with an operator-valued output process [24,30]. If $\hat{\rho}_t$ is the Gaussian state conditioned on a continuous measurement of $\mathbf{z}_t$, we have $\hat{\mathbf{x}}_t = \mathrm{tr}\{\mathbf{x}\hat{\rho}_t\}$ and $P_t = \mathrm{Re}(\mathrm{tr}\{\mathbf{x}\mathbf{x}^T\hat{\rho}_t\}) - \hat{\mathbf{x}}_t \hat{\mathbf{x}}_t^T$, i.e., the symmetrized covariance matrix of $\mathbf{x}_t$ with respect to $\hat{\rho}$. In other words, the conditional Gaussian state $\hat{\rho}_t$ is parametrized by $\hat{\mathbf{x}}_t$ and P_t [32]. By averaging over all possible trajectories of $\hat{\mathbf{x}}_t$, we recover the unconditional state, whose covariance matrix $\mathrm{Re}(\mathbb{E}[\mathbf{x}_t \mathbf{x}_t^T])$ we can extract from the
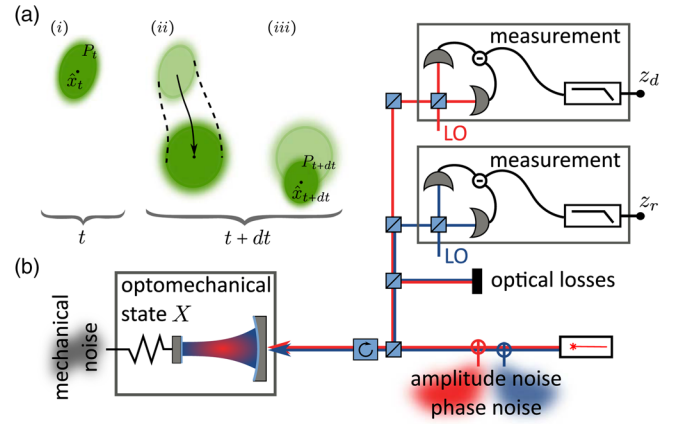

FIG. 1 (color online). Kalman filter for cavity optomechanical systems. (a) Working principle of the Kalman filter: (i) The conditional state is depicted by the green phase-space ellipse, which (ii) evolves in time according to the system dynamics. (iii) After a time dt, a Bayesian update is applied based on the measurement outcome to find the new conditional state. This procedure minimizes the mean-square estimation error, which makes the Kalman filter optimal for real-time state estimation. (b) Schematic of the experiment: The optomechanical cavity is driven by two laser beams each of which carry amplitude and phase noise. The mechanical motion is typically driven by Brownian noise. After their interaction with the cavity, the optical fields are detected by two independent homodyne measurements (signals z_d and z_r), which themselves are subject to optical losses and noise. Building an accurate Kalman filter requires appropriate modeling of all relevant noise sources.

estimated data by noting that $\mathbb{E}[(\mathbf{x}_t - \hat{\mathbf{x}}_t)\hat{\mathbf{x}}_t^T] = 0$, and, thus, $\mathrm{Re}(\mathbb{E}[\mathbf{x}_t \mathbf{x}_t^T]) = P_t + \mathbb{E}[\hat{\mathbf{x}}_t \hat{\mathbf{x}}_t^T]$ [2,33].

The Kalman-filter equations (2) describe how the conditional state is iteratively updated [Fig. 1(a)]. First, the estimate $\hat{\mathbf{x}}_t$ and the covariance P_t are propagated for an infinitesimal time interval dt [the first term in Eq. (2a) and the first three terms in Eq. (2b)] according to the state-space model Eq. (1a). Second, the measurement outcome is incorporated as a Bayesian update that corrects the value of the estimate $\hat{\mathbf{x}}_t$ and contracts the covariance ellipse [last terms in Eqs. (2a) and (2b)]. The updated values are again propagated by dt, and the procedure is repeated.

