Black Holes in Higher Derivative Gravity

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Extensions of Einstein gravity with higher-order derivative terms arise in string theory and other effective theories, as well as being of interest in their own right. In this Letter we study static black-hole solutions in the example of Einstein gravity with additional quadratic curvature terms. A Lichnerowicz-type theorem simplifies the analysis by establishing that they must have vanishing Ricci scalar curvature. By numerical methods we then demonstrate the existence of further black-hole solutions over and above the Schwarzschild solution. We discuss some of their thermodynamic properties, and show that they obey the first law of thermodynamics.

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The well-known problem of the nonrenormalizability of Einstein gravity has given rise to many attempts to view it as an effective low-energy theory that will receive higher-order corrections that become important as the energy scale increases (see, for example, Ref. [1]). In string theory, the Einstein-Hilbert action is just the first term in an infinite series of gravitational corrections built from powers of the curvature tensor and its derivatives. In other approaches, only a finite number of additional terms might be added. It was shown in Ref. [2] that if one adds all possible quadratic curvature invariants to the usual Einstein-Hilbert action one obtains a renormalizable theory, albeit at the price of introducing ghostlike modes. Arguments have been given for why these might not be fatal to the theory (for example, see Ref. [3] for a recent discussion). In any case, it is worthwhile to study in detail the properties of the theory of Einstein gravity with added quadratic curvature terms, in order to shed light on the question of whether it has irredeemable pathologies or whether they can be controlled in some manner.

Black holes are the most fundamental objects in a theory of gravity, and they provide powerful probes for studying some of the more subtle global aspects of the theory. It is therefore of considerable interest to investigate the structure of black-hole solutions in theories of gravity with higher-order curvature terms. In this Letter, we report on some investigations of the static, spherically symmetric black-hole solutions in four-dimensional Einstein-Hilbert gravity with added quadratic curvature terms, for which the most general action can be taken to be

$$I = \int d^4x \sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2), \qquad (1)$$

where α, β, and γ are constants and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. We shall work in units where we set $\gamma = 1$, and the equations of motion following from Eq. (1) are then

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 4\alpha B_{\mu\nu} + 2\beta R\left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}\right) + 2\beta(g_{\mu\nu}\Box R - \nabla_\mu \nabla_\nu R) = 0, \qquad (2)$$

where $B_{\mu\nu} = (\nabla^\rho \nabla^\sigma + \frac{1}{2} R^{\rho\sigma}) C_{\mu\rho\nu\sigma}$ is the Bach tensor, which is trace free.

In general, the theory describes a system with a massive spin-2 mode with mass squared $m_2^2 = 1/(2\alpha)$ and a massive spin-0 mode with mass squared $m_0^2 = 1/(6\beta)$, in addition to the massless spin-2 graviton. These massive modes will be associated with rising and falling Yukawa type behavior in the metric modes near infinity [4], of the form $(1/r)e^{\pm m_2 r}$ and $(1/r)e^{\pm m_0 r}$. In particular, one can expect that if the generic initial data are set at some small distance, the rising exponentials will eventually dominate, leading to singular asymptotic behavior. In seeking black-hole solutions, the question then arises as to whether the rising exponentials can be avoided for appropriately finely tuned initial data.

It can easily be seen that any solution of pure Einstein gravity will also be a solution of Eq. (2), and so in particular the usual Schwarzschild black hole continues to be a solution in the higher-order theory. The question we wish to address, then, is whether there exist any other static black hole solutions, over and above the Schwarzschild solution.

Static, spherically symmetric black-hole solutions have been investigated in Ref. [5], using generalizations of the Lichnerowicz and Israel theorems for Einstein gravity. Since we will arrive at somewhat different conclusions, we shall briefly summarize the key elements in Ref. [5], although derived in a different notation. We consider static metrics of the form $ds_4^2 = -\lambda^2 dt^2 + h_{ij} dx^i dx^j$, where λ and h_{ij} are functions only of the three spatial coordinates x^i. Taking the trace of the field equations (2) gives $\beta(\Box - m_0^2)R = 0$. We then multiply this by λR and integrate over the spatial domain from a putative horizon out to infinity. Expressed in terms of the covariant derivative D_i with respect to the spatial 3-metric h_{ij}, this gives

$$\int \sqrt{h} d^3x [D^i(\lambda R D_i R) - \lambda(D_i R)^2 - m_0^2 \lambda R^2] = 0. \quad (3)$$

