Effective Field Theories from Soft Limits of Scattering Amplitudes

Clifford Cheung,[1] Karol Kampf,[2] Jiri Novotny,[2] and Jaroslav Trnka[1]

[1]*Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA*
[2]*Institute of Particle and Nuclear Physics, Charles University in Prague, Prague, Czech Republic*

We derive scalar effective field theories—Lagrangians, symmetries, and all—from on-shell scattering amplitudes constructed purely from Lorentz invariance, factorization, a fixed power counting order in derivatives, and a fixed order at which amplitudes vanish in the soft limit. These constraints leave free parameters in the amplitude which are the coupling constants of well-known theories: Nambu-Goldstone bosons, Dirac-Born-Infeld scalars, and Galilean internal shift symmetries. Moreover, soft limits imply conditions on the Noether current which can then be inverted to derive Lagrangians for each theory. We propose a natural classification of all scalar effective field theories according to two numbers which encode the derivative power counting and soft behavior of the corresponding amplitudes. In those cases where there is no consistent amplitude, the corresponding theory does not exist.

Introduction.—Infrared dynamics are inextricably linked to symmetry. For example, soft limits in gauge and gravity theories are fixed by conservation laws [1], while soft limits of pion amplitudes secretly encode underlying patterns of symmetry breaking [2]. Tacitly, symmetries are considered primary and the corresponding soft theorems secondary. In this letter we argue for precisely the opposite: by constructing scattering amplitudes directly and imposing various soft behaviors, we instead derive the theories and their symmetries.

The idea of building a theory from its scattering amplitudes rather than its Lagrangian is not new. Famously, tree amplitudes in gauge and gravity theories can be constructed solely from considerations of Lorentz invariance and factorization. The same is true of nonlinear σ models [3], albeit with the crucial and additional assumption of the so-called Adler zero [4], which describes the vanishing of pion scattering amplitudes in the soft limit.

The present work is a generalization of this prescription with the aim of enumerating all possible effective field theories of a massless scalar. We focus here on on-shell tree amplitudes in four dimensions, but our methods apply to diverse dimensions and loop integrands. In the soft limit of an external leg, $p \to 0$, the tree amplitude is

$$A(p) = O(p^\sigma), \tag{1}$$

where σ is a non-negative integer characterizing the soft limit degree. Larger values of σ imply cancellations in the amplitude enforced by relations among the coupling constants of the underlying theory, i.e., more symmetry.

A massless scalar has the schematic Lagrangian

$$\mathcal{L} = (\partial\phi)^2 \sum_{m,n=0}^{\infty} \lambda_{m,n} \partial^m \phi^n, \tag{2}$$

where m is even by Lorentz invariance [5]. In general, the soft limit will enforce cancellations among diagrams of different topologies. For example, an $n + 2$ particle amplitude includes diagrams with a single $\lambda_{m,n}$ vertex as well as diagrams with a single propagator connecting a $\lambda_{m',n'}$ vertex to a $\lambda_{m'',n''}$ vertex. By dimensional analysis, cancellations can only occur if $m = m' + m''$ and $n = n' + n''$, corresponding to all $\lambda_{m,n}$'s for which

$$\rho = m/n, \tag{3}$$

for a fixed non-negative rational number ρ characterizing a particular power counting order in derivatives. For fixed ρ, Eq. (2) takes the schematic form

$$\mathcal{L}_{(\rho)} = (\partial\phi)^2 F(\partial^m \phi^n), \tag{4}$$

for a general function F, where m and n are the smallest numbers satisfying Eq. (3). Some familiar examples are

$$\mathcal{L}_{(0)} = (\partial\phi)^2 F(\phi), \qquad \mathcal{L}_{(1)} = (\partial\phi)^2 F(\partial\phi), \tag{5}$$

corresponding to theories of free fields and Nambu-Goldstone bosons, respectively.

For a theory $\mathcal{L}_{(\rho)}$ we can impose the soft limit in Eq. (1) to constrain F, yielding a new theory $\mathcal{L}_{(\rho,\sigma)}$. Thus, all scalar effective theories can be classified by two numbers, (ρ, σ), which specify the derivative power counting of a theory together with the degree of its soft limits. With an explicit Lagrangian, it is straightforward to compute the scattering amplitude and its soft limit, but a more interesting exercise is the reverse: assume a value of (ρ, σ) and derive the corresponding theory.