The model.—We consider a typical cavity optomechanical architecture [Fig. 1(b)], in which a Fabry-Pérot cavity (resonance frequency ω_c) coupled to a single mechanical mode [34] is driven by two laser fields (at frequencies $\omega_{0,d}$, $\omega_{0,r}$). The "resonant" beam ($\omega_r = \omega_c$) acts as a weak probe of the cavity length to stabilize the laser frequency with respect to the cavity resonance; the "detuned" beam ($\omega_d \neq \omega_c$) induces dynamical backaction effects, e.g., for laser cooling. This captures all relevant scenarios applied in typical optomechanics experiments. The mechanical element has a resonance frequency ω_m and energy damping rate γ_m. Both cavity modes exhibit decay at a (half width at half maximum) rate $\kappa = \kappa_1 + \kappa_2$, where κ_1 describes the

input coupler, and κ_2 accounts for spurious photon losses. The system is described by the (linearized) quantum Langevin equations [35–38]

$$\dot{q} = \omega_m p, \tag{3a}$$

$$\dot{p} = -\omega_m q - \gamma_m p + \sum_{i=r,d} g_i (\cos\theta_i x_i - \sin\theta_i y_i) + \xi, \tag{3b}$$

$$\dot{x}_i = -\kappa x_i + \Delta_i y_i + g_i \sin\theta_i q + \sqrt{2\kappa_1} x_{i,1}^{\text{in}}$$
$$+ \sqrt{2\kappa_2} x_{i,2}^{\text{in}} + 2\sqrt{\kappa_1}\delta\beta_i + |\alpha_{0,i}|\sin\theta_i \dot{\phi}_i, \tag{3c}$$

$$\dot{y}_i = -\kappa y_i - \Delta_i x_i + g_i \cos\theta_i q + \sqrt{2\kappa_1} y_{i,1}^{\text{in}}$$
$$+ \sqrt{2\kappa_2} y_{i,2}^{\text{in}} + |\alpha_{0,i}|\cos\theta_i \dot{\phi}_i, \tag{3d}$$

where q, p ($[q,p]=i$) describe the position and momentum of the mirror, and x_k, y_k with $[x_l, y_k] = i\delta_{lk}$ for $l, k \in \{r, d\}$, respectively, denote the amplitude and the phase quadrature of the cavity modes of the resonant and detuned beam. The optomechanical coupling to the cavity mode i is given by $g_i = \sqrt{2}g_0|\alpha_{0,i}|$ with $\alpha_{0,i} = \sqrt{2\kappa_1 P_i/\hbar\omega_{0,i}}/(\kappa + i\Delta_i)$, where g_0 is the single-photon coupling strength, P_i is the corresponding driving laser power, and $\Delta_i = \omega_{0,i} - \omega_c$ is the detuning of the respective driving laser (at $\omega_{0,i}$) with respect to the cavity resonance frequency (ω_c). The coupling of the mechanics to a thermal bath is modeled by a self-adjoint noise term ξ with $\langle\xi(t)\xi(s) + \xi(s)\xi(t)\rangle = 2\gamma_m(2\bar{n}+1)\delta(t-s)$ and $\bar{n} \approx k_B T/\hbar\omega_m$ (the mean occupation number of the bath at temperature T). Optical shot noise is denoted by x_i^{in}, y_i^{in} with variances $\langle x_i^{\text{in}}(t)x_j^{\text{in}}(s)\rangle = \langle y_i^{\text{in}}(t)y_j^{\text{in}}(s)\rangle = \frac{1}{2}\delta_{ij}\delta(t-s)$. Terms proportional to $\delta\beta_i$ and $\dot{\phi}_i$ describe the classical amplitude and phase noise of the driving lasers [36–38].

