Constraining Quantum Critical Dynamics: $(2+1)$D Ising Model and Beyond

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Quantum critical (QC) phase transitions generally lead to the absence of quasiparticles. The resulting correlated quantum fluid, when thermally excited, displays rich universal dynamics. We establish nonperturbative constraints on the linear-response dynamics of conformal QC systems at finite temperature, in spatial dimensions above 1. Specifically, we analyze the large frequency or momentum asymptotics of observables, which we use to derive powerful sum rules and inequalities. The general results are applied to the $O(N)$ Wilson-Fisher fixed point, describing the QC Ising model when $N=1$. We focus on the order parameter and scalar susceptibilities, and the dynamical shear viscosity. Connections to simulations, experiments, and gauge theories are made.

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The quantum Ising model in two spatial dimensions $(2+1)$D, e.g., on a square lattice, undergoes a quantum critical (QC) phase transition as the ratio of the transverse magnetic field to the exchange coupling is tuned. It is the archetypal example of a nontrivial $(2+1)$D QC point, possibly the simplest one with Z_2 symmetry, but lacks an exact solution contrary to its lower dimensional counterpart. Rather than having quasiparticle excitations, present in the para- or ferromagnetic phases, the spectrum at the QC point is *continuous*. Various methods such as Monte Carlo simulations [1], field theory expansions [1–3], and recently conformal bootstrap [4], have shed light on the critical exponents characterizing its thermodynamics and ground state correlations. In contrast, little is known about its quantum *dynamical* properties at finite temperature [3,5], which are not only important to understand the nature of this strongly correlated quantum fluid but also of clear relevance to experiments.

In this article we study QC dynamics, with a focus on the quantum $O(N)$ Wilson-Fisher fixed point which describes the QC transition for the quantum Ising $(N=1)$ and XY $(N=2)$ models, and the Néel transition in certain antiferromagnets $(N=3)$. Focusing on a large class of experimentally relevant observables, we establish nonperturbative results for the large frequency or momentum asymptotic behavior and sum rules. These provide strong constraints on the universal scaling functions characterizing the system's low-energy responses. The exact sum rules can be seen as generalizations of the celebrated f-sum rule to scale invariant systems. Our results provide rigorous means to assess approximations, constrain numerical results, and ultimately assist with the analysis of experimental data. The methods we use partly rely on the conformal symmetry of the QC point, present for the $O(N)$ Wilson-Fisher fixed point. However, the key ideas are more general, and they greatly generalize the recent analysis [6] for the dynamical conductivity of $(2+1)$D conformal field theories (CFTs). The Letter is organized as follows: We first establish general properties regarding the asymptotics and sum rules of CFTs, and subsequently apply them to the Wilson-Fisher theory, and finally give a broad outlook, including a discussion regarding the implications for Monte Carlo simulations.

Asymptotics and OPE.—We consider a thermally excited system tuned to a QC point via a nonthermal parameter g. In the phase diagram Fig. 1(a), this corresponds to the line in the QC fan at $g=g_c$ and $T\geq 0$. We are interested in the linear-response dynamics at finite temperature, more precisely in the retarded dynamical susceptibility associated with a bosonic observable $\mathcal{O}$, such as the energy or charge density: $\chi^R(t,\mathbf{x}) = -i\Theta(t)\langle[\mathcal{O}(t,\mathbf{x}),\mathcal{O}(0,\mathbf{0})]\rangle_T$, where the average is taken over the thermal ensemble. We set $\hbar = k_B = c = 1$; c is the characteristic speed near the QC point. We will often work in Fourier space: $\chi^R(\omega,\mathbf{k}) = \int dt d^d\mathbf{x}\chi^R(t,\mathbf{x})e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$, where ω is the real frequency and $\mathbf{k}$ the momentum. Using T, the only energy scale available, χ^R can be rewritten to make its scaling properties manifest:

