Entropic Tests of Multipartite Nonlocality and State-Independent Contextuality

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(Received 27 February 2015; published 22 May 2015)

We introduce a multipartite extension of an information-theoretic distance introduced by Zurek [Nature (London) 341, 119 (1989)]. We use it to develop a new tool for studying quantum correlations from an information-theoretic perspective. In particular, we derive entropic tests of multipartite nonlocality for three qubits and for an arbitrary even number of qubits as well as a test of state-independent contextuality. In addition, we rederive the tripartite Mermin inequality and a state-independent noncontextuality inequality by Cabello [Phys. Rev. Lett. 101, 210401 (2008)]. This suggests that the information-theoretic distance approach to multipartite nonlocality and state-independent contextuality can provide a more general treatment of nonclassical correlations than the orthodox approach based on correlation functions.

DOI: 10.1103/PhysRevLett.114.200401 PACS numbers: 03.65.Ud, 03.65.Ta, 03.67.Mn

Introduction.—In classical information theory, if some binary property A is correlated with B', B' with A', and A' with B, then A must be correlated with B. This is not necessarily true in nonclassical information theories where correlations can be nontransitive. For instance, A can be anticorrelated with B [1] (see Fig. 1). If one looks only at the outcomes of random variables, the classical and nonclassical scenarios are dramatically different; however, from the entropic point of view they do not differ at all [2]. More precisely, the Shannon entropies $H(A)$, $H(B)$, and $H(AB)$, where $H(A) = -\sum_a P(A=a)\log_2 P(A=a)$, are the same regardless of whether the system is classical or not.

In order to detect nonclassicality via an entropic test, one should look for other types of nonclassical correlations [see Fig. 1(b), right]. These can be found by either looking for a different set of measurements [3] or by a postprocessing of a measured data, e.g., mixing of nonclassical and classical distributions [2].

The entropic tests of the bipartite nonlocality were introduced in the late 1980s [3] and further investigated recently in Ref. [4]. These tests were extended to include the state-dependent contextuality [5–7], which is a more general concept than nonlocality since it studies systems that are not necessarily spatially separated. However, the entropic approach to the state-independent contextuality has not been proposed before.

We use the previously developed information-theoretic distance approach to nonclassical correlations [8–10] and propose a new multipartite distance that can be applied to binary ± 1 measurements. This provides a powerful tool for studying the distinction between quantum and classical correlations. We quantify multipartite correlations in terms of Shannon entropy and derive an entropic tripartite inequality. This inequality is consistent with a similar one in Ref. [4], and its structure resembles the tripartite Mermin inequality [11]. Next, we derive an entropic inequality to test the multipartite nonlocality of an arbitrary even number of qubits. Finally, we derive an entropic inequality to test the state-independent contextuality, which resembles the correlation-based inequality by Cabello [12]. All these inequalities are satisfied by correlations for which the information-theoretic distance is properly defined. This is true for local realistic and noncontextual systems, but not for quantum systems.

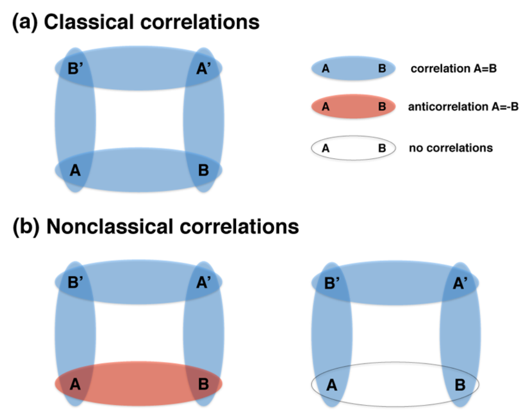

FIG. 1 (color online). Bipartite correlations between two observers. Each local measurement is maximally random, i.e., $P(A=1) = P(A=-1) = 1/2$, etc. Classical correlations (a) are transitive: if $A = B'$, $B' = A'$, and $A' = B$, then $A = B$. However, nonclassical correlations can be nontransitive: $A = -B$ [(b), left], or no correlations $H(AB) = H(A) + H(B)$ [(b), right]. Such extreme nonclassical bipartite correlations cannot be observed between quantum systems.