Homodyne detection is used to independently measure the generalized quadratures z_i of the reflected optical modes [Fig. 1(b)]. The cavity input-output relations yield

$$z_i' = (\sqrt{2\kappa_1} x_i + x_i^{\text{in}} + \delta\beta_i)\cos\varphi_i$$
$$+ (\sqrt{2\kappa_1} y_i + y_i^{\text{in}})\sin\varphi_i, \tag{4}$$

where $\delta\beta_i$ describes classical amplitude noise. We model optical losses and inefficient detection as beam-splitter losses parametrized by η. The measured quantities are rescaled to $z_i = \sqrt{1-\eta}z_i' + \sqrt{\eta}z_i^{\text{in}}$, where z_i^{in} describes additional quantum noise independent of x_i^{in} and y_i^{in}, i.e., $\langle x_i^{\text{in}}(t)z_j^{\text{in}}(s)\rangle = \langle y_i^{\text{in}}(t)z_j^{\text{in}}(s)\rangle = 0$. Defining the vectors $\mathbf{x}_t = [q(t), p(t), x_d(t), y_d(t), x_r(t), y_r(t)]^T$ and $\mathbf{z}_t = [z_d(t), z_r(t)]$, Eqs. (3) and (4) can be rewritten in the compact form (1).

Contrary to the idealizing assumptions made above, many of the noise sources in an actual experiment are frequency dependent, here the laser amplitude and phase

noise. This needs to be taken into account by properly extending the state-space model. We incorporate three types of laser noise: (i) broadband laser noise originating from the laser itself, (ii) narrow-band Pound-Drever-Hall phase modulation in the resonant beam required for locking the laser to the cavity frequency, and (iii) narrow-band laser noise originating from the feedback loop of the laser lock. Each of these noise sources is experimentally characterized and is modeled independently to match the overall spectral characteristics [39]. Furthermore, we extend the state-space model to incorporate higher-order mechanical modes.

Measurements and innovations.—We use the recorded homodyne signals z_i as input to the Kalman filter for estimation of the optomechanical state, which is done off-line. Figure 2 shows a 2 μs trace of the detector signals (corresponding to 100 sample points), along with the optimal measurement prediction. The prediction shows excellent qualitative agreement with the measured data both in the weak ($g_d < \kappa$) and strong coupling regime ($g_d > \kappa$). Quantitatively, the validity of the estimation is assessed by the innovation sequence $\boldsymbol{\nu}_t = \mathbf{z}_t - C_t\hat{\mathbf{x}}_t$, i.e.,
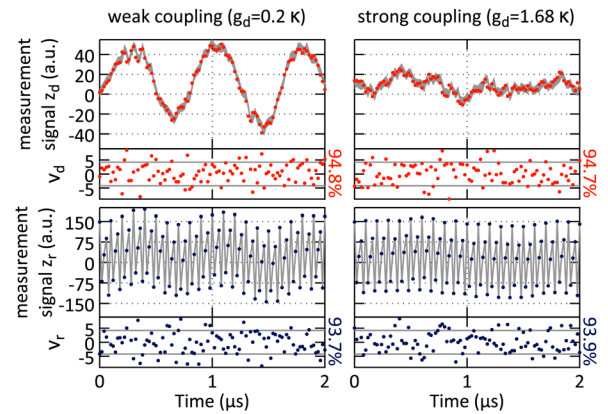