Since λ vanishes on the horizon, it follows that if $D_i R$ goes to zero sufficiently rapidly at spatial infinity the total derivative (i.e., surface term) gives no contribution, and the nonpositivity of the remaining terms then implies $R = 0$. In other words, as shown in Ref. [5], any static black-hole solution of Eq. (1) must have a vanishing Ricci scalar. This leads to a great simplification, and it means that one can, without loss of generality, study the case of pure Einstein-Weyl gravity [i.e., Eq. (1) with $\beta = 0$], since obviously the term quadratic in R makes no contribution to the field equations for a configuration with $R = 0$. Furthermore, the trace of the field equations (2) for Einstein-Weyl gravity immediately implies $R = 0$. In fact, the two differential equations for h and f are both now of only second order in derivatives.

The second stage of the discussion in Ref. [5] then involved looking at the remaining content of Eq. (2), i.e., the nontrace part. According to Ref. [5], this led to another integral identity that then implied, under certain assumptions, that $R_{\mu\nu} = 0$. If this were correct, then the conclusion would be that the usual Schwarzschild solution is the only static black hole solution of the theory described by Eq. (1). However, we find that there are sign errors in the expression given in Ref. [5]. Setting $R = 0$, as already argued above, multiplying Eq. (2) by $\lambda R^{\mu\nu}$, and then integrating over the spatial region outside the horizon gives

$$\int \sqrt{h} d^3x \bigg[D^i W_i - \frac{1}{4}\lambda(D_i \bar{R} - 4D^j R_{ij})^2 + 4\lambda(D^j R_{ij})^2 - 4\lambda(D_{[i} R_{j]k})^2 + \lambda(D_i R_{jk})^2 - \frac{1}{4}\lambda \bar{R}^2 (m_2^2 + \bar{R}) - \lambda(m_2^2 R^{ij} R_{ij} - 2R^{ij} R_{jk} R^k{}_i) \bigg] = 0, \quad (4)$$

where $W_i = \lambda R^{jk} D_i R_{jk} + \frac{1}{4}\lambda \bar{R} D_i \bar{R} - 2\lambda R^{jk} D_j R_{ik} - \lambda \bar{R} D_j R_i{}^j$ and $\bar{R}$ is the Ricci scalar of the spatial metric h_{ij}. Although the surface term will give zero, the mix of positive and negative signs in the bulk terms precludes one from obtaining any kind of vanishing theorem for the Ricci tensor of the four-dimensional metric. This raises the intriguing possibility that there might in fact exist static, spherically symmetric black-hole solutions over and above the Schwarzschild solution.

The equations of motion following from Eq. (1) are too complicated to be able to solve explicitly, even for the case of the static, spherically symmetric ansatz

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (5)$$

In our work, we have therefore carried out a numerical investigation of the solutions. To do this, we begin by supposing that there exists a black-hole horizon at some radius $r = r_0 > 0$, at which the metric functions h and f vanish, and we then obtain near-horizon Taylor expansions for $h(r)$ and $f(r)$, of the form

$$h(r) = c[(r - r_0) + h_2(r - r_0)^2 + h_3(r - r_0)^3 + \cdots],$$
$$f(r) = f_1(r - r_0) + f_2(r - r_0)^2 + f_3(r - r_0)^3 + \cdots. \quad (6)$$

Substituting into the equations of motion (2), with β set to zero for the reasons discussed above, the coefficients h_i and f_i for $i \geq 2$ can be solved for in terms of the two nontrivial free parameters r_0 and f_1. There is also a "trivial" parameter, corresponding to the freedom to rescale the time coordinate, which we have accordingly written in the form of an overall scaling of $h(r)$. Thus, we have

$$h_2 = \frac{1 - 2f_1 r_0}{f_1 r_0^2} + \frac{1 - f_1 r_0}{8\alpha f_1^2 r_0},$$
$$f_2 = \frac{1 - 2f_1 r_0}{r_0^2} - \frac{3(1 - f_1 r_0)}{8\alpha f_1 r_0},$$

and so on. [We used Taylor expansions to $\mathcal{O}((r - r_0)^9)$ in our numerical integrations.] The Schwarzschild solution corresponds to $f_1 = 1/r_0$, and so it is convenient to parametrize f_1 as

$$f_1 = \frac{1 + \delta}{r_0}, \quad (7)$$

with nonvanishing δ characterizing the extent to which the near-horizon solution deviates from the Schwarzschild solution.