To begin, we construct general *Ansätze* for on-shell tree amplitudes consistent with Lorentz invariance and factorization but restricted to a particular (ρ, σ) derivative power

counting and soft limit. This generates the span of all possible amplitudes describing massless scalars. For many values of (ρ, σ) there is no consistent scattering amplitude, so there is no corresponding theory. Even if a consistent amplitude exists, however, this may not be so interesting if the soft limit is obvious from counting the number of derivatives per field. By this logic a soft limit $\sigma \leq (m+2)/(n+2)$ is automatic, so the interesting case is in the opposite regime,

$$\sigma > \frac{\rho n + 2}{n + 2}, \qquad (6)$$

after plugging it into Eq. (3). Table I summarizes those theories which have enhanced soft limits which exceed the degree expected from naive derivative power counting. Also listed are the number of physical parameters which define each theory. Here Galileon$_{4,5}$ denotes the original Galileon theory on a basis where the three-point interaction vertex has been removed by a field redefinition, and Galileon$_4$ denotes Galileon$_{4,5}$ theory truncated to just the four-point interaction. For Galileon$_4$ we have strong evidence—up to 12-point amplitudes—for an intriguing $O(z^3)$ enhanced soft limit.

Afterwards, we show that fixing (ρ, σ) places constraints on the Noether currents which can be used to derive the Lagrangians for Dirac-Born-Infeld (DBI) and Galileon theories.

Amplitudes from Ansätze.—To begin, we construct an *Ansatz* for on-shell tree amplitudes constrained by Lorentz invariance, factorization, and a specified derivative power counting and soft limit degree, (ρ, σ). When an *Ansatz* exists, the corresponding theory can exist.

The on-shell three-point amplitude vanishes in any theory due to kinematics, so here we focus on the case where the leading nonzero on-shell amplitude is four point. An analogous discussion applies for theories in which the leading nonzero amplitude is higher point.

Definition of Ansätze.—Any scalar n-point on-shell scattering amplitude can be written in terms of the kinematical invariants

$$s_{ij} = (p_i + p_j)^2 = 2(p_i \cdot p_j), \qquad i, j \in \{1, ..., n\}, \quad (7)$$

which is a redundant basis. First of all, by momentum conservation we can always eliminate all dependence on

TABLE I. Theories with (ρ, σ) derivative power counting and soft limit degree that have enhanced soft limits.

(ρ, σ)	Theory	Parameters
$(0, \infty)$	Free	0
$(1,2)$	Dirac-Born-Infeld	1
$(2,2)$	Galileon$_{4,5}$	2
$(2,3)$	Galileon$_4$	1

the momentum of particle n, so we can restrict to s_{ij}, where $i, j \neq n$. Second, there is an additional constraint because particle n is on shell, so $\sum_{i,j\neq n} s_{ij} = 0$. Last of all, in four dimensions, five generic momenta are necessarily linearly dependent, leading to the so-called Gram-determinant relations. Since these are nonlinear constraints, a truly independent set of s_{ij} is difficult to compute analytically. Instead it is much simpler to use a redundant basis of kinematic invariants and mod out by the redundancy at the end of the calculation.

Locality of the underlying theory enforces stringent analyticity conditions on the tree amplitude, fixing it to be a rational function of momenta. This is required so that all nonanalyticities in the amplitude come from kinematic singularities corresponding to factorization channels. The general *Ansatz* for the $(n + 2)$-point amplitude in a theory with derivative power counting $\rho = m/n$ is then

$$A_{n+2} = \sum_\alpha c_\alpha^{(0)}(s_{\alpha_1} \ldots s_{\alpha_{m/2+1}}) + \sum_{\alpha,\beta} \frac{c_\alpha^{(1)}(s_{\alpha_1} \ldots s_{\alpha_{m/2+2}})}{s_\beta}$$

$$+ \sum_{\alpha,\beta} \frac{c_\alpha^{(2)}(s_{\alpha_1} \ldots s_{\alpha_{m/2+3}})}{s_{\beta_1} s_{\beta_2}} + \cdots, \qquad (8)$$

where α labels pairs of external legs that enter into the numerator factors and β labels factorization channels whose corresponding off-shell propagators are $s_\beta = \sum_{i,j\in\beta} s_{ij}$. Symmetries of the corresponding Feynman diagrams relate many coefficients $c_\alpha^{(k)}$, and moreover the *Ansatz* is kinematically redundant due to reasons mentioned above.