FIG. 2 (color online). Measurement signals and Kalman-filter predictions. Shown are the measurement signals of the two homodyne detections of the detuned and resonant beam, $z_d(t)$ (red dots) and $z_r(t)$ (blue dots), respectively, and their Kalman-filter predictions $\hat{z}_i(t)$ (gray line) both for the weak (left) and the strong coupling regime (right). Error bars of the prediction ($\pm 2\sigma$) are indicated by the width of the gray line. Kalman-filter innovations $\nu_i(t) = z_i(t) - \hat{z}_i(t)$ are plotted below each data set and demonstrate the accuracy of the implemented filter. To assess the performance of the filter, we calculate the fraction of normalized innovations that are contained in a two-sided 95% confidence region ($\pm 2\sigma$ indicated by the lines) of a zero-mean Gaussian distribution (beside each plot). The experiment was performed at room temperature with a micromechanical oscillator of $\omega_m = 2\pi \times 1.278$ MHz, $\gamma_m = 2\pi \times 265$ Hz, and optomechanical parameters $\kappa = 0.34\omega_m$, $g_0 = 2\pi \times 7.7$ Hz [39]. For both coupling strengths of the detuned beam ($\Delta_d = \omega_m$), we use $\varphi_d \approx 0$, $\Delta_r = 0$, $g_r = 0.2\kappa$, and $\varphi_r = \pi/2$. Note that the fast oscillation of $z_r(t)$, which is due to the 20 MHz Pound-Drever-Hall phase modulation for frequency locking, is taken into account by the Kalman filter.
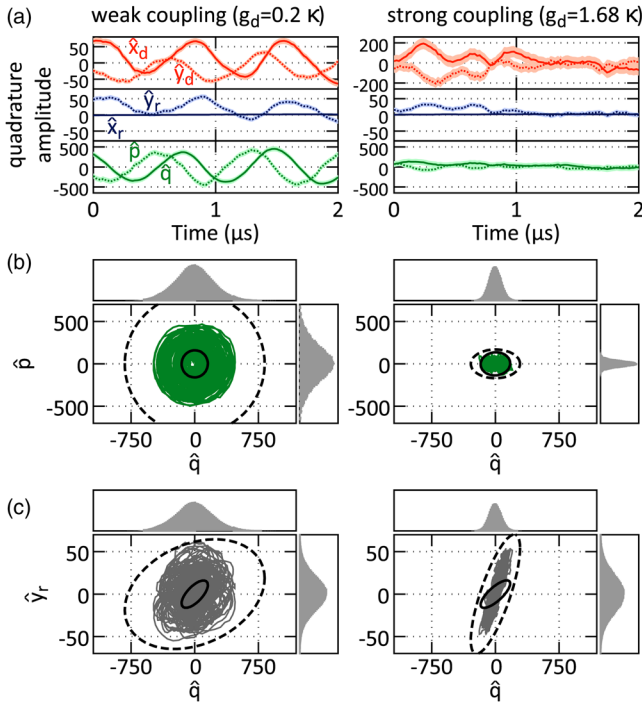
FIG. 3 (color online). Estimating optomechanical quadratures.
(a) Shown are Kalman-filter real-time estimates for the optical
amplitude ($\hat{x}_i$, straight line) and phase ($\hat{y}_i$, dashed line) quadrature
of the detuned (red, top) and resonant (blue, middle) beam along
with the mechanical position ($\hat{q}$, straight green line) and momen-
tum quadrature ($\hat{p}$, dashed green line) of the optomechanical
system for the weak and strong coupling regime. The mechanical
[(b), green line] and optomechanical [(c), gray line] phase-space
trajectories are estimated over a period of 100 μs. A histogram
along each quadrature is shown as a side panel and estimated over
10 ms. The uncertainty ellipse of the unconditional (conditional)
state is shown as dashed (straight) line. Note that the length of the
shown trajectory is not sufficient to adequately represent the state's
statistics. All units are given in terms of quadrature zero-point
fluctuations (zpf). For our experimental parameters $q_{\mathrm{zpf}} =
2.73 \times 10^{-16}$ m, $p_{\mathrm{zpf}} = 3.87 \times 10^{-19}$ kgm/s.

the difference between the predicted measurement $\hat{\mathbf{z}}_t =
C_t \hat{\mathbf{x}}_t$ of the Kalman filter and the actual measurement
outcome $\mathbf{z}_t$. For an optimally working filter, $\boldsymbol{\nu}_t$ must be a
Gaussian zero-mean white-noise process with a variance
given by $\mathbb{E}[\boldsymbol{\nu}_t \boldsymbol{\nu}_t^T] = C_t P_t C_t^T + V$. We use this fact to fine-
tune model parameters starting from their independently
determined values. The statistics of $\boldsymbol{\nu}_t$ of the resulting
Kalman filter closely matches these criteria, hence, dem-
onstrating the accuracy of the filter (Fig. 2; see, also,
Ref. [39] for further statistical analysis).