We use the expansions to set the initial data at a radius r_i just outside the horizon, and then use numerical routines in MATHEMATICA to integrate the equations out to a large radius. Generically, one finds that for a given choice of the parameters r_0 and δ the solution rapidly becomes singular as one integrates outwards from $r = r_i$, as expected in view of our earlier observations about the rising Yukawa terms in the asymptotic form for the metric. If we fix a particular value for r_0, we can then use the shooting method to try to home in on a special value of δ for which the outward integration can proceed without encountering a singularity. Of course, in practice, because of accuracy limitations in the integrations, the solution will always eventually become singular at large enough r. The signal for a good black-hole solution is that $f(r)$ and $h(r)$ should approach constants as

r increases [in fact Ricci scalar flatness implies $f(r)$ must approach 1], and that by stepping up the accuracy and precision goals in the calculations one can extend at will the maximum upper limit $r = r_f$ for which the smooth behavior can be achieved. In practice, by running the routines with accuracy and precision goals of order 20 decimal places, we have been able to obtain very clean and trustworthy solutions out to at least 60 times the horizon radius.

Our findings are that there exists a range of values for the horizon radius, bounded below by a certain multiple of the length $\sqrt{\alpha}$, for which we can obtain precisely one static black-hole solution in addition to the Schwarzschild solution. In order to make the statement of our results in the most concise possible way, it is convenient, without loss of mathematical generality, to make a specific choice for the value of α in Eq. (1). We shall take

$$\alpha = \frac{1}{2}. \tag{8}$$

We then find that for each choice of $r_0 > r_0^{\rm min}$, where

$$r_0^{\rm min} \approx 0.876, \tag{9}$$

we can find a non-Schwarzschild static black hole. For each such r_0, there is a corresponding value $\delta = \delta^*$ of the "non-Schwarzschild parameter" that yields the black-hole solution without a singularity at spatial infinity. As r_0 is taken closer and closer to the value $r_0^{\rm min}$, the required value δ^* becomes smaller and smaller, tending to zero at $r_0 = r_0^{\rm min}$. Thus, the Schwarzschild and the non-Schwarzschild black holes "coalesce" as $r_0 = r_0^{\rm min}$ is approached.

As r_0 is increased above $r_0^{\rm min}$, the Schwarzschild and non-Schwarzschild black-hole solutions separate more from one another (and in particular the required value of δ increases). The mass of the Schwarzschild black hole is simply $\frac{1}{2} r_0$, and thus it increases linearly as r_0 increases. By contrast, the mass of the non-Schwarzschild black hole decreases as r_0 increases, until at $r_0 = r_0^{m=0}$ it becomes massless, where

$$r_0^{m=0} \approx 1.143. \tag{10}$$

[The definition of mass in higher-derivative theories was discussed in Refs. [6,7]. For asymptotically flat black holes it is just $\frac{1}{2}$ the coefficient of $1/r$ in g_{tt} (assuming g_{tt} is normalized canonically at infinity).] Interestingly, if r_0 is increased beyond $r_0^{m=0}$, one can still obtain a non-Schwarzschild black-hole solution for an appropriate choice of δ, but now the mass is actually negative. In other words there is still a regular horizon, and the metric is asymptotically flat at large distances, but the metric function f now rises above 1 as r increases from r_i, before sinking down to 1 again in the asymptotic region. Figure 1 shows the masses of the Schwarzschild and non-Schwarzschild black holes as a function of r_0, and their masses as functions of their Hawking temperatures.

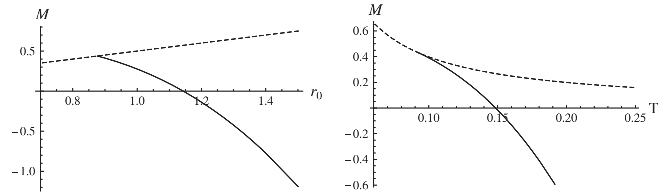

FIG. 1. The masses as functions of r_0 (left plot) and as functions of the Hawking temperatures (right plot) for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.

