Angular Statistics of Lagrangian Trajectories in Turbulence

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The angle between subsequent particle displacement increments is evaluated as a function of the time lag in isotropic turbulence. It is shown that the evolution of this angle contains two well-defined power laws, reflecting the multiscale dynamics of high-Reynolds number turbulence. The probability density function of the directional change is shown to be self-similar and well approximated by an analytically derived model assuming Gaussianity and independence of the velocity and the Lagrangian acceleration.

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Advances in experimental devices and numerical simulations over the last two decades have opened the way to a Lagrangian characterization of turbulent flows [1–3]. The structural description of the statistical dynamics of turbulence has thereby shifted from the investigation of spatial correlations of instantaneous velocity fields to the study of temporal correlations along fluid particle trajectories. In the Lagrangian reference frame, the spatiotemporal complexity of turbulence manifests itself through the spiraling chaotic motion of fluid particles, changing direction at every time scale. This directional change of Lagrangian tracers, as a function of the time lag between two observations, is the subject of the present Letter. Instantaneous measures of the curvature in turbulence have been investigated in the past ten years for academic turbulent flows, both in three [4–7] and in two space dimensions [8,9]. Curvature is dominated by the small-scale structures and contains only little information on the multiscale dynamics of turbulent flows. Multiscale dynamics can be measured by Lagrangian structure functions [1,3], but those do not contain any direct information on the curvature of the trajectories.

A time scale dependent measure which is related to the curvature was only recently introduced by Burov *et al.* [10]. More precisely, in this last work, the directional change of a particle was introduced, and the characteristics of this new measure were investigated in various types of random walks. In the present Letter, we will show how this measure can characterize the time correlation of the direction of a fluid particle in a turbulent flow. In particular, we will show how the multiscale character of a turbulent flow can be revealed by considering the time lag dependence of the directional change.

We define the Lagrangian spatial increment as

$$\delta\boldsymbol{X}(\boldsymbol{x}_0,t,\tau)=\boldsymbol{X}(\boldsymbol{x}_0,t)-\boldsymbol{X}(\boldsymbol{x}_0,t-\tau), \qquad (1)$$

where $\boldsymbol{X}(\boldsymbol{x}_0,t)$ is the position of a fluid particle at time t, passing through point $\boldsymbol{x}_0$ at the reference time $t=t_0$ and advected by a velocity field $\mathbf{u}$, i.e., $d\boldsymbol{X}/dt=\mathbf{u}$. The cosine of the angle $\Theta(t,\tau)$ between subsequent particle increments, introduced in [10], is

$$\cos(\Theta(t,\tau))=\frac{\delta\boldsymbol{X}(\boldsymbol{x}_0,t,\tau)\cdot\delta\boldsymbol{X}(\boldsymbol{x}_0,t+\tau,\tau)}{|\delta\boldsymbol{X}(\boldsymbol{x}_0,t,\tau)||\delta\boldsymbol{X}(\boldsymbol{x}_0,t+\tau,\tau)|}. \qquad (2)$$

The angle is illustrated in Fig. 1 (top). Rather than considering its instantaneous evolution, its averaged absolute value is of particular interest in an isotropic random