Definition of soft limit.—A priori, the soft limit of A_n is obtained by rescaling one of the external momenta by $p \to zp$ with $z \to 0$, but this procedure does not conserve total momentum. Instead, to compute the soft limit we apply a complex momentum shift to the external particles, chosen so as to conserve total momentum and maintain the on-shell conditions. The complex deformation is controlled by a number z that labels a one-parameter family of on-shell amplitudes $A_n(z)$ where $z \to 0$ corresponds to a soft limit $p \to zp$ and the deficit momenta are channeled into the remaining hard particles. While any number of legs may be shifted, we make the minimal choice of three, dubbed the "soft shift" in Ref. [6]. We then expand in powers of z,

$$A_n(z) = \sum_{s=0}^{\infty} A_{n,s} z^s, \qquad (9)$$

where we assume that the soft limit is nonsingular. To enforce the soft limit $A_n = O(z^\sigma)$, we solve for the coefficients $c_\alpha^{(k)}$ of the *Ansatz* in Eq. (8) subject to $A_{n,s} = 0$ for $s < \sigma$. As noted earlier, the s_{ij}'s satisfy complicated nonlinear constraints, so these equations cannot be solved in closed form. Instead, we evaluate A_n numerically many

times at arbitrary kinematical points and expand around $z \to 0$, yielding linear equations in $c_\alpha^{(k)}$ which are then easy to solve. We then plug the solution back into the *Ansatz*, and the number of independent parameters determines the number of physical parameters of the theory.

Results.—The analysis of three- and four-point amplitudes is simple for any ρ. The three-point amplitude vanishes by kinematics, while the four-point amplitude is nontrivial and has a soft limit fixed by the number of derivatives per field, independent of the explicit forms of operators. In particular, for generic ρ we have $m + 2 = 2\rho + 2$ derivatives and, therefore, the kinematical *Ansatz* is

$$A_4 = \sum_{a_1 a_2 a_3} c_{a_1 a_2 a_3} (s_{12})^{a_1} (s_{23})^{a_2} (s_{31})^{a_3}, \qquad (10)$$

for $a_1 + a_2 + a_3 = \rho + 1$. However, in the soft limit $s_{12}, s_{23}, s_{31} = O(z)$, so regardless of the particular derivative structure of the amplitude, $A_4 = O(z^{\rho+1})$. No further cancellations are possible, so we cannot obtain stronger behavior by relating the parameters. Next, we consider higher-point amplitudes for various values of ρ.

Case: $\rho = 0$.—The soft limit is ill defined because the Lagrangian secretly describes a free field theory. This is manifest after a well-chosen field redefinition, $\phi \to \phi'(\phi)$, which takes $\mathcal{L}_{(0)} = (\partial\phi)^2 F(\phi) = \partial\phi' \partial\phi'$. Hence, while off-shell Feynman diagrams are nontrivial, they all vanish on shell.

Note that this is not generally true if ϕ carries a flavor, which is why $O(z)$ behavior is possible in the nonlinear σ model [4]. Indeed, this is true even if you consider only flavor-stripped amplitudes [7].

Case: $0 < \rho < 1$.—Amplitudes do not vanish in the soft limit, so $A_n = O(1)$. This can be derived by contradiction. A vanishing soft limit requires that, for each leg, $A_n \to 0$ when $p \to 0$. Enforcing this on each leg sequentially and demanding a permutation invariant amplitude yields a unique *Ansatz*,

$$A_n = p_1^{\mu_1} p_2^{\mu_2} \dots p_n^{\mu_n} L_{\mu_1 \mu_2 \dots \mu_n}, \qquad (11)$$

where $L_{\mu_1 \mu_2 \dots \mu_n}$ is a completely symmetric tensor constructed from factors of momenta and the metric. This implies that the number of derivatives cannot be less than the number of fields, so $\rho \geq 1$. This can be easily understood from the symmetry point of view: the theory must be derivatively coupled to have a vanishing soft limit.