Multipartite distance.—The entropy-based information-theoretic distance was originally proposed by Zurek [13]. The essence of distance is to measure how far apart two points are. Points can be represented by coordinates in a Cartesian space or by more abstract objects, such as functions or operators. In the information-theoretic framework these objects are two random variables—in our case, two jointly measurable observables. Can we extend the notion of distance to more than two points? We show that this can be done for binary, jointly measurable random variables. For a similar approach, see Ref. [14].

Consider the following function defined for binary observables A and B:

$$d(A,B) = H(A\cdot B). \tag{1}$$

The outcome of $A\cdot B$ is the product ab.

The function in Eq. (1) satisfies distance properties: (i) non-negativity, $d(A,B)\geq 0$, because $H(X)\geq 0$ and it equals zero only if the entropy of $A\cdot B$ is zero, i.e., in case A and B are maximally correlated or anticorrelated, (ii) symmetry, $d(A,B)=d(B,A)$, and (iii) triangle inequality, $H(A\cdot B)\leq H(B\cdot C)+H(A\cdot C)$.

The triangle inequality is satisfied because $H(A\cdot B|A\cdot C, B\cdot C)=0$; i.e., if the outcomes of the two measurements $A\cdot C$ and $B\cdot C$ are known, then the outcome of $A\cdot B$ is the product of the two outcomes and is therefore known. More precisely, $H(A\cdot B)\leq H(A\cdot B, B\cdot C, A\cdot C) = H(A\cdot B|A\cdot C, B\cdot C) + H(B\cdot C, A\cdot C) = H(B\cdot C, A\cdot C)\leq H(B\cdot C)+H(A\cdot C)$. We used $H(AB)=H(A|B)+H(B)$, $H(AB)\leq H(A)+H(B)$, and $H(A)\leq H(AB)$.

Equation (1) can be extended to multipartite measurements: for a set of binary ± 1 variables $\{A_1, A_2, \ldots, A_n\}$, one can define

$$\delta(A_1, A_2, \ldots, A_n) = H(A_1\cdot A_2\cdot\cdots\cdot A_n). \tag{2}$$

The function δ is non-negative, symmetric, and associative:

$$\begin{aligned}\delta(A_1,\ldots,A_k,A_{k+1},\ldots,A_n)\\ = \delta((A_1\cdot\ldots\cdot A_k),(A_{k+1}\cdot\ldots\cdot A_n)).\end{aligned} \tag{3}$$

Because of the symmetry, any two A_i can be associated. Moreover, the associativity also implies that δ obeys the following version of the triangle inequality:

$$\begin{aligned}&\delta(A_1,\ldots,A_k,A_{k+1},\ldots,A_n)\\ &= \delta((A_1\cdot\ldots\cdot A_k),(A_{k+1}\cdot\cdots\cdot A_n))\\ &\leq \delta((A_1\cdot\cdots\cdot A_k)\cdot(B_1\cdot\cdots\cdot B_m))\\ &\quad + \delta((B_1\cdot\cdots\cdot B_m)\cdot(A_{k+1}\cdot\cdots\cdot A_n))\\ &= \delta(A_1,\ldots,A_k,B_1,\ldots,B_m)\\ &\quad + \delta(B_1,\ldots,B_m,A_{k+1},\ldots,A_n).\end{aligned} \tag{4}$$

In Ref. [14], Vitanyi considered the following quantity,

$$E_{\max}(X) = \max_{x\in X} K(X|x), \tag{5}$$

where X is a set, x are its elements, and K stands for Kolmogorov complexity. However, $E_{\max}(X)$ with K replaced by the Shannon entropy cannot be used to detect the difference between classical and nonclassical correlations. This motivated us to look for Eq. (2).

