Geometrically Induced Magnetic Catalysis and Critical Dimensions

Antonino Flachi,[1] Kenji Fukushima,[2] and Vincenzo Vitagliano[1]
[1]*Multidisciplinary Center for Astrophysics and Department of Physics, Instituto Superior Técnico, University of Lisbon, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal*
[2]*Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*
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We discuss the combined effect of magnetic fields and geometry in interacting fermionic systems. At leading order in the heat-kernel expansion, the infrared singularity (that in flat space leads to the magnetic catalysis) is regulated by the chiral gap effect, and the catalysis is deactivated by the effect of the scalar curvature. We discover that an infrared singularity is found in higher-order terms that mix the magnetic field with curvature, and these lead to a novel form of geometrically induced magnetic catalysis. The dynamical mass squared is then modified not only due to the chiral gap effect by an amount proportional to the curvature, but also by a magnetic shift $\propto (4-D)eB$, where D represents the number of space-time dimensions. We argue that $D=4$ is a critical dimension across which the behavior of the magnetic shift changes qualitatively.

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The phenomenon of spontaneous symmetry breaking, originally inspired by the BCS theory of superconductivity, was initially established to account for the masses of nucleons and light pions [1]. In modern terminology, we say that chiral symmetry, i.e., a symmetry between left- and right-handed sectors of massless fermions, is spontaneously broken by a nonvanishing chiral condensate $\langle\bar{\psi}\psi\rangle \neq 0$. When this happens, a dynamical mass for the fermions, denoted here as M, is generated by such a condensation phenomenon. Since the early days, theorists have been trying to develop ways to compute the effective potential as a function of the chiral condensate in order to be able to unfold the nature of symmetry breaking and of its associated phase transitions in a wide variety of contexts.

The chiral condensate plays an essential role in the characterization of the phase structure of quark matter in extreme conditions, for instance, at high temperature or density, as well as in the presence of external (magnetic, gravitational, etc.) fields (see Refs. [2–4] for reviews). However, theoretical applications are not limited to quark matter, and concepts and methods spread over a wide range of subjects. A well-known example in condensed matter physics, where the concept of the chiral condensates or the Dirac and Haldane masses plays a central role, is in the case of graphene in a strong magnetic field [5], which exhibits an anomalous quantum Hall effect [6]. These phenomena (and the role that the chiral condensate plays) can be understood intuitively from the following argument: The chiral condensate consists of a left-handed (or right-handed) fermion and a right-handed (or left-handed) antifermion with zero net momenta, so the momentum directions should be opposite and the spin directions antiparallel to each other. Since a strong magnetic field tends to align the spins of the fermion and antifermion, it enhances the chance for dynamical breaking of chiral symmetry to occur.

Symmetry breaking is often modeled by an interacting low-energy effective theory, expressed in terms of fermion degrees of freedom, like in the BCS theory. If we denote the coupling constant that characterizes the strength of the interaction with λ, then spontaneous symmetry breaking usually occurs when the coupling constant exceeds some critical value λ_c, i.e., $\lambda > \lambda_c$. When the presence of an external magnetic field B triggers the condition $\lambda_c \to 0$, we say that "magnetic catalysis" is realized. This phenomenon, originally discovered in Refs. [7,8] (see Ref. [3] for reviews), is caused by an infrared (IR) singular contribution to the effective potential, strengthened by the effect of the dimensional reduction due to the magnetic field. In the following, the term magnetic catalysis will always be used in this sense (and not in reference to the increasing behavior of $\langle\bar{\psi}\psi\rangle$ as a function of increasing B, as it is sometimes found in the literature [9]).

The magnetic catalysis is naturally lost at finite temperature, since there are no Matsubara zero modes and therefore no IR singularity, explaining the presence of a finite-T phase transition even when $B \neq 0$ [10]. Interestingly, a similar mechanism works when the spatial geometry is curved. Such similarities and some potential implications have recently been discussed in the context of the "chiral gap effect," leading to an intuitive understanding of the fate of chiral symmetry in the presence of nonzero curvature [11]. The chiral gap effect is a robust consequence arising from the quasinonperturbative (R-resummed) form of the propagator in curved space originally formulated in Ref. [12] (see Ref. [13] for a discussion in relation to strongly interacting fermions and Refs. [14,15] for additional discussion on heat-kernel resummations). The situation we wish to consider in the present Letter concerns the combined effect of curvature and magnetic fields, and the question we intend to address is whether the magnetic catalysis can be deactivated in the presence of curvature. As we will show, something novel happens due to the interplay of magnetic fields and geometry. This is an intriguing question, leading to profound consequences for many physical systems. For instance, it is of relevance to astrophysical systems (e.g., neutron stars) that involve not only gravity but also strong magnetic fields. In condensed matter physics, it is also possible to induce a curvature locally by the insertion of defects in the lattice of strongly coupled materials and use the combined effects of geometry and external magnetic fields to probe the nature of chiral symmetry. Motivated by these applications, in the following we shall focus our attention to the physically relevant case of $eB \gg R$, where R indicates the Ricci scalar.