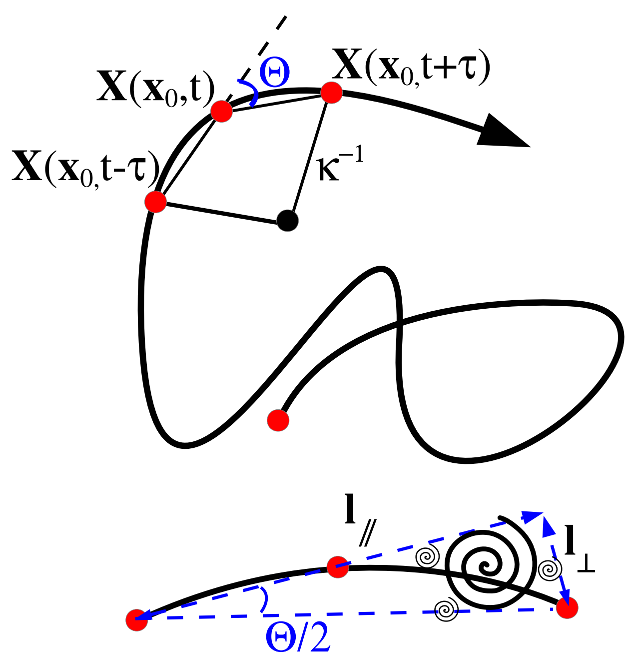

FIG. 1 (color online). Top: definition of the angle between subsequent Lagrangian particle increments. Bottom: short time evolution and definition of the length scales $l_\perp$ and $l_\parallel$. These lengths are defined by fitting a right triangle through the three measurement points, with its hypotenuse passing through $\boldsymbol{X}(\boldsymbol{x}_0,t-\tau)$ and $\boldsymbol{X}(\boldsymbol{x}_0,t+\tau)$. The length of the leg that passes through $\boldsymbol{X}(\boldsymbol{x}_0,t)$ and $\boldsymbol{X}(\boldsymbol{x}_0,t-\tau)$ defines $l_\parallel$ and the length of the leg that passes through $\boldsymbol{X}(\boldsymbol{x}_0,t+\tau)$ defines $l_\perp$.

velocity field. The ensemble average will be denoted in the following by

$$\theta(\tau) \equiv \langle|\Theta(t,\tau)|\rangle. \quad (3)$$

We omitted the time dependence since we will consider statistically stationary flow in the following. For short time lags, $\theta(\tau)$ should be close to zero, whereas for times long compared to the correlation time associated with the spiraling motion $\theta(\tau)$ should tend to $\pi/2$ by symmetry.

For short times, the instantaneous angle $\Theta(\tau, t)$ is related to the curvature κ [see Fig. 1 (top)] by the relation

$$\kappa(t) = \lim_{\tau\to 0} \frac{|\Theta(t,\tau)|}{2\tau\|\boldsymbol{u}(t)\|}, \quad (4)$$

with $\boldsymbol{u}$ being the velocity. How the angle varies in between the short and long-time limits is the main subject of the present Letter, and we will show that the dependence of $\theta(\tau)$ on the time lag contains the signature of the multiscale dynamics of a turbulent flow.

The database used to investigate the behavior of $\theta(\tau)$ is described in [11,12]. The simulation was carried out using standard pseudospectral techniques, following 8×10^6 fluid particles in a statistically stationary isotropic turbulent flow during 5.8 integral time scales in a periodic cube of dimension 2π. The resolution is 1024^3 gridpoints. The integral time scale is 2.1 and the Kolmogorov time scale $\tau_K = (\nu/\epsilon)^{1/2} = 0.036$, where $\epsilon = 0.31$ is the mean dissipation rate and $\nu = 4\times10^{-4}$ the kinematic viscosity. The Lagrangian integral time scale is of the order of the Eulerian integral timescale. The Taylor-scale Reynolds number is $R_\lambda = 225$.

Figure 2 shows $\theta(\tau)$ in double-logarithmic representation. The angle increases monotonically from zero to $\pi/2$, and this latter value is approached for values of τ of the order of the Lagrangian integral timescale. Two power laws

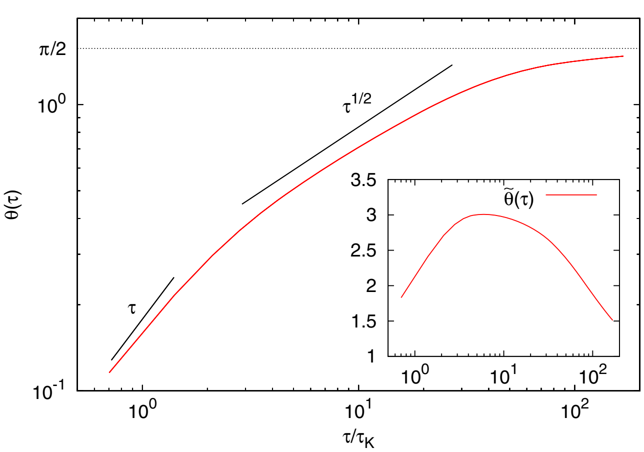

FIG. 2 (color online). The average angle θ as a function of the time lag τ normalized by the Kolmogorov time scale τ_K. In the inset, the compensated angle, $\tilde{\theta}(\tau) \equiv \theta(\tau)\sigma_u/(\epsilon\tau)^{1/2}$, is plotted.