Tripartite information-theoretic Bell inequality.—Let us examine the properties of Eq. (2) for tripartite measurements. We have

$$\begin{aligned}&\delta(A_1,B_1,C_1)\\ &\leq d(A_1,(B_2.C_2)) + d((B_2.C_2),(B_1.C_1))\\ &= d(A_1,B_2.C_2) + \delta(B_2,C_1,B_1,C_2)\\ &\leq \delta(A_1,B_2,C_2) + d(A_2,B_2\cdot C_1) + d(A_2,B_1\cdot C_2)\\ &= \delta(A_1,B_2,C_2) + \delta(A_2,B_2,C_1) + \delta(A_2,B_1,C_2),\end{aligned} \tag{6}$$

where A_i, B_j, and C_k $(i,j,k=1,2)$ are measurements of Alice, Bob, and Charlie, respectively. This inequality is similar to the one in Ref. [4].

The inequality (6) was derived using the classical properties of Shannon entropy; therefore, it must hold in any theory that obeys them. In particular, all local realistic theories have a joint probability distribution for all observables [15] and their joint entropy. However, Eq. (6) is violated by quantum mechanics (see Fig. 2).

Let us consider a three-qubit system in the Greenberger-Horne-Zeilinger (GHZ) state $|\mathrm{GHZ}\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$ shared between Alice, Bob, and Charlie. Each of them performs one of the two possible local ± 1 measurements on their subsystem: $A_1, A_2, B_1, \ldots$ We choose $\delta(A_i,B_j,C_k) = H(A_i\otimes B_j\otimes C_k)$ (for $i,j,k=1,2$) and obtain

$$\begin{aligned}H(A_1\otimes B_1\otimes C_1) &\leq H(A_1\otimes B_2\otimes C_2)\\ &\quad + H(A_2\otimes B_1\otimes C_2) + H(A_2\otimes B_2\otimes C_1).\end{aligned} \tag{7}$$

Quantum theory violates the inequality (7) if Alice, Bob, and Charlie choose

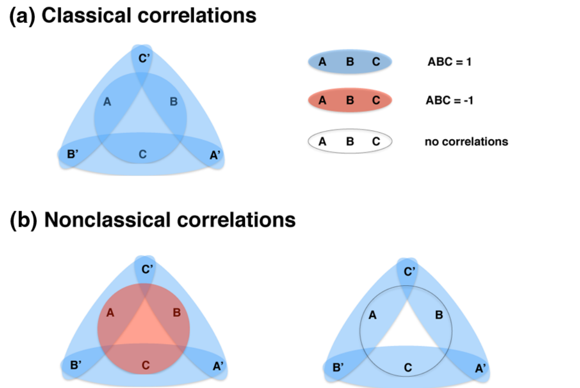

FIG. 2 (color online). Tripartite correlations between three observers. Each local measurement is maximally random, i.e., $P(A=1)=P(A=-1)=1/2$, etc. Classical correlations (a) imply that if $AB'C'=A'BC'=A'B'C=1$, then $ABC=1$. However, for nonclassical tripartite correlations [(b), left] one can observe $ABC=-1$ (GHZ paradox). The above nonclassical correlations cannot be distinguished from the classical ones with entropic inequalities. No tripartite correlations between A, B, and C can be observed [(b), right]. These can be detected via entropic inequalities. In contrast to the bipartite scenario, such extreme nonclassical tripartite correlations can be observed between quantum systems.

$$A_1 = B_1 = C_1 = \cos\left(\frac{\pi}{6}\right)X + \sin\left(\frac{\pi}{6}\right)Y,$$
$$A_2 = B_2 = C_2 = \cos\left(\frac{\pi}{12}\right)X - \sin\left(\frac{\pi}{12}\right)Y, \quad (8)$$

where X and Y are Pauli operators. We have $H(A_1 \otimes B_2 \otimes C_2) = H(A_2 \otimes B_1 \otimes C_2) = H(A_2 \otimes B_2 \otimes C_1) = 0$ and $H(A_1 \otimes B_1 \otimes C_1) = 1$. This achieves maximal algebraic violation of Eq. (7).