Curvature-induced chiral gap and magnetic inhibition.—It is an immediate consequence of the chiral gap effect that the IR singularity responsible for the magnetic catalysis disappears unless we take account of higher-order terms. Since it will turn out to be quite instructive, we shall explicitly check this. The partially resummed heat-kernel expansion reads

$$\mathrm{Tr}e^{-t\mathcal{D}} = \frac{1}{(4\pi t)^{D/2}} e^{-M_R^2 t} \frac{eBt}{\sinh(|eB|t)} e^{(it/2)eF_{\mu\nu}\sigma^{\mu\nu}} \sum_{k=0}^{\infty} a_k t^k, \tag{1}$$

where $\mathcal{D}$ represents the Dirac operator.

The above expression is obtained by means of a double resummation over the scalar curvature and of the purely magnetic contributions. The first one gives rise to the first exponential factor, while the second one resums the purely magnetic contributions. The first kind of resummation has been derived in Ref. [12], while the second one has been discussed, for example, in Ref. [16] (see also Chap. 5 of Ref. [14] for a nice derivation). The shift in the mass due to the scalar curvature resummation, i.e., $M_R^2 \equiv M^2 + R/12$, represents the essence of the curvature-induced chiral gap. We can describe a qualitative mechanism for this effect by directly looking at the spectra of the Dirac operator. For concreteness, let us consider the case of D-dimensional de Sitter space S^D for which the eigenvalues of the *free* Dirac operator are [17]

$$\lambda_n^{(\pm)} = m \pm i\sqrt{\frac{R}{D(D-1)}}\left(n + \frac{D}{2}\right). \tag{2}$$

Similarly to the case of a box with periodic boundary conditions, the spectra, labeled by an integer index $n \geq 1$, become discrete in the present case. For sufficiently large D and taking into account the degeneracy for each n, we can recover a continuum spectra (labeled by continuum momenta), except for the presence of a gap proportional to $\sqrt{R}$. This gap prevents the Dirac eigenvalues from accumulating around the zero mode, disfavoring the formation of a nonvanishing chiral condensate when $R > 0$, as it follows straightforwardly from the Banks-Casher relation. In fact, Eq. (2) allows for a deeper understanding of such a curvature-induced chiral gap: If we take the product $\lambda_n^{(+)}\lambda_n^{(-)}$, then the leading effect of R naively looks like a shift in m^2 by R, but in the original eigenvalues $\lambda_n^{(\pm)}$ of the first-order Dirac operator it is obvious that this R-induced shift is not on the real axis, but it occurs along the imaginary axis, which explains how such a curvature-generated mass gap can be consistent with chiral symmetry.

Picking up the first contribution with $a_0 = 1$ from Eq. (1), we can express the one-loop effective potential as

$$V_{R\text{-resum}}[M] = \frac{M^2}{2\lambda_D} + \kappa_D(eB)\int_{1/\Lambda^2}^{\infty} dt t^{-D/2} e^{-M_R^2 t}\coth(eBt), \tag{3}$$

where we have used proper-time regularization. In the above expression, we have denoted the coupling constant in D dimensions with λ_D and defined $\kappa_D \equiv 2^{\lfloor (D+1)/2\rfloor -1}/(4\pi)^{D/2}$. In obtaining Eq. (3), we could have adopted any other regularization scheme (ζ function, Pauli-Villars, etc.) as long as gauge invariance is preserved. Different prescriptions would not change the IR singular structure. In the limit of $eB \gg \Lambda^2$, $\coth(eBt) \approx 1$, and, for $R = 0$, we can approximate the potential as

$$V_{\text{catalysis}}[M] \approx \frac{M^2}{2\lambda_4} + \kappa_4(eB)M^2\left(-1 + \gamma_E + \ln\frac{M^2}{\Lambda^2}\right) \tag{4}$$

for $D = 4$. We may then define a "renormalized" coupling as $1/(2\lambda_4') \equiv 1/(2\lambda_4) + \gamma_E\kappa_4(eB)$ and observe that the dynamical mass always takes a finite value: $M^2 = \Lambda^2\exp[-1/2\kappa_4\lambda_4'(eB)]$. This is how magnetic catalysis works.