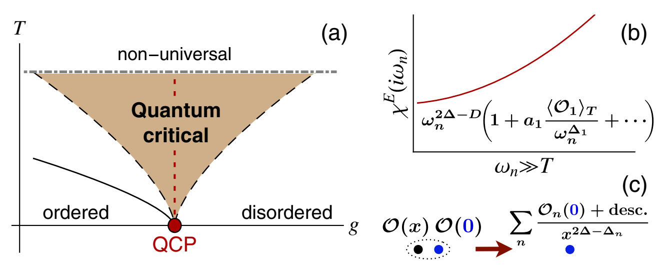

FIG. 1 (color online). (a) Phase diagram near a quantum critical point (QCP). (b) Asymptotic behavior of the Euclidean susceptibility associated with an operator $\mathcal{O}$ of scaling dimension Δ: $\chi^E(i\omega_n) = \langle\mathcal{O}(-\omega_n)\mathcal{O}(\omega_n)\rangle_T$. (c) Schematic operator product expansion (OPE) determining the asymptotics of χ. "desc." denotes the descendants of the primary $\mathcal{O}_n$ (dimension Δ_n).

$$\chi^R(\omega,\mathbf{k}) = T^{2\Delta_{\mathcal{O}}-D}\Phi\left(\frac{\omega}{T},\frac{\mathbf{k}}{T^{1/z}}\right), \tag{1}$$

where $\Delta_{\mathcal{O}}$ is the scaling dimension of $\mathcal{O}$, Φ is a universal scaling (complex) function, and z the dynamical critical exponent. This scaling structure emerges at low energies, i.e., $\omega, |\mathbf{k}|, T \ll \Lambda_{\rm UV}$, where $\Lambda_{\rm UV}$ is a microscopic lattice energy scale, represented by the horizontal dot-dashed line in Fig. 1(a). We emphasize that in this regime the ratios $(\omega, |\mathbf{k}|^z)/T$ can be arbitrary. We introduce the corresponding universal response function

$$\mathcal{R}(\omega,\mathbf{k}) = \frac{\chi^R(\omega,\mathbf{k})}{i\omega - 0^+}, \tag{2}$$

using the Kubo prescription. E.g., if $\mathcal{O} = J_x$ is a conserved current, χ^R_{xx} is the xx-polarization function and $R_{xx}(\omega,\mathbf{0}) = \sigma_{xx}(\omega)$ the dynamical conductivity. Because of the strong interactions and the resulting absence of quasiparticles in generic QC systems, little is known about these universal responses, and our goal is to unravel some of their robust properties. First, let us begin with the large-frequency regime, $\omega \gg T, |\mathbf{k}|$, where the dynamics are near those of the ground state. These can be elegantly studied via the operator product expansion [7] of $\mathcal{O}$ with itself. The OPE is an operator relation and does not depend on temperature. For a general QFT, it is a short time or distance expansion that captures the behavior of the operator product $\mathcal{O}_1(t,\mathbf{x})\mathcal{O}_2(0,\mathbf{0})$ as $t, |\mathbf{x}| \to 0$, which by locality can be expressed as an infinite sum of operators evaluated at $t, |\mathbf{x}| = 0$. We will mostly focus on CFTs, which have $z = 1$ and describe a large class of experimentally relevant QC phase transitions such as those in the quantum Ising and XY models. In a CFT, the $\mathcal{O}\mathcal{O}$ OPE [8,9] of a primary operator $\mathcal{O}$ with scaling dimension $\Delta_{\mathcal{O}}$ reads [Fig. 1(c)]

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_{\mathcal{O}_n \text{ primary}} \frac{C_n(x,\frac{\partial}{\partial y})}{|x|^{2\Delta_{\mathcal{O}}-\Delta_n}}\mathcal{O}_n(y)|_{y=0}, \tag{3}$$

which is expressed in imaginary time τ: $|x|^2 = \tau^2 + \mathbf{x}^2$. A primary operator transforms homogeneously under conformal transformations, e.g., conserved currents and the order parameter in the $O(N)$ model. The sum in Eq. (3) is over primaries $\mathcal{O}_n$ with scaling dimensions Δ_n; it includes the identity (dimension 0). The differential operator $C_n(x, \partial/\partial x)$ is homogeneous under $x \to bx$, and encodes the contributions from the descendants of $\mathcal{O}_n$ (obtained by applying derivatives to $\mathcal{O}_n$). Going to Fourier space and taking a thermal expectation value (TEV) we obtain a key result [Fig. 1(b)]: the $|k| \gg T$ behavior of the Euclidean susceptibility,