can be identified in this graph, with a crossover around twice the Kolmogorov time scale. The origin of these power laws will now be elucidated.

For our phenomenological explanation, we consider high-Reynolds-number isotropic turbulence, containing flow structures on a wide range of different scales. We consider short time lags $\tau \ll T$, where T is the Lagrangian integral time scale of the flow. In this limit, the angle $\Theta(t,\tau)$ can be approximated using a Taylor expansion by,

$$\frac{l_\perp}{l_\parallel} \approx |\tan(\Theta/2)| \approx |\Theta/2|. \quad (5)$$

The definitions of $l_\perp$ and $l_\parallel$ are given in Fig. 1 (bottom) and correspond to the absolute values of the distance traveled parallel with, and perpendicular to, the initial displacement increment, respectively, over a time interval 2τ. The values of $l_\parallel$ and $l_\perp$ can be estimated, again using a Taylor expansion, to be

$$l_\parallel \approx 2U(t,\tau)\tau, \qquad l_\perp \approx 2\tau^2 a_\perp(t,\tau), \quad (6)$$

with $U(t,\tau)$ and $a_\perp(t,\tau)$ the absolute values of the velocity and the acceleration perpendicular to the velocity, respectively, coarse grained over a time τ along the fluid particle trajectory. Without loss of generality, we will write $U(t,\tau)$ and $a_\perp(t,\tau)$ as

$$U(t,\tau) = \sigma_u(\tau)\xi_u(t,\tau), \qquad a_\perp(t,\tau) = \sigma_a(\tau)\xi_a(t,\tau), \quad (7)$$

where $\sigma_u^2(\tau)$ and $\sigma_a^2(\tau)$ are the variance of the coarse-grained velocity and perpendicular acceleration, respectively. The quantities $\xi_u(t,\tau)$ and $\xi_a(t,\tau)$ are positive random variables with unit mean value and unit mean variance. We thereby obtain,

$$|\Theta(t,\tau)| \approx 2\tau\frac{\sigma_a(\tau)\xi_a(t,\tau)}{\sigma_u(\tau)\xi_u(t,\tau)}. \quad (8)$$

We assume the velocity and the acceleration *uncorrelated*, a reasonable assumption at very high Reynolds numbers, as long as $\tau \ll T$. Without coarse graining, this assumption was also used in Ref. [5] to model the curvature in isotropic turbulence. Using this assumption, we find

$$\theta(\tau) \approx 2\tau\frac{\sigma_a(\tau)}{\sigma_u(\tau)}. \quad (9)$$

Since for $\tau \ll T$, the velocity is roughly constant over the time interval, $\sigma_u(\tau) \approx U_{\rm rms}$. However, $a_\perp$ is dominantly determined by the small scales and fluctuates rapidly. Only for τ small with respect to the smallest Lagrangian time scale, the Kolmogorov scale, can we consider $\sigma_a(\tau) \approx (a_\perp)_{\rm rms}$, i.e., independent of τ. For these very short time scales, we have thus

$$\theta(\tau) \approx 2\tau\frac{\sigma_a}{\sigma_u} \quad \text{for } \tau \ll \tau_K, \quad (10)$$

where σ_a and σ_u are the total rms perpendicular acceleration and velocity, respectively. The linear relation between $\theta(\tau)$ and τ is well observed in Fig. 2. We can further express this in terms of quantities which are easy to determine experimentally. Assuming classical scaling [13], the acceleration variance for sufficiently high Reynolds number is given by the relation