The derivation of Eq. (6) holds for any distance with the associativity property. For instance, applying the generalization of the covariance distance [9,10] $\delta(A_1, A_2, A_3) = 1 - \langle A_1 \cdot A_2 \cdot A_3 \rangle$ to Eq. (6) gives the original tripartite Mermin inequality [11]

$$\langle A_1 \cdot B_2 \cdot C_2 \rangle + \langle A_2 \cdot B_1 \cdot C_2 \rangle + \langle A_2 \cdot B_2 \cdot C_1 \rangle - \langle A_1 \cdot B_1 \cdot C_1 \rangle \leq 2. \quad (9)$$

Multipartite information-theoretic Bell inequality.—To extend the previous result to more than three parties, we follow the approach of Ref. [16] and use more than two measurements per observer. For simplicity, we consider an even number of qubits, which requires three measurements per observer.

Let us consider an even number N of observers sharing a multipartite system. Each observer measures three randomly chosen binary ± 1 observables $M_i^{(j)}$, where $i = 1, 2, 3$ labels measurements and $j = 1, \ldots, N$ labels observers. We have

$$\begin{aligned}
&\delta(M_1^{(1)}, M_1^{(2)}, M_1^{(3)}, M_1^{(4)}, \ldots, M_1^{(N)}) \\
&\quad \leq \delta(M_1^{(1)}, M_2^{(2)}, M_3^{(3)}, M_3^{(4)}, \ldots, M_3^{(N)}) \\
&\qquad + \delta(M_3^{(1)}, M_1^{(2)}, M_2^{(3)}, M_3^{(4)}, \ldots, M_3^{(N)}) \\
&\qquad + \delta(M_3^{(1)}, M_3^{(2)}, M_1^{(3)}, M_2^{(4)}, \ldots, M_3^{(N)}) + \cdots \\
&\qquad + \delta(M_2^{(1)}, M_3^{(2)}, M_3^{(3)}, M_3^{(4)}, \ldots, M_1^{(N)}) \\
&\qquad + \delta(M_2^{(1)}, M_2^{(2)}, M_2^{(3)}, M_2^{(4)}, \ldots, M_2^{(N)}). \quad (10)
\end{aligned}$$

The term on the left-hand side contains only measurements $M_1^{(j)}$, whereas the first N terms on the right-hand side are cyclic permutations of one measurement $M_1^{(j)}$, one measurement $M_2^{(j)}$, and $N-2$ measurements $M_3^{(j)}$. The remaining term on the right-hand side contains only measurements $M_2^{(j)}$.

The derivation is as follows. We start with the multipartite distance $\delta(M_1^{(1)}, M_1^{(2)}, M_1^{(3)}, M_1^{(4)}, \ldots, M_1^{(N)})$ and apply the triangle inequality (together with symmetry and associativity) to obtain

$$\begin{aligned}
&\delta(M_1^{(1)}, M_1^{(2)}, M_1^{(3)}, M_1^{(4)}, \ldots, M_1^{(N)}) \\
&\quad \leq \delta(M_1^{(1)}, M_2^{(2)}, M_3^{(3)}, M_3^{(4)}, \ldots, M_3^{(N)}) \\
&\qquad + \delta(M_1^{(2)}, \ldots, M_1^{(N)}, M_2^{(2)}, M_3^{(3)}, \ldots, M_3^{(N)}). \quad (11)
\end{aligned}$$