The maximum possible mass for the non-Schwarzschild black hole, attained when $r_0 = r_0^{\rm min}$, is given by $M^{\rm max} = \frac{1}{2} r_0^{\rm min} \approx 0.438$. From the slope of $M(T)$ it can be seen that the specific heat $C = \partial M/\partial T$ is negative for both black holes, and more negative for the non-Schwarzschild black hole at a given temperature.

The plots of the metric functions f and h for the examples of a positive-mass black hole with $r_0 = 1$ and a negative-mass black hole with $r_0 = 2$ are shown in Fig. 2.

Having established the existence of the non-Schwarzschild black holes, it is instructive to study some of their thermodynamic properties, and to compare these with the properties of the Schwarzschild black holes. In order to do this, we collected the numerical results for a sequence of black-hole solutions with r_0 in the range $r_0^{\rm min} \approx 0.876 < r_0 < 1.5$, and then fitted the data to appropriate polynomials. Because we are working with a higher-derivative theory, the entropy is not simply given by one quarter of the area of the event horizon, and instead we need to use the formula derived by Wald [8,9]. This has been evaluated for the ansatz (5) in quadratic curvature gravities in Ref. [10], and applied to our case with $\beta = 0$ and $\gamma = 1$ in Eq. (1) this gives $S = \pi r_0^2 + 4\pi\alpha(1 - f_1 r_0) = \pi r_0^2 - 4\pi\alpha\delta^*$. (There is a freedom to add a constant multiple of the Gauss-Bonnet invariant to the Lagrangian, which shifts the entropy by a parameter-independent constant without affecting the equations of motion. We have used this to ensure the entropy of the Schwarzschild black hole vanishes when the mass vanishes.) We then find that the mass and the temperature of these non-Schwarzschild black holes, as a function of the entropy, take the form

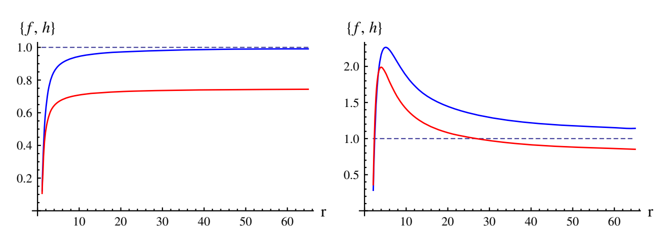

FIG. 2 (color online). The non-Schwarzschild black hole for $r_0 = 1$ (left plot) and $r_0 = 2$ (right plot). In each plot the upper curve is $f(r)$ and the lower curve is $h(r)$. For clarity we have chosen a rescaling of h so that it approaches $\frac{3}{4}$, rather than 1, to avoid an asymptotic overlap of the curves.

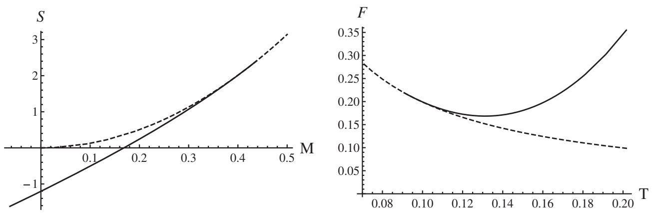

FIG. 3. The first plot shows the entropy as a function of mass and the second shows the free energy $F = M - TS$ as a function of T for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.

$$M \approx 0.168 + 0.131S - 0.00749S^2 - 0.000139S^3 + \cdots,$$
$$T \approx 0.131 - 0.0151S - 0.000428S^2 + \cdots. \tag{11}$$

It can be seen that $\partial M/\partial S \approx 0.131 - 0.0150S - 0.000417S^2$, which is very close to the expression for the temperature. Thus, the non-Schwarzschild black holes are seen to obey the first law $dM = TdS$ to quite a high precision. Note that the expressions for M and T as a function of S for the Schwarzschild black holes are very different in form, with $M = (S/4\pi)^{1/2}$ and $T = \frac{1}{4}(\pi S)^{-1/2}$.

It is interesting to note that the