$$\sigma_a^2 = \frac{\epsilon^{3/2}}{\nu^{1/2}} f(R_\lambda), \tag{11}$$

where $f(R_\lambda)$ is constant in the absence of intermittency. Taking into account intermittency corrections [14–16], the function f is well described by the expression

$$f(R_\lambda) = 2.5 R_\lambda^{0.25} + 0.8 R_\lambda^{0.11}. \tag{12}$$

Equations (10) and (11) then yield,

$$\theta(\tau) \sim \frac{\tau}{T} R_\lambda^{1/2} f(R_\lambda)^{1/2}, \quad \text{for } \tau \ll \tau_K. \tag{13}$$

This relation could be checked by varying the Reynolds numbers in experiments or simulations.

At time scales larger than τ_K, but smaller than T, i.e., in the inertial interval, the above approximations to obtain Eq. (9) are still valid. However, the subsequent approximation, that $\sigma_a(\tau)$ is independent of τ is not valid anymore. Indeed, the orientation of the acceleration fluctuates rapidly in time [17], on a time scale of the order of τ_K. Coarse graining the acceleration over an interval $\tau > \tau_K$, the influence of the more rapidly fluctuating scales is filtered out. Indeed, even if their contribution to the rms acceleration is dominant, if the coarse graining is performed before considering the norm, positive and negative contributions will cancel each other. The remaining variance will be predominantly caused by scales with a time scale larger than, or comparable to τ. Following classical Kolmogorov phenomenology [18,19], the acceleration induced by inertial range structures with typical time scale τ will be of the order

$$\sigma_a(\tau) \sim (\epsilon/\tau)^{1/2}. \tag{14}$$

This estimate is obtained by neglecting the viscous contribution to the acceleration, a reasonable assumption even near the dissipation range scales [15], and realizing that the acceleration is due to pressure forces, which satisfy, at inertial range scales, to a good approximation Kolmogorov scaling [20,21]. Indeed, intermittency does not seem to affect the scaling of the pressure spectrum significantly, but it does change the prefactor [22].

The reciprocal dependence of the acceleration variance on τ in Eq. (14) illustrates that the smallest scales are most efficient in accelerating the fluid particles. After the influence of the scales smaller than τ is removed by the coarse graining, the remaining dominant contribution is caused by the smallest scales still present, i.e., with time scale τ. It is therefore those scales, with correlation-time τ which will deviate particles from their trajectory over a length scale of the order of the correlation length of the structures. The scale of such inertial range eddies with a correlation time τ is proportional to

$$l(\tau) \sim \tau^{3/2} \epsilon^{1/2}. \tag{15}$$

This phenomenological picture is illustrated in Fig. 1 bottom, where it can be understood intuitively that scales of the size $l \ll l(\tau)$ are too small to efficiently contribute to a perpendicular displacement averaged over a time interval τ.

Combining Eqs. (14) and (8) we obtain in the inertial range

$$\theta(\tau) \sim \tau^{1/2} \frac{\epsilon^{1/2}}{\sigma_u} \sim \left(\frac{\tau}{T}\right)^{1/2}, \quad \text{for } \tau_K \ll \tau \ll T. \tag{16}$$

Again, this scaling is observable in Fig. 2, even though the power law is less well present than in the dissipation range. This is better appreciated by considering the compensated angle, $\tilde{\theta}(\tau) \equiv \theta(\tau)\sigma_u/(\epsilon\tau)^{1/2}$, plotted in the inset of the figure. The slow emergence of inertial ranges with the Reynolds number in Lagrangian statistics is fairly common [23], and is was recently even argued that they might be nonexistent [24]. In the present case, the emergence of a plateau is strongly suggestive. This might be because the inertial range scaling of the mean-angle $\theta(\tau)$ is not directly related to the Lagrangian structure functions. Indeed, the scaling is induced by considering the coarse-grained Lagrangian acceleration, a quantity of which the scaling is related to that of the Eulerian pressure gradient. The wave number spectrum of the pressure fluctuations, and thereby of its gradient, is in the inertial range, as mentioned above, rather well described by Kolmogorov scaling.