The term on the left-hand side and the first term on the right-hand side correspond to measurable quantities in the inequality (10), whereas the second term on the right-hand side cannot be experimentally verified. We expand the last term as

$$\begin{aligned}
&\delta(M_1^{(2)}, \ldots, M_1^{(N)}, M_2^{(2)}, M_3^{(3)}, \ldots, M_3^{(N)}) \\
&\quad \leq \delta(M_3^{(1)}, M_1^{(2)}, M_2^{(3)}, M_3^{(4)}, \ldots, M_3^{(N)}) \\
&\qquad + \delta(M_1^{(3)}, \ldots, M_1^{(N)}, M_2^{(2)}, M_2^{(3)}, M_3^{(1)}, M_3^{(3)}). \quad (12)
\end{aligned}$$

Again, we generated a term that is observable and an additional term that requires further application of the triangle inequality. The following pattern emerges. After k repetitions of the above procedure, one generates the measurable term

$$\delta(M_3^{(1)}, \ldots, M_3^{(k-1)}, M_1^{(k)}, M_2^{(k+1)}, M_3^{(k+2)}, \ldots, M_3^{(N)}) \quad (13)$$

and the nonmeasurable term

$$\begin{aligned}
&\delta(M_1^{(k+1)}, \ldots, M_1^{(N)}, M_2^{(2)}, \ldots, M_2^{(k+1)}, M_3^{(2)}, \ldots, \\
&\qquad M_3^{(k)}, M_3^{(k+2)}, \ldots, M_3^{(N)}), \quad (14)
\end{aligned}$$

if k is odd, or

$$\delta(M_1^{(k+1)},\ldots,M_1^{(N)},M_2^{(2)},\ldots,M_2^{(k+1)},M_3^{(1)},M_3^{(k+1)}), \quad (15)$$

if k is even. After $k=N-1$ repetitions, we obtain the right-hand side terms of Eq. (10) except the last two, and an additional nonmeasurable term. This term can be expanded into two missing measurable ones,

$$\begin{aligned}\delta(M_1^{(N)},M_2^{(2)},\ldots,M_2^{(N)},M_3^{(2)},\ldots,M_3^{(N-1)})\\ \le \delta(M_2^{(1)},M_3^{(2)},M_3^{(3)},\ldots,M_3^{(N-1)},M_1^{(N)})\\ +\delta(M_2^{(1)},M_2^{(2)},M_2^{(3)},M_2^{(4)},\ldots,M_2^{(N)}), \quad (16)\end{aligned}$$

which ends the derivation.

The inequality (10) can be maximally violated in quantum mechanics by the N-partite GHZ state $|\mathrm{GHZ}\rangle_N = (|0\ldots0\rangle + |1\ldots1\rangle)/\sqrt{2}$ and for $\delta(M_1^{(1)},M_1^{(2)},M_1^{(3)},M_1^{(4)},\ldots,M_1^{(N)}) = H(M_1^{(1)}\otimes M_1^{(2)}\otimes\cdots\otimes M_1^{(N)})$, etc. In this case, the local measurements are $M_i^{(j)} = \cos\alpha_i X + \sin\alpha_i Y$, where $\alpha_1=\pi/2N$, $\alpha_2=0$, and $\alpha_3=-[\pi/2N(N-2)]$. The choice of the angles stems from the following observation,

$$(M_i^{(1)}\otimes\cdots\otimes M_j^{(N)})\frac{|0\ldots0\rangle+|1\ldots1\rangle}{2} = \frac{e^{-i(\alpha_i+\cdots+\alpha_j)}|0\ldots0\rangle + e^{i(\alpha_i+\cdots+\alpha_j)}|1\ldots1\rangle}{2} = |\overline{\mathrm{GHZ}}\rangle_N, \quad (17)$$

where the overlap $\langle\mathrm{GHZ}|\overline{\mathrm{GHZ}}\rangle_N = \cos(\alpha_i+\cdots+\alpha_j)$. In every case, except $H(M_1^{(1)}\otimes M_1^{(2)}\otimes\cdots\otimes M_1^{(N)})$, the overlap is one and the corresponding entropy is zero. On the other hand, the entropy on the left-hand side of Eq. (10) is one because the overlap is zero.