Case: $\rho = 1$.—The first nontrivial case for a single scalar is $\rho = 1$, for which $m = n$ and we have one derivative per field. If we want to impose $O(z)$ behavior in the soft limit, the theory must necessarily be derivatively coupled; i.e., the corresponding Lagrangian is $\mathcal{L}_{(1,1)} \sim \sum \lambda_{2n} (\partial\phi)^{2n}$. This simplifies the *Ansatz* for the amplitude (all labels must appear in s_{ij}). For example, for $n = 4$ and $n = 6$, we get

$$A_4 = c_4(s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}) \qquad (12)$$

$$A_6 = 2c_4^2 \left[\frac{\widetilde{s_{123}}\ \widetilde{s_{456}}}{s_{123}} + \cdots \right] + c_6(s_{12}s_{34}s_{56} + \cdots), \qquad (13)$$

where the ellipses denote the sum over all permutations and $s_{123} = s_{12} + s_{23} + s_{31}$, $\widetilde{s_{123}} = s_{12}s_{23} + s_{23}s_{31} + s_{31}s_{12}$.

Now we impose an enhanced soft limit by demanding that $A_{n,1} = 0$, so $A_n = O(z^2)$. The four-point case is trivial as $s_{12}, s_{23}, s_{31} \sim z$ in the soft limit, and $O(z^2)$ is trivially satisfied and there is no condition on c_4. At six point this is a highly nontrivial constraint which is satisfied if we set $c_6 = 2c_4^2$.

The same argument can be applied to each higher-point amplitude, so $A_8 = O(z^2)$ can be used to fix the new coupling coefficient c_8 in terms over lower order couplings. By induction it is then obvious that this infinite system of equations has, at most, one solution. Indeed, there is exactly one solution, and the corresponding c_{2n} conspires to be the series expansion of

$$\mathcal{L}_{(1,2)} = -\frac{1}{g}\sqrt{1 - g(\partial\phi)^2}, \qquad (14)$$

where $g = 2c_4$ and we ignore vacuum energy. This is the scalar part of the DBI action, which describes a fluctuation of a brane in an extra dimension. The hidden symmetry is a nonlinearly realized higher-dimensional Lorentz symmetry. Later on, we give an analytical derivation of the DBI Lagrangian from the soft limit.

Case: $\rho = 2$.—The inequality in Eq. (6) implies that $\sigma \geq 2$ is an enhanced soft limit. The general action has $2n - 2$ derivatives on n fields ϕ. The theory must have at least one derivative per field, but the remaining $n - 2$ derivatives can be distributed in various ways among fields, so schematically $\mathcal{L}_{(2,1)} \sim \sum_{n=2}^{\infty} F_n(\partial^{2n-2}\phi^n)$, where F_n denotes a collection of operators with free coefficients which have $2n - 2$ derivatives on n fields. We construct the *Ansatz* for the amplitude for a given n and impose the condition that $A_{n,1} = 0$, so $A_n = O(z^2)$. For $n = 4$ there are two independent kinematical structures,

$$A_4 = c_1(s_{12}^3 + s_{23}^3 + s_{31}^3) + c_2(s_{12}s_{23}s_{31}), \qquad (15)$$

whose behavior is $O(z^3)$ for arbitrary c_1 and c_2, as argued earlier. Going to higher points we find unique solutions for the constraints: $A_5 = O(z^2)$, $A_6 = O(z^3)$, $A_7 = O(z^2)$, while for $n = 8$ we get two solutions for $\sigma = 2$ and one solution for $\sigma = 3$. It is easy to see the amplitude is generated by the Lagrangian

$$\mathcal{L}_{(2,2)} = \lambda_4 \mathcal{O}_4 + \lambda_5 \mathcal{O}_5, \qquad (16)$$

where $\mathcal{O}_4 \sim \partial^6\phi^4$ and $\mathcal{O}_5 \sim \partial^8\phi^5$ are four- and five-point interaction vertices. Indeed, the derivative counting and structure is precisely that of the four- and five-point

interaction vertices of the four-dimensional Galileon theories studied in Ref. [8]. The Galileon Lagrangian exhibits a second order shift symmetry $\phi \rightarrow \phi + a + b_\mu x^\mu$ and has equations of motion that are second order in derivatives of ϕ. The missing three-point interaction can be eliminated via the Galileon duality (for a detailed discussion, see Ref. [9]), yielding just the four-and five-point interactions, which we denote by Galileon$_{4,5}$.