The above arguments and results considered the average value $\theta(\tau)$ only. Further information, in particular, on higher order moments, is contained in the probability density function (PDF) of the instantaneous angle and its evolution with τ. Those PDFs of the angle $\Theta(t, \tau)$ and its cosine are shown in Fig. 3. It is observed that the $P_\tau(\Theta)$ for small τ consists of a peak near zero, whereas for long times, a symmetric distribution between 0 and π is obtained. This latter distribution corresponds to the distribution between two randomly chosen vectors in three dimensions. Its distribution is given by $P_\infty(\Theta) = \sin(\Theta)/2$. The distribution of the cosine of the angle between two random vectors is thus equidistributed so that $P_\infty(\cos(\Theta)) = 1/2$. It is observed that these two long-time distributions are approached for long times in Fig. 3. We will now show how we can predict the short-time, small $\Theta(\tau, t)$ behavior of the PDFs. In particular, we will consider the PDF of $1 - \cos(\Theta)$. Since the cosine of small deviations in Θ gives values near unity, $1 - \cos(\Theta)$ directly measures the

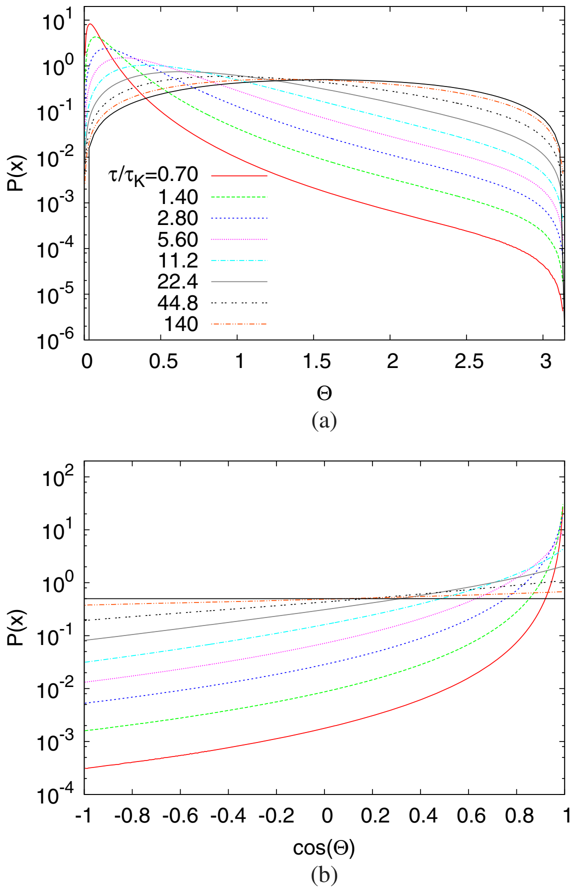

FIG. 3 (color online). PDFs of (a) Θ and (b) $\cos(\Theta)$ for different time lags. Solid black lines indicate the long time asymptotic form of the PDFs.