Information-theoretic state-independent contextuality.— Contextuality is a form of nonclassicality more general than nonlocality. In this case, all measurements can be performed on a single localized system. The crucial assumption is based on a classical intuition that an outcome of one measurement does not depend on what other compatible (nondisturbing) measurement is performed at the same time. This assumption is known as noncontextuality and systems violating it are called contextual. Interestingly, in quantum theory contextuality can be exhibited by any state of the system with the dimension larger than two, whereas nonclassicality in nonlocal scenarios can be exhibited only by entangled states.

We will now consider an entropic version of the state-independent contextuality proof commonly known as the Peres-Mermin square [17–19]. Consider nine ±1 observables that can be measured on a single system. Because of compatibility relations, these measurements can be performed in the following triples: $\{A,a,\alpha\},\{B,b,\beta\},\{C,c,\gamma\},\{A,B,C\},\{a,b,c\},\{\alpha,\beta,\gamma\}$. The classical reasoning based on noncontextuality hypothesis implies that, for measurable products $q_1=A\cdot a\cdot\alpha$, $q_2=B\cdot b\cdot\beta$, $q_3=C\cdot c\cdot\gamma$, $q_4=A\cdot B\cdot C$, $q_5=a\cdot b\cdot c$, and $q_6=\alpha\cdot\beta\cdot\gamma$, one has $\prod_{i=1}^6 q_i = 1$.

However, in nonclassical theories we have $\prod_{i=1}^6 q_i=-1$ (see Fig. 3). In quantum theory this is achieved by a set of two-qubit measurements: $A=X\otimes\mathbb{1}$, $a=\mathbb{1}\otimes X$, $\alpha=X\otimes X$, $B=\mathbb{1}\otimes Y$, $a=Y\otimes\mathbb{1}$, $\beta=Y\otimes Y$, $C=X\otimes Y$, $c=Y\otimes X$, $\gamma=Z\otimes Z$, where X, Y, and Z are Pauli operators [17–19].

Next, we derive

$$\begin{aligned}&\delta(\alpha,\beta,\gamma)\\ &\le\delta(A,a,\alpha)+\delta(A,a,\beta,\gamma)\\ &\le\delta(A,a,\alpha)+\delta(B,b,\beta)+\delta(A,a,B,b,\gamma)\\ &\le\delta(A,a,\alpha)+\delta(B,b,\beta)+\delta(A,B,C)+\delta(C,a,b,\gamma)\\ &\le\delta(A,a,\alpha)+\delta(B,b,\beta)+\delta(A,B,C)+\delta(a,b,c)\\ &\quad+\delta(C,c,\gamma). \quad (18)\end{aligned}$$

The corresponding entropic inequality is

$$\begin{aligned}H(\alpha\cdot\beta\cdot\gamma)\le H(A\cdot a\cdot\alpha)+H(B\cdot b\cdot\beta)\\ +H(A\cdot B\cdot C)+H(a\cdot b\cdot c)+H(C\cdot c\cdot\gamma). \quad (19)\end{aligned}$$

For the above quantum observables on a localized two-qubit system, one finds that $q_1=q_2=\ldots=q_5=1$ and $q_6=-1$ for any quantum state. This distribution of outcomes does not violate the inequality (19); however, if we follow the method of Ref. [2] and equally mix it with the classical distribution $q_1=\cdots=q_6=1$, we get the maximal violation [see Fig. 3(b), right].