Estimation of optomechanical quadratures.—Kalman
filtering provides direct, real-time access both to the optical
intracavity quadratures and to the mechanical degree of
freedom in a cavity optomechanical system [Fig. 3(a)]. In the
weak coupling regime, the thermally driven mechanical
motion and its coupling to the optical intracavity fields is
visible. Clearly, the mechanical motion modulates both

quadratures x_d, y_d of the detuned beam [14], which couples
to the mirror via the optomechanical beam-splitter interac-
tion. The situation is different for the resonant beam, whose
amplitude quadrature x_r contains shot noise only, while its
phase quadrature y_r couples to the mechanical position.

The phase-space representation captures the essence of
Kalman filtering. The estimated mechanical quadratures
rotate in phase space [Fig. 3(b)]. Their probability distri-
bution along each mechanical quadrature is shown as
histograms beside each axis and demonstrates the
Gaussian nature of the micromirror motion. We compare
the uncertainty ellipse of the unconditional (dashed line)
and conditional (solid line) mechanical state, i.e., the area in
which we expect with 95% probability to find the mechani-
cal quadratures. For a purely thermal state, the area of the
unconditional ellipse is proportional to the thermal occu-
pation number $\bar{n}$. In the weak coupling regime, the
information provided by the measurement update leads
to a clear reduction in the uncertainty (a factor of 27 in
effective temperature), which is the optimal one for the
given coupling strength [52]. In the strong coupling regime,
laser cooling has already significantly diminished the
thermally induced uncertainty of the unconditional state.
In addition, the cavity dynamics introduces a notable
ellipticity in the phase-space distribution [53]. The condi-
tional state uncertainty is similar to the weak coupling
situation. This is because for technical reasons the signal
power at the homodyne detectors was kept constant for
both coupling strengths, which means that the stronger
detuned optical drive beam does not provide more infor-
mation on the system state.

Figure 3(c) shows real-time estimates of the optome-
chanical correlations between mechanical position and the
phase quadrature of the resonant beam. Analogous to the
mechanical phase space, the conditional state uncertainties
are strongly reduced, reflecting the real-time information
gain on the optomechanical correlations.

Conclusion.—We have successfully implemented
Kalman filtering for optimal state estimation of cavity
optomechanical systems. Its accuracy crucially relies on
an accurate state-space model of the specific experiment.
The applications of this method in the domain of opto-
mechanics are manifold. For example, Kalman filtering
enables mechanical feedback control in the quantum regime.
While in this work we operate the filter off-line, its real-time
application in the frequency range investigated here is
feasible using current field programmable gate array hard-
ware [39]. The optimality of the filter guarantees that the
reduction in conditional state uncertainty corresponds to the
maximal cooling one can achieve through active feedback at
this specific coupling strength. As a consequence, ground-
state cooling is readily achievable by combining Kalman
filtering with measurements in the strong cooperativity
regime [33]. This regime has been reached in current
experiments [21,54–56]. In our case, it requires cryogenic

cooling of the mechanical environment to 300 mK and quality factors above 10^6. As another example, mechanical sensing requires precise knowledge of the system dynamics in the absence of the external impetus, which is equivalent to the task of implementing the optimal estimator, i.e., the Kalman filter. The same is true for the task of characterizing or reconstructing an optomechanical quantum state (for example, in terms of entanglement), where the relevant information is often encoded in the covariance matrix P_t. One fascinating prospect there is the generation of entanglement of macroscopic test masses through measurement [29,57]. In summary, Kalman filtering adds a significant performance advantage for classical and quantum control of cavity optomechanical systems.

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[32] One can also adopt a quantum-optical interpretation of $\hat{\mathbf{x}}_t$. Formally integrating (and assuming vanishing initial conditions) gives $\hat{\mathbf{x}}_t = \int_{-\infty}^{t} \mathcal{K}(t,s) \mathbf{z}_s ds$ with an integral kernel $\mathcal{K}$ depending on A_t, C_t, and K_t. Thus, $\hat{\mathbf{x}}_t$ is formally equivalent to a (unnormalized) bosonic mode extracted from the output process $\mathbf{z}_t$. In a quantum-optical setting, this could be, for example, a temporal light mode extracted from the output light of a cavity.

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