$$\chi^E(k) = |k|^{2\Delta_{\mathcal{O}}-D} \sum_{\mathcal{O}_n \text{ primary}} \left(c_n(k)\frac{\langle\mathcal{O}_n\rangle_T}{|k|^{\Delta_n}} + \cdots\right), \tag{4}$$

where $|k|^2 = \omega_n^2 + \mathbf{k}^2$, $\omega_n = 2\pi T n$ is a Matsubara frequency. The dimensionless functions c_n encode the appropriate k-space tensor structure (and can contain logarithms). The dots correspond to higher powers of $T/|k|$ arising from the descendants of $\mathcal{O}_n$. Crucially, a scaling operator will acquire a TEV, $\langle\mathcal{O}_n\rangle_T = d_n T^{\Delta_n}$, since T is the only energy scale. d_n is a universal real number. Substituting this into Eq. (4) we obtain a general expression for the large-k asymptotic expansion of χ. We see that the lowest dimension operators appearing in the OPE dictate how the susceptibility approaches its ground state value as $T/|k| \to 0$. To obtain the real quantum dynamics, we can analytically continue the imaginary frequency expansion Eq. (4) to real frequencies [10] termwise, with the replacement $i\omega_n \to \omega + i0^+$. This follows from the structure of the OPE and the spectral representation connecting the Euclidean and retarded susceptibilities (see Supplemental Material [11] for an extension of the proof in [12]).

Interestingly, unitarity and conformal symmetry constrain the scaling dimensions of these operators [13]: $\Delta_n \geq (D-2)/2$. This leads to important inequalities for dynamical susceptibilities. Let us work in $(2+1)$D and consider a putative low-energy susceptibility $\chi(\omega)$ obtained from an experiment or simulation, and express it as

$$\chi(\omega) \overset{\omega\gg T}{=} \omega^{2\delta_1-3}\left[a + b\left(\frac{T}{\omega}\right)^{\delta_2} + \cdots\right], \tag{5}$$

at large frequencies. Finding either $\delta_i < 1/2$ would violate unitarity bounds and thus rule out a conformal QC point [14]. For Wilson-Fisher QC points, we shall see that the stronger condition, $\delta_2 > 1.4$, holds. Before applying the above general results to those CFTs, we discuss how the asymptotics can be used to prove sum rules for any susceptibility (2-point function).

Sum rules.—We put forth a powerful sum rule for the real frequency quantum dynamical response function:

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\mathrm{Re}\,\delta\mathcal{R}(\omega,\mathbf{k}) = -\delta\chi^\infty. \tag{6}$$

$\delta\mathcal{R}$ is defined as in Eq. (2), with a modified susceptibility $\chi \to \delta\chi$, defined below. The sum rule is independent of small frequency details, and fundamentally relies on the retarded causal structure of χ^R. More precisely, it is the zero-frequency limit of the Kramers-Kronig transform for the modified susceptibility, $\delta\chi^R(\omega,\mathbf{k}) - \delta\chi^\infty$, which we now discuss. In the QC scaling regime, χ does not usually decay at large frequencies unlike on the lattice because it encodes excitations at all scales. To formulate the sum rule, we thus generally need to subtract terms, denoted by $\tilde{\chi}$, from χ to remove its large-ω divergence [15–18]: $\delta\chi(\Omega,\mathbf{k}) = \chi(\Omega,\mathbf{k}) - \tilde{\chi}(\Omega)$, where Ω is a complex frequency in the upper half plane. In some cases, one further needs to subtract a remaining constant: $\delta\chi^\infty = \delta\chi^{\rm E}(|k| \to \infty)$, where the limit is taken at fixed T. We emphasize that $\tilde{\chi}(\Omega)$ is *momentum* *independent* because the asymptotic behavior Eq. (4) depends on powers of $T/\sqrt{\omega_n^2+|\mathbf{k}|^2}$ due to the asymptotic reemergence of Lorentz invariance (broken by T), and we fix $\mathbf{k}$ as we take $\omega_n \gg T$. We note that the correlation functions studied in this Letter only depend on the magnitude of the momentum $|\mathbf{k}|$, which implies that $\mathrm{Im}\chi^R$,$\mathrm{Re}\mathcal{R}$ are ω-even functions, so that the integral Eq. (6) can be written for $\omega \geq 0$.