It should be noted that the IR singularity possibly remains if $R < 0$ (see Ref. [18] for the analysis with $R < 0$). Then it is natural to expect that effects of negative curvature may be compensated by those of temperature leading to a restoration of the catalysis. For $R > 0$, however, M_R^2 never reaches zero and $\ln(M_R^2/\Lambda^2)$ is no longer IR singular, which could be regarded as a mechanism of "magnetic inhibition" induced geometrically (having an origin totally different from the inhibition at finite temperature [19]).

Magnetic catalysis from higher-order terms.—The above is not the whole story. As done in the derivation of the chiral gap effect, below we shall assume a geometrical structure such that we can neglect Ricci and Riemann tensors as compared to the Ricci scalar: $|R_{\mu\nu}| \ll |R|$ and $|R_{\mu\nu\rho\sigma}| \ll |R|$, as happens, for instance, for maximally symmetric geometries with large D. However, in the present case, since higher-order contributions coming from a_k with $k \geq 2$ involve terms that mix the magnetic field with curvature, these cannot be discarded but may dominate over the purely gravitational tensorial combinations. Before proceeding with detailed calculations including higher-order terms, it is useful to obtain a parametric form for the chiral condensate or the dynamical mass (correction) by using dimensional analysis. Let us suppose that higher-order terms give rise to an IR-singular correction δM^2 to the dynamical mass. Then $(eB)^2 R$ would be the most natural combination from which the correct mass dimensionality arises. This is because a_k should vanish (except for subleading contributions involving $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$) when we take either $B = 0$ or $R = 0$. Since B is generated from a Lorentz-invariant contraction of $F_{\mu\nu}$, the lowest order should be $(eB)^2$. Finally, multiplying by the four-fermion coupling constant λ_D that has a mass dimension, we conclude that δM^2 should be a function of $\lambda_D (eB)^2 R$.

In D-dimensional space-time, the mass dimension of λ_D is $[\lambda_D] = 2 - D$, and so $[\lambda_D (eB)^2 R] = 8 - D$. Thus, dimensional analysis allows us to write

$$\delta M^2 \sim [\lambda_D (eB)^2 R]^{2/(8-D)}. \qquad (5)$$

The above expression indicates that $D = 8$ is a critical dimension. For the $D = 8$ case, $\lambda_8 (eB)^2 R$ becomes dimensionless, and the parametric dependence of δM^2 should become logarithmic, i.e., $\delta M^2 \sim \Lambda^2 \exp[-C/\lambda_8 (eB)^2 R]$, which reminds us of the standard magnetic catalysis.

Let us now look into the concrete calculations of the chiral condensate and of the effective potential. Since we are working in the regime where $eB \gg R$, in the heat-kernel expansion we need to resum all the terms of the form $(eB)^2 R (eBt)^k$ with $k \geq 2$. Here, we adopt the same strategy as when performing the R resummation that leads to an exponential dependence. Namely, we postulate a resummed expansion based on the following reorganisation of the various terms:

$$\sum_{k=0}^{\infty} a_k t^k = 1 + R \sum_{k=1}^{\infty} \alpha_k (eB)^{k-1} t^k e^{\beta_k eBt}, \qquad (6)$$

where α_k and β_k are dimensionless coefficients that can be computed explicitly from the heat-kernel coefficients. The above expression can be obtained by direct construction of the corrections to the zeta function the higher-order mixed terms induce (the first few coefficients can be explicitly obtained from those reported in Ref. [12] by keeping only the terms that mix the magnetic field with curvature), by ordering according to their mass dimensionality and exponentiating.

We explicitly obtained a_2 and a_3 ($a_1 = 0$ from an obvious reason of dimensionality) and identified the coefficients α_2 and β_2 (and $\alpha_1 = 0$ corresponding to $a_1 = 0$). Similar computations of heat-kernel coefficients are reported in Refs. [12,16]. After lengthy calculations, we obtained the effective potential that reads

$$V_{B\text{-resum}}[M] = \frac{M^2}{2\lambda_D} + \kappa_D (eB) \int_{1/\Lambda^2}^{\infty} dt t^{-D/2} e^{-M_R^2 t} + \frac{\kappa_D (eB)^2 R}{6D(D-1)} \int_{1/\Lambda^2}^{\infty} dt t^{-D/2+2} e^{-M_B^2 t}. \qquad (7)$$

Since we are interested in whether the magnetic catalysis occurs in the presence of curvature when eB is the largest scale in the system, we have approximated $\coth(eBt) \approx 1$ in the above expression. We defined $M_B^2 \equiv M_R^2 + [2(4-D)/15] eB$, which represents a B-induced correction to the chiral gap effect. This expression implies that the curvature-induced chiral gap $R/12$ could be compensated by the effect of the magnetic field for $D - 4 > 0$ if $eB \gg R$.