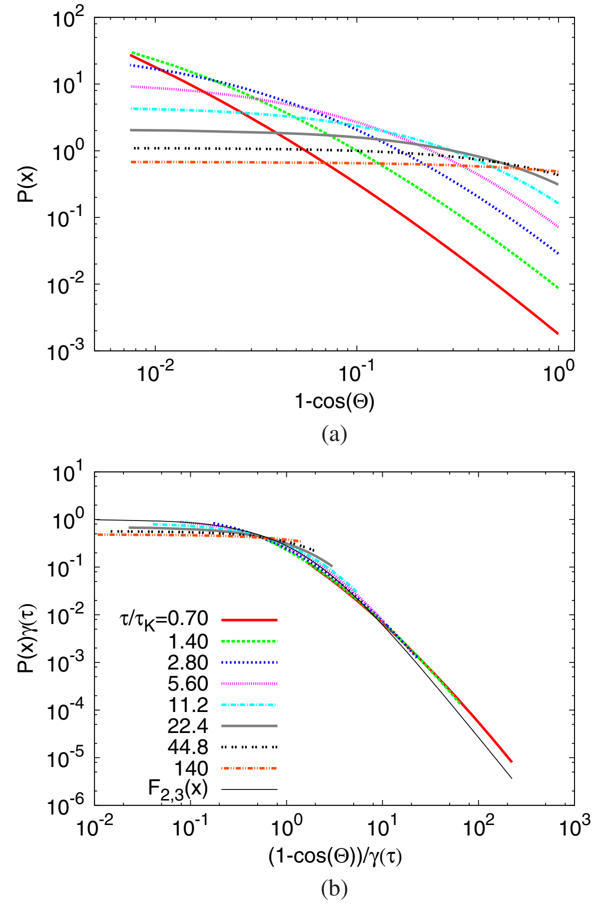

FIG. 4 (color online). (a) PDFs of $1 - \cos(\Theta)$. (b) The same PDFs normalized and compared to the analytical prediction.

magnitude of the directional change. In addition, it is easy to compare the PDF of this quantity to the long-time limit consisting of a straight line. The PDFs of $1 - \cos(\Theta)$ are shown in Fig. 4(a) in double logarithmic representation. To explain their shape and their evolution with τ, we use a Taylor expansion for small Θ,

$$1 - \cos\left(\Theta(t,\tau)\right) \approx \frac{1}{2}\Theta(t,\tau)^2 \approx 2\tau^2 \frac{\sigma_a^2(\tau)\xi_a^2(t,\tau)}{\sigma_u^2(\tau)\xi_u^2(t,\tau)}, \quad (17)$$

where we used Eq. (8). If we assume $\boldsymbol{a}_\perp$ to satisfy a Gaussian distribution, which is only a good approximation for the core of the PDF, and if we further assume $\boldsymbol{u}$ to be uncorrelated with $\boldsymbol{a}_\perp$, and also multivariate Gaussian [5], then both ξ_a^2 and ξ_u^2 satisfy a χ-squared distribution. For a given velocity vector in $3D$, having 3 degrees of freedom, the perpendicular acceleration is confined to the plane perpendicular to the velocity and is a 2-component vector. The ratio of two properly normalized independent χ^2-distributed quantities with n, m degrees of freedom, respectively, is given by an $F_{n,m}$ Fischer distribution. We expect the PDF of $1 - \cos(\Theta)$ therefore to be given by an $F_{2,3}$ distribution. More precisely,

$$\gamma(\tau)P_{1-\cos(\Theta(t,\tau))}[x/\gamma(\tau)] = F_{2,3}(x), \quad (18)$$

where $\gamma(\tau) = \theta(\tau)^2/3$ and $\theta(\tau)$ is shown in Fig. 2. It is observed in Fig. 4(b) that the agreement with the prediction is fairly satisfactory, considering the assumptions we made in the derivation of the shape of the PDF and the fact that no adjustable parameters were used to fit the PDF to the F distribution.

The results obtained in the present investigation show that the time series of the Lagrangian position can reveal the inertial range structure of turbulence through the time lag dependence of the quantity $\theta(\tau)$. In particular, we show how Kolmogorov's inertial range theory is linked to the angular statistics of Lagrangian fluid particle trajectories.

The present framework will allow experimentalists to verify the scaling of Lagrangian statistics in very-high-Reynolds numbers flows, even if the measurement techniques are not sufficiently rapid to resolve down to the Kolmogorov scale. Indeed, no measurements of the instantaneous velocity or acceleration are needed, only Lagrangian position measurements sufficiently sampled to resolve the inertial range time scales. A further come out of this investigation are the scale-dependent measures for the mean-angle and the probability density functions, which will allow us to more accurately model the topology of Lagrangian trajectories in dispersion models.

The measure we investigated in the foregoing allows a different approach to simultaneously characterize the multiscale character of turbulence and the curvature of Lagrangian fluid particle trajectories. In this light, an interesting perspective is to clarify the link between the current work and the results obtained using the recently introduced longitudinal and transversal Lagrangian structure functions [25].

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