The highly nontrivial and theory dependent information is contained in the subtraction terms $\tilde{\chi}, \delta\chi^\infty$ which are determined from the large-frequency behavior, i.e., from the leading terms in OPE, Eq. (3). We now derive some general properties of the subtractions. First, the main subtraction $\delta\chi = \chi - \tilde{\chi}$ is generally required because the leading asymptotic $|k| \gg T$ behavior of $\chi^E(k)$ is $|k|^{2\Delta_\mathcal{O}-D}$, and most operators have $2\Delta_\mathcal{O} > D$. In contrast, in almost all cases the subtraction of a constant is not needed, i.e., $\delta\chi^\infty = 0$. Indeed, from Eq. (4) this constant can be nonzero only if the $\mathcal{O}\mathcal{O}$ OPE contains an operator $\mathcal{O}_*$ with dimension $\Delta_* = 2\Delta_\mathcal{O} - D$. Moreover, $\mathcal{O}_*$ needs to have a nonzero TEV. In which case, $\delta\chi^\infty \propto \langle \mathcal{O}_* \rangle_T$ and the constant of proportionality is the corresponding OPE coefficient. A further necessary condition for $\delta\chi^\infty \neq 0$ is $\Delta_\mathcal{O} \geq (3D-2)/4$ because of the unitarity bound [13] on Δ_*. A *generic* case where the subtraction $\delta\chi^\infty$ appears is for a 2-point function of $T_{\mu\nu}$, the stress tensor, because the latter has scaling dimension D. In this case $\mathcal{O}_* = T_{\mu\nu}$, since the stress tensor generally appears in the $T_{\mu\nu}T_{\lambda\varepsilon}$ OPE. Below we will the consequences of this for the shear viscosity.

$O(N)$ model.—We now apply the above general results to the QC point of the quantum $O(N)$ model [3,19] in dimensions $2 < D < 4$. This is the famous Wilson-Fisher conformal fixed point. It describes a variety of experimentally relevant quantum phase transitions: Ising ($N=1$), XY ($N=2$), etc. An exact solution exists at $N=\infty$, which we will use to perform nontrivial checks. As a field theory, the $O(N)$ (nonlinear sigma) model is defined by the action $S = \int d^Dx (1/g)\partial_\mu\varphi_a\partial_\mu\varphi_a$, where $\varphi_a(x)$ is a real N-component vector field of fixed norm $\varphi_a\varphi_a = 1$. As the coupling g is increased the system undergoes a QC phase transition at $g = g_c$ from a broken symmetry phase to a symmetric one for $g > g_c$ [Fig. 1(a)].

For our asymptotics and sum rule analysis we need the list of operators $(\mathcal{O}_n, \Delta_n)$ with low dimensions $\Delta_n \leq D$. These are known from large-N and small $(4-D)$ expansions [1,2], Monte Carlo calculations [1], nonperturbative bootstrap [4,20,21], etc. The first one being the order parameter field ϕ_a with dimension $\Delta_\phi = (D-2+\eta_\phi)/2$, where η_ϕ is the field's anomalous dimension. The following $O(N)$-invariant operators will also appear: the "thermal" operator $(\mathcal{O}_g, \Delta_g)$, the conserved currents $(J^\mu_{ab}, D-1)$, and the stress tensor $(T_{\mu\nu}, D)$. The dimensions of the currents and stress tensor receive no anomalous corrections because they are protected by symmetries. The operator $\mathcal{O}_g$ (often denoted by ε in the context of the Ising model) is associated with the Lagrange multiplier field $\lambda(x)$ that constrains $\varphi_a\varphi_a = 1$ in the $O(N)$ model. It has dimension $\Delta_g = D - 1/\nu$, where ν is the correlation length exponent; for the $D=3$ Ising case [1,4], $\Delta_g = 1.413$. It is directly related to the singlet ϕ^2, and tunes the system away from the QC point. Being the only relevant $O(N)$-symmetric scalar, it is the most important operator as it dominates the asymptotic quantum dynamics: we will see that it generally gives the first finite-T correction. This was recently shown [6] to be the case for the conductivity of the $O(N)$ model, and observed numerically [6] for $N=2$. Given the generality of our OPE analysis, we infer that this "dominance" of the relevant symmetric scalar is a generic property of QC transitions.