The last term in Eq. (7) represents the correction to the potential coming from higher-order terms mixing curvature tensors with magnetic field and that can be obtained, at the price of a long calculation, by starting from the general expression of the zeta function and from the explicit knowledge of the heat-kernel coefficients.

The term that carries the potential IR singularity is the third term in the effective potential (7). This can be explicitly evaluated, leading to

$$\int_{1/\Lambda^2}^{\infty} dt t^{-D/2+2} e^{-M_B^2 t} = M_B^{D-6} \Gamma(3 - D/2, M_B^2/\Lambda^2), \qquad (8)$$

from which it is easy to estimate the parametric dependence of the chiral condensate, which turns out to be consistent with Eq. (5). In what follows below let us look at Eq. (5) for specific choices of D.

($D = 3$).—This is probably the most relevant to systems in condensed matter physics. We should remark here that, while the condition of large D allows for a simple hierarchy between curvature invariants, it is not a necessary condition for the validity of the above statements. In this case, interestingly, the magnetic shift and the curvature-induced chiral gap accumulate. Because the correction (8) is infrared singular as $\propto 1/M_B^3$, it would be dominant near $M_B \sim 0$, and we can easily solve the gap equation to find

$$M_B^2 = M^2 + \frac{R}{12} + \frac{2}{15} eB = \left[\frac{\sqrt{\pi} \kappa_3 \lambda_3 (eB)^2 R}{24} \right]^{2/5}, \qquad (9)$$

which is of course consistent with Eq. (5). Under the condition of $eB \gg R$, thus, the magnetic shift term $\sim eB$ would oversaturate M_B^2. This means that there is no stable solution near $M_B^2 \sim 0$ with $M^2 > 0$, and M^2 is actually determined by the balance with the second term in the potential (7) rather than the third term. In summary, for

$D = 3$, the chiral gap effect overwhelms the magnetic catalysis, and higher-order terms would not override this situation.

($D = 4$).—Since there is no B-induced correction to the mass squared, $D = 4$ is an exceptional (and also realistic in relativistic systems) case. It is obvious from Eq. (8) that a singular term $\propto 1/M_B^2$ appears in the effective potential. Then we can easily locate the minimum of the effective potential at

$$M_B^2 = M^2 + \frac{R}{12} = \sqrt{\frac{\lambda_4 \kappa_4 (eB)^2 R}{36}}. \qquad (10)$$

We note that M^2 is nonvanishing if $(eB)^2 > R/(4\lambda_4\kappa_4)$, which means that the magnetic catalysis is recovered in this case for $D = 4$. Strictly speaking, for an arbitrary value of B, magnetic catalysis is not necessarily guaranteed, since this inequality also determines a critical λ_4. We should, however, note that we postulated $eB \gg R$ in our analysis; however, it is obvious that the term $R/12$ turns out to be subdominant.

It is quite natural that M^2 is proportional to the density of states of the Landau levels, $\sim eB$, although a rather unconventional combination of $\lambda_4 R$ appears in addition. Because $\lambda_4 R$ is dimensionless in $D = 4$, the dependence on $\lambda_4 R$ could have any functional form, in principle, as long as it vanishes for $R \to 0$. The square root dependence as seen in Eq. (5) is very interesting, because the right-hand side $\sim eB\sqrt{\lambda_4 R}$ could dominate over the chiral gap effect contribution $\sim R$ not only for $eB \gg R$ but also for $R \ll \lambda_4^{-1}$ (and $B \neq 0$). The latter is usually the case (and $\lambda_4^{-1/2}$ should be interpreted as a typical scale in theory, e.g., $\sim\Lambda_{\mathrm{QCD}}$ in the case of quarks).