We have checked up to 12 particles in which the amplitudes derived from $\mathcal{O}_4$ alone yield $A_n = O(z^3)$, which suggests an even simpler theory,

$$\mathcal{L}_{(2,3)} = \lambda_4 \mathcal{O}_4, \tag{17}$$

which we will refer to as the Galileon$_4$ theory.

Case: $\rho > 2$.—We have done some partial analyses for $\rho = 3$ and $\rho = 4$ for $n = 5$ and for $\rho = 3$ for $n = 6$, and indeed there are unique amplitudes there with nontrivial soft-limit behavior, i.e., $A_5, A_6 = O(z^3)$ for $\rho = 3$ and $A_5 = O(z^4)$ for $\rho = 4$. It is very suggestive that these are exactly the theories found in Ref. [10], i.e., theories with higher shift symmetries.

Case: $1 < \rho = fractional$.—As discussed earlier, ρ is a non-negative rational number. Restricting to derivatively coupled theories, $\rho \geq 1$, so we should consider all theories with $\rho = m/n$ for integers m, n with $m \geq n$. For example, for $\rho = 3/2$ we have $\sigma \geq 2$, and the schematic Lagrangian is

$$\mathcal{L}_{(\rho,\sigma)} \sim (\partial\phi)^2 + (\partial^8\phi^6) + (\partial^{14}\phi^{10}) + \cdots \tag{18}$$

For this case we have checked to see that $O(z^2)$ soft behavior is impossible with $(\partial^8\phi^6)$ and first becomes possible with $(\partial^{14}\phi^{10})$. We have done this check for $\rho = \frac{3}{2}, \frac{4}{3}, \frac{6}{5}, \frac{8}{5}$ which rules out all theories with operators $n < 8$ for $\sigma = 2$.

Amplitudes from equations of motion.—Our analysis thus far does not prove the existence of theories: as such, a claim would require an accounting of an infinite number of amplitudes. However, we can finish the job by using the prescribed soft limits to derive the Lagrangians for these theories explicitly. The action for a massless scalar field is

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial\phi) = \int d^4x \frac{1}{2}(\partial\phi)^2 + S_{\text{int}}[\phi], \tag{19}$$

where S_{int} contains all nonlinear interactions. The equations of motion are

$$\Box\phi(x) = \frac{\delta S_{\text{int}}[\phi]}{\delta\phi(x)}. \tag{20}$$

Equations of motion are identically satisfied when evaluated inside a time-ordered product. Hence, Eq. (20) is valid when sandwiched between in and out states such that define the on-shell scattering amplitude $A = \langle f|i\rangle$. We thus obtain $\langle f|\delta S_{\text{int}}[\phi]/\delta\phi(x)|i\rangle = \langle f|\Box\phi(x)|i\rangle$. Transforming to the

Fourier conjugate field $\tilde{\phi}(p)$ where $p \equiv p_f - p_i$ and taking the on-shell limit, $p^2 \rightarrow 0$, we obtain the Lehmann-Symanzik-Zimmermann reduction formula for the amplitude with the scalar emitted with momentum p,

$$A(p) \equiv \langle f + \tilde{\phi}(p)|i\rangle = \lim_{p^2 \rightarrow 0} i\langle f|\frac{\delta S_{\text{int}}[\tilde{\phi}]}{\delta\tilde{\phi}(p)}|i\rangle. \tag{21}$$

Here $A(p)$ satisfies Eq. (1) in the soft limit if and only if

$$\langle f|\frac{\delta S_{\text{int}}[\phi]}{\delta\phi(x)}|i\rangle = \partial_{\mu_1}...\partial_{\mu_\sigma}\langle f|K^{\mu_1...\mu_\sigma}(x)|i\rangle, \tag{22}$$

for some local operator $K^{\mu_1...\mu_n}(x)$, in which case

$$A(p) = \lim_{p^2 \rightarrow 0} i^{\sigma+1} p_{\mu_1}...p_{\mu_\sigma}\langle f|\tilde{K}^{\mu_1...\mu_\sigma}(p)|i\rangle. \tag{23}$$