Order parameter susceptibility.—We first study $\chi_{ab}(k) = \langle \phi_a(-k)\phi_b(k)\rangle_T$, i.e., the order parameter susceptibility. It is one of the simplest observables, and yields the low-energy staggered spin susceptibility of quantum antiferromagnets with transitions in the $O(N)$ universality class. We begin by analyzing its asymptotics. By symmetry, and from the knowledge of the operators with low dimensions we can write the leading terms in the $\phi_a(x)\phi_b(0)$ OPE:

$$\frac{C_\phi}{x^{2\Delta_\phi}} + \frac{C_{\phi\phi g}\mathcal{O}_g(0)}{x^{2\Delta_\phi - \Delta_g}} + \frac{C_{\phi\phi T} x_\mu x_\nu T_{\mu\nu}(0)}{x^{2\Delta_\phi - D + 2}} + \cdots, \tag{7}$$

where we focus on $a, b = 1$ since χ_{ab} is diagonal by virtue of the $O(N)$ symmetry. We have omitted the contribution from the currents J^μ_{ab} because they have vanishing TEV (no excess charge or net current in the thermal ensemble). Taking the TEV of Eq. (7) gives the asymptotic behavior

$$\chi^E_{11}(i\omega_n, \mathbf{k}) = |k|^{2\Delta_\phi - D}\left[\mathcal{C}_\phi + \mathcal{C}_{\phi\phi g} d_g \left|\frac{T}{k}\right|^{\Delta_g} + \mathcal{C}_{\phi\phi T}\frac{k_\mu k_\nu}{k^2} d_T^{\mu\nu}\left|\frac{T}{k}\right|^D + \cdots\right]. \tag{8}$$

$C_\#/\mathcal{C}_\#$ in Eqs. (7) and (8) are real OPE coefficients in position and momentum space, respectively, which can be obtained from ground state 3-point functions. As anticipated, the first subleading term comes from the relevant scalar $\mathcal{O}_g$. The next term arises from the stress tensor, where $\langle T_{\mu\nu}\rangle_T = d_T^{\mu\nu} T^D$ is diagonal. At $N=\infty$, $\Delta_\phi = (D-2)/2$ saturates the unitarity bound, but finite N fluctuations lead to a small anomalous dimension $\eta_\phi \ll 1$ [1,2,4,20]. The OPE coefficients $\mathcal{C}_{\phi\phi g}, \mathcal{C}_{\phi\phi T}$ are generally finite and can be computed using a $1/N$ expansion for instance. $C_{\phi\phi g}$ has been computed using bootstrap [22] and Monte Carlo calculations [23] for $N=1$. From the above expansion, we can derive the sum rule for χ_{ab}. First, for any N, χ_{ab}

decays sufficiently fast at large frequencies so that the subtractions vanish, $\tilde{\chi}_{ab} = \delta\chi^{\infty}_{ab} = 0$, and the sum rule takes its simplest form:

$$\int_0^\infty d\omega \mathrm{Re}\mathcal{R}_{ab}(\omega, \mathbf{k}) = 0, \tag{9}$$

where $R_{ab}(\omega, \mathbf{k}) = \chi^R_{ab}(\omega, \mathbf{k})/(i\omega - 0^+)$ is the response.