($D = 5$).—The IR singularity is weakened as $\sim 1/M_B$ in Eq. (8), but it is still sufficient to lead to a chiral condensate. In this case we find that M^2 is affected by a B-induced term:

$$M_B^2 = M^2 + \frac{R}{12} - \frac{2eB}{15} = \left(\frac{\sqrt{\pi}\lambda_5 \kappa_5 (eB)^2 R}{120}\right)^{2/3}. \qquad (11)$$

Interpreting this expression requires care, particularly due to the presence of the third term $\sim eB$ in the left-hand side. Equation (11) may imply that $M^2 \sim eB$ (regardless of λ_5) even in the limit of vanishing R. Going back to Eq. (7), however, the third term in the effective potential is vanishing for $R = 0$, and so M^2 cannot have any such correction. Thus, the behavior of the effective potential itself smoothly changes from $R = 0$ to $R \neq 0$, but the position of the minimum discontinuously jumps once B is switched on. This is not so unusual if the potential shape is shallow enough. In principle, such a discontinuous behavior at $R = 0$ would be made milder by further resummation of IR singular terms around $M_B^2 \sim 0$.

We should emphasise that this "geometrically induced magnetic catalysis" is absent in flat space and can occur even in the case of a weakly curved geometry. For this reason, its effect should be more easily observable than the chiral gap effect itself. In fact, even though the correction to the potential height is negligibly small due to suppression by R, a sizable shift in M^2 is possible. We did not include the IR-safe terms in the above discussion, since they, obviously, do not change our qualitative conclusion.

($D = 8$).—Qualitatively, the cases $D = 5$, 6, and 7 are similar. So, let us consider the next nontrivial case, $D = 8$. As we have already mentioned, $\lambda_8 (eB)^2 R$ is dimensionless in this case, and explicit calculations lead to

$$M_B^2 = M^2 + \frac{R}{12} - \frac{16eB}{15} = \Lambda^2 \exp\left[-\frac{168}{\lambda_8' \kappa_8 (eB)^2 R}\right], \qquad (12)$$

where, for notational convenience, we have defined $\lambda_8'^{-1} \equiv \lambda_8^{-1} + (-1 + \gamma_E)\kappa_8 (eB)^2 R/168$.

Although this nonanalytic form in terms of the coupling λ_8' is peculiar, physical consequences are dominated by a shift of $-16eB/15$ in M_B^2 that induces $M^2 \sim eB$ as long as M_B^2 is non-negative. The same argument can be applied to larger D. Hence, regarding the behavior of M^2, we can generalize our study for any $D > 4$ and conclude that the essential feature of the geometrically induced magnetic catalysis should be common if D is larger than the critical dimension $D = 4$.

Conclusions and outlook.—In this Letter, we have investigated the two competing effects of curvature and magnetic field with the intent to clarify the interplay between the chiral gap effect and the magnetic catalysis. In conformity with the chiral gap effect, quasinonperturbative contributions (due to R resummation in the heat kernel) induce a curvature correction to the mass and regulate, as it may be intuitively expected, the infrared singularity deactivating the strict magnetic catalysis at this level of approximation.

However, we have discovered (using both dimensional analysis and by means of explicit evaluation of the effective potential) that $D = 4$ is a critical dimension. Above $D = 4$, next-to-leading-order corrections in the heat-kernel expansion become relevant, restoring the magnetic catalysis in a novel, geometrically induced fashion. We have also seen that below $D = 4$ these corrections become irrelevant for the magnetic catalysis, and the curvature and magnetic field cooperate. More precisely, for $D > 4$, the infrared singularity induces a dynamical mass of the order of eB, though the singular contribution to the effective potential is suppressed by small R. In the critical $D = 4$ case, the dynamical mass squared is proportional to $eB\sqrt{\lambda_4 R}$, where λ_4 is the four-Fermion coupling constant and the combination $\lambda_4 R$ is dimensionless.

Possible applications of the geometrically induced magnetic catalysis discussed here are of relevance for a wide variety of physical setups. Perhaps the most natural environment is that of neutron stars where both a curvature and a magnetic field (that could be as strong as 10^{12} T [20]) could lead to a realization of what we have discussed here. Micro black holes in the early Universe or in high-energy particle collisions accelerator experiment could also be an interesting playground for the chiral gap effect and the geometrically induced magnetic catalysis. In an ultrarelativistic nucleus-nucleus collision, a pulsed and strong magnetic field is generated, and the space-time geometry is nontrivial due to expansion and flowing fluids (that may cause horizon formation or an acoustic metric; see Refs. [21,22] for related discussions). Finally, although for lower dimensionality the magnetic field and the curvature cooperate (rather than compete), the present discussion may be of relevance in the context of strongly coupled layered materials, where curvature can be generated locally by the insertion of defects.

In order to be able to discuss the problem in a model-independent way, in this work we have focused only on the strict characterization of the magnetic catalysis. It would be intriguing to use chiral perturbation theory, that is, a theoretical approach complementary to the fermionic description adopted here. Also, studying the behavior of the chiral condensate as a function of various external parameters including the temperature effect would be interesting. These remain as future problems.

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