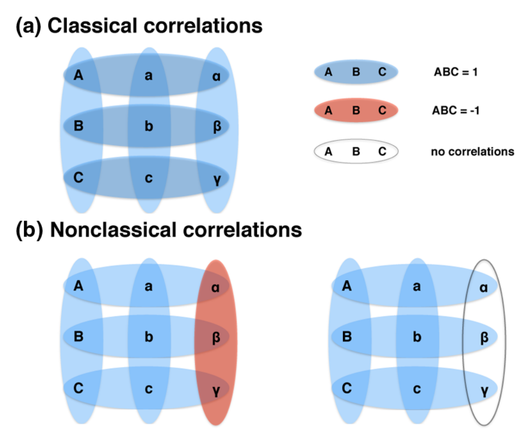

FIG. 3 (color online). Peres-Mermin square. Classical correlations (a) imply $(Aa\alpha)(Bb\beta)(Cc\gamma)(ABC)(abc)(\alpha\beta\gamma)=1$. Nonclassical contextual correlations [(b), left] result in the product equal to -1. These nonclassical correlations cannot be distinguished from classical ones with entropic inequalities. This can be done by mixing them with the classical correlations [(b), right] since $H(\alpha\cdot\beta\cdot\gamma)=1$.

As in the case of the multipartite nonlocality, the derivation of Eq. (18) holds not only for the function δ, but for any distance with the associativity property. Applying $\delta(A_1, A_2, A_3) = 1 - \langle A_1 \cdot A_2 \cdot A_3 \rangle$ to Eq. (18) gives the noncontextuality inequality by Cabello [12], which is violated by any quantum state

$$\langle A \cdot a \cdot \alpha \rangle + \langle B \cdot b \cdot \beta \rangle + \langle A \cdot B \cdot C \rangle + \langle a \cdot b \cdot c \rangle + \langle C \cdot c \cdot \gamma \rangle - \langle \alpha \cdot \beta \cdot \gamma \rangle \leq 4. \quad (20)$$

Discussion.—We have developed a new tool for studying quantum correlations. In particular, we used it to derive entropic inequalities to test the multipartite nonlocality and the state-independent contextuality. In both cases, quantum mechanics allows for the maximal violation of these inequalities. On the other hand, the bipartite quantum nonlocality was unable to maximally violate entropic inequalities based on a finite number of measurements [3]. This brings us to an interesting analogy. It was shown (see, for example, Ref. [11]) that correlation-based Bell inequalities admit the maximal quantum violation only for multipartite systems. Here we show that the same is true for multipartite entropic Bell inequalities.

Another important observation is the fact that for the multipartite nonlocality we could find a state and measurements leading to a direct violation, whereas for the state-independent contextuality we had to mix the measured nonclassical data with a classical probability distribution to observe the violation. We attribute this to the state-independence property. The state-independent contextuality is a property of measurements, not the states. To obtain a direct violation of an entropic inequality, one would have to look for a product of some observables for which the entropy in any state is larger than entropies of other products. Although we do not provide a proof, we speculate that such observables do not exist.

We also rederived two known inequalities. This suggests that the multipartite information-theoretic distance provides a more general treatment of nonclassical correlations than the standard correlation functions approach. We have already proposed this idea for bipartite nonclassical correlations in Ref. [9]. However, to fully prove it one needs to show that all multipartite Bell inequalities and noncontextuality inequalities can be derived from some multipartite distance.

There are several open problems that require further investigation: (i) extension to nonlocality of an odd number of qubits, (ii) finding an information-theoretic distance suitable to investigate multipartite nonlocality of higher-dimensional quantum systems, and (iii) derive multipartite monogamy relations from the properties of information-theoretic distances, perhaps using ideas in Ref. [9].

S. R. was supported by Canada's NSERC, MPrime, CIFAR, and CFI and IQC. P. K. and D. K. are supported by the Foundational Questions Institute (FQXi) and by the National Research Foundation and Ministry of Education in Singapore. D. K. is also supported by the Singapore Ministry of Education through Academic Research Fund Tier 3 "Random numbers from quantum processes" Grant No. MOE2012-T3-1-009.

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