When $N = \infty$, we have the exact solution for $2 < D < 4$: $\chi^{\mathrm{E}}_{ab}(i\omega_n, \mathbf{k}) = \delta_{ab}/(\omega_n^2 + \mathbf{k}^2 + m_T^2)$, where $m_T = \Theta_d T$ is the thermal mass, and Θ_d is a positive number [11,24]. Expanding for $|k| \gg T$, we get $\chi^{\mathrm{E}}_{ab}(k) = (1/k^2)[1 - (m_T/k)^2 + (m_T/k)^4 + \cdots]$. In agreement with the OPE, Eq. (7), the subleading term $-m_T^2/k^4$ has $\Delta_g^{N=\infty} = 2$ (i.e., $1/\nu = D - 2$) and is proportional to $\langle \mathcal{O}_g\rangle_T = \sqrt{N} m_T^2$. This later TEV is evaluated [6] in the $N = \infty$ limit. We note the absence of a contribution from the stress tensor, $\sim m_T^3/|k|^5$. Although the real-space OPE coefficient $C_{\phi\phi T}$ in Eq. (7) is nonzero, upon Fourier transforming to k space, that term does not contribute to the large-k behavior. This is an artifact of $N = \infty$, where ϕ has no anomalous dimension. Finally, the sum rule Eq. (9) can be easily checked as the spectral function is a sum of (quasiparticle) delta functions.

Scalar susceptibility.—The scalar susceptibility χ_s is the 2-point function of the thermal operator, $\langle \mathcal{O}_g(-k)\mathcal{O}_g(k)\rangle_T$. It has recently been the focus of attention in the study of the amplitude “Higgs” mode [25–30]. Again, we first examine the $\mathcal{O}_g\mathcal{O}_g$ OPE. The terms relevant here are given, *mutatis mutandis*, by Eq. (7). This then leads to the large-k expansion Eq. (8) with (ϕ_a, Δ_ϕ) replaced by $(\mathcal{O}_g, \Delta_g)$. With these information, we can derive the sum rule for χ_s. First, $\delta\chi_s^\infty = 0$ since there is no $O(N)$ singlet with dimension $\Delta_* = 2\Delta_g - D$ in the spectrum. The other ingredient needed to build the sum rule is the term removing the large-ω divergence, $\tilde{\chi}_s$. In this case, it is simply the ground state value of χ_s at $\mathbf{k} = \mathbf{0}$: $\tilde{\chi}_s(\Omega) = \chi_s^{T=0}(\Omega, \mathbf{0}) = C_g \Omega^{2\Delta_g - D}$. The sum rule reads

$$\int_0^\infty d\omega \mathrm{Re}[\mathcal{R}_s(\omega, \mathbf{k}) - \mathcal{R}_s^{T=0}(\omega, \mathbf{0})] = 0. \tag{10}$$

We can again carry out the asymptotic analysis exactly for $N \to \infty$. The result is [11] $\chi_s^E(k) = -(N/a_0)|k|^{4-D}[1 - a_g(T/k)^2 - a_T(T/|k|)^D]$, where $a_\#$ are D-dependent constants. Interestingly, the coefficient of the subleading term, a_g, vanishes exactly for $D = 3$. This comes from the somewhat surprising fact that the thermal operator $\mathcal{O}_g$ does not appear by itself in the $\mathcal{O}_g\mathcal{O}_g$ OPE when $D = 3$ in the $N = \infty$ limit. In other words, the $C_{ggg}^{D=3}$ OPE coefficient vanishes. This does not happen for $D \neq 3$, and we do not expect it to hold at finite N in $D = 3$. Indeed, for the Ising case this coefficient was recently computed using Monte Carlo methods and found to be finite [23]. Finally, the sum rule Eq. (10) can be checked numerically [11] at $N = \infty$.