The assumption of locality is crucial: if the Fourier transformed operator $\tilde{K}^{\mu_1...\mu_n}(p)$ is singular as p goes to zero, then this will compensate for the $p_{\mu_1}...p_{\mu_\sigma}$ factors in the numerator. However, regularity of the operator at $p \rightarrow 0$ is, in principle, violated if the theory has cubic vertices, in which case the soft limit can generate collinear singularities which produce inverse powers of z.

The condition in Eq. (22) is satisfied provided

$$\frac{\delta S_{\text{int}}[\phi]}{\delta\phi(x)} = \partial_{\mu_1}...\partial_{\mu_\sigma}K^{\mu_1...\mu_\sigma}(x), \tag{24}$$

on the support of any equations which hold when evaluated inside the time-ordered product, i.e., when sandwiched between the in and out states. *A priori*, Eq. (24) can be true due to algebraic identities or conservation equations. Let us classify each theory in turn.

Case: $(\rho, \sigma) = (1, 1)$.—These theories have exactly one derivative per field, so the Lagrangian is $\mathcal{L}(X) = X/2 + \mathcal{L}_{\text{int}}(X) = \sum_n c_n X^n$, where $X = (\partial\phi)^2$. The Noether current is

$$J^\mu = \frac{\partial\mathcal{L}(X)}{\partial(\partial_\mu\phi)} = 2\mathcal{L}'(X)\partial^\mu\phi, \tag{25}$$

and the variation of the action yields

$$\frac{\delta S_{\text{int}}}{\delta\phi} = -\partial_\mu \frac{\partial\mathcal{L}_{\text{int}}(X)}{\partial(\partial_\mu\phi)} = \partial_\mu(\partial^\mu\phi - J^\mu), \tag{26}$$

which is of the form of Eq. (24), with $\sigma = 1$ for any c_n.

Case: $(\rho, \sigma) = (1, 2)$.—Extending to an enhanced soft limit $\sigma = 2$ implies additional constraints on $\mathcal{L}(X)$, so the c_n's are constrained. Plugging Eq. (26) into Eq. (22) for $\sigma = 2$ implies that

$$\langle f|J^\mu|i\rangle = \partial_\nu\langle f|L^{\mu\nu}|i\rangle. \tag{27}$$

However, $J^\mu = \partial_\nu L^{\mu\nu}$ cannot be true algebraically, simply because $L^{\mu\nu}$ involves $\partial\phi$, so $\partial_\nu L^{\mu\nu}$ involves $\partial\partial\phi$, which can

never match J^μ, which only involves $\partial\phi$. Consequently, we need a supplemental equation that holds when evaluated between in and out states. The natural candidate equation is conservation of the energy-momentum tensor,

$$T^{\mu\nu} = 2\mathcal{L}'(X)\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}(X). \qquad (28)$$

By derivative counting, there is a unique *Ansatz* for $L^{\mu\nu}$ for which $\partial_\nu L^{\mu\nu}$ does not involve $\partial\partial\phi$ when evaluated between in and out states: $L^{\mu\nu} = g\phi T^{\mu\nu}$ for some constant g. Plugging this into Eq. (27), we obtain

$$\langle f|J^\mu|i\rangle = \partial_\nu\langle f|L^{\mu\nu}|i\rangle = g\langle f|T^{\mu\nu}\partial_\nu\phi|i\rangle, \qquad (29)$$

where we have used $\partial_\nu T^{\mu\nu} = 0$ when evaluated between in and out states. This formula is automatically true if $J^\mu = gT^{\mu\nu}\partial_\nu\phi$ applies algebraically, which implies the differential equation

$$2\mathcal{L}'(X)/g = 2\mathcal{L}'(X)X - \mathcal{L}(X), \qquad (30)$$

whose solution is $\mathcal{L}(X) \propto \sqrt{1 - gX}$, which is precisely the DBI action for a single scalar field. In principle, there could be solutions which do not satisfy $J^\mu = gT^{\mu\nu}\partial_\nu\phi$ algebraically. However, our earlier scattering amplitude analysis found that there exists only one or zero theories with a $\sigma = 2$ soft limit, so the one and only solution is the DBI theory.

Case: $(\rho, \sigma) = (2, 2)$.—These theories are of the form $\mathcal{L} = \sum_n c_n \partial^{2(n-1)}\phi^n$. As before, $\sigma = 2$ implies the condition in Eq. (27). As shown in Ref. [11], this constraint is algebraically satisfied for the Noether current associated with the Galileon. A subtlety in this case is that the Galileon theory has a cubic vertex, in which case $L^{\mu\nu}$ can, in principle, not be regular at $p \to 0$. However, a field redefinition of the Galileon can be used to eliminate the cubic vertex [9], truncating down to Galileon$_{4,5}$. Note that in an explicit evaluation of amplitudes that shows the further restriction to only four-point interaction vertices, Galileon$_4$ satisfies stronger soft-limit behavior, $\sigma = 3$. We do not yet have a general proof for this.

Finally, we note that for $(\rho, \sigma) = (1, 2), (2, 2)$, one can construct a fully nonperturbative derivation of $O(z^2)$ scaling from symmetries without *a priori* knowledge of the Lagrangian. We present the full details in Ref. [12].

Conclusion.—In this Letter we have shown that scattering amplitudes can be used to derive and classify scalar effective field theories. Soft limits and derivative power counting uniquely fix the Lagrangian of the corresponding effective field theory, and we have derived DBI and Galileon theories as examples. This work is part of a more general program to construct and classify all possible effective field theories and symmetries, with the possibility of discovering new ones.

[1] S. Weinberg, Infrared photons and gravitons, Phys. Rev. **140**, B516 (1965).

[2] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, What is the simplest quantum field theory?, J. High Energy Phys. 09 (2010) 016.

[3] J. A. Cronin, Phenomenological model of strong and weak interactions in chiral $U(3) \times U(3)$, Phys. Rev. **161**, 1483 (1967); S. Weinberg, Dynamical Approach to Current Algebra, Phys. Rev. Lett. **18**, 188 (1967); S. Weinberg, Nonlinear realizations of chiral symmetry, Phys. Rev. **166**, 1568 (1968).

[4] L. Susskind and G. Frye, Algebraic aspects of pionic duality diagrams, Phys. Rev. D **1**, 1682 (1970); H. Osborn, Implications of Adler zeros for multipion processes, Lett. Nuovo Cimento **2**, 717 (1969); J. R. Ellis and B. Renner, On the relationship between chiral and dual models, Nucl. Phys. **B21**, 205 (1970).

[5] For Lorentz invariants built from Levi-Civita tensors, any odd number of insertions vanishes by Bose symmetry while any even number of insertions simplifies to a product of metrics. Thus there are no parity-violating theories of a single scalar.

[6] C. Cheung, C.-H. Shen, and J. Trnka, Simple recursion relations for general field theories, arXiv:1502.05057.

[7] K. Kampf, J. Novotny, and J. Trnka, Recursion relations for tree-level amplitudes in the $SU(N)$ nonlinear sigma model, Phys. Rev. D **87**, 081701 (2013).

[8] G. R. Dvali, G. Gabadadze, and M. Porrati, 4-D gravity on a brane in 5-D Minkowski space, Phys. Lett. B **485**, 208 (2000); A. Nicolis, R. Rattazzi, and E. Trincherini, Galileon as a local modification of gravity, Phys. Rev. D **79**, 064036 (2009).

[9] K. Kampf and J. Novotny, Unification of Galileon dualities, J. High Energy Phys. 10 (2014) 006.

[10] K. Hinterbichler and A. Joyce, Goldstones with extended shift symmetries, Int. J. Mod. Phys. D **23**, 1443001 (2014); T. Griffin, K. T. Grosvenor, P. Horava, and Z. Yan, Scalar field theories with polynomial shift symmetries, arXiv:1412.1046.

[11] K. Hinterbichler, M. Trodden, and D. Wesley, Multifield Galileons and higher codimension branes, Phys. Rev. D **82**, 124018 (2010).

[12] C. Cheung, K. Kampf, J. Novotny, and J. Trnka (to be published).