Dynamical shear viscosity.—Finally, we examine a correlator involving the stress tensor. Not only is this of fundamental interest because it can be defined for any CFT, but it will also reveal the full complexity of the sum rule. We consider the dynamical shear viscosity, $\eta(\omega, \mathbf{k}) = \chi_\eta^R(\omega, \mathbf{k})/(i\omega - 0^+)$, obtained from the T_{xy} 2-point function, χ_η^R. T_{xy} measures the flux of x momentum in the y direction, and η probes the system’s resistance against momentum gradients. The asymptotic behavior of η follows from the $T_{xy}T_{xy}$ OPE, which we here formulate in momentum space:

$$\begin{aligned}\lim_{|k|\gg|p|} T_{xy}(k)T_{xy}(-k+p) &= \mathcal{C}_T|k|^D\delta(p) + \mathcal{C}_{TTg}|k|^{D-\Delta_g}\mathcal{O}_g(p)\\ &\quad + \mathcal{C}^{\mu\nu}_{TTT}T_{\mu\nu}(p) + \cdots.\end{aligned} \tag{11}$$

This can then be used to derive a sum rule for η, which is more involved than for the response functions considered above. For one, $\delta\chi_\eta^\infty = \mathcal{C}^{\mu\nu}_{TTT}\langle T_{\mu\nu}\rangle_T$ is nonzero, as was explained above on general grounds for 2-point functions involving $T_{\mu\nu}$. Second, the subtraction involved in $\delta\chi_\eta$ is *temperature dependent* because $\mathcal{O}_g$ is relevant. This leads to the following sum rule for the shear response:

$$\int_0^\infty d\omega\, \mathrm{Re}[\eta(\omega, \mathbf{k}) - \mathcal{C}_T\omega^d - \mathcal{A}(\omega/i)^{d-\Delta_g}] = c_\eta P, \tag{12}$$

where $d = D - 1$, $\mathcal{A} = \mathcal{C}_{TTg}\langle\mathcal{O}_g\rangle_T$, $P = \langle T_{xx}\rangle_T$ is the pressure of the CFT, and $c_\eta = -\pi\,\mathcal{C}^{\mu\nu}_{TTT}\langle T_{\mu\nu}\rangle_T/(2P)$ is a dimensionless constant. The second term in the integrand is $\eta^{T=0}(\omega, \mathbf{0})$ and mirrors the subtraction in the scalar sum rule. The third one depends on temperature via $\mathcal{A} \propto T^{\Delta_g}$ and scales with a nontrivial ω power depending on the correlation length exponent ν via $\Delta_g = D - 1/\nu$. Some QC theories are simpler in that they lack a relevant scalar that condenses at $T > 0$, as we now discuss.

We contrast the above shear sum rule with the simpler ones obtained [12,31] for $\mathcal{N} = 4$ super Yang-Mills and pure Yang-Mills theory, which are gauge theories in $D = 4$. In those cases, the result is as in Eq. (12) except that the third term in the integrand is absent. This stems from the fact that those theories do not contain a symmetric relevant scalar like $\mathcal{O}_g$; i.e., they are not obtained by fine tuning a symmetric “mass” term to zero. The massless version of QED in $D = 3$ with many Dirac fermions coupled to a $U(1)$ gauge field also satisfies this property, being a stable *phase*. It will thus have a shear sum rule of the same form as super Yang-Mills theory. Finally, we note that shear sum rules analogous to Eq. (12) were derived in the context of strongly interacting ultracold Fermi gases [32–34], which generally do not have emergent Lorentz symmetry.

Outlook.—Our nonperturbative results, via the operator product expansion, for the asymptotics and sum rules apply to a wide class of conformal QC points, many of which describe experimentally relevant systems. It will be interesting to apply the program described in this article to theories other than to the $O(N)$ Wilson-Fisher fixed point, treated here, or even to nonconformal QC systems. The strong constraints we have derived will also be useful for the analysis of numerical and experimental data. For instance, quantum Monte Carlo is a powerful tool to study QC dynamics in *imaginary* time [6,35–39], and can be used to study the asymptotic regime where the OPE analysis applies, as was recently shown [6] for the conductivity. The asymptotics and sum rules will also help with the difficult task of analytically continuing the imaginary time data to real time by constraining the allowed scaling functions. Along those lines, our results can be used with a novel method [6,36] of analytic continuation based on the AdS/CFT holographic principle [40]: Specific data about a QC theory can be encoded in holographic physically motivated *Ansätze* for the scaling functions. These can then be used to perform the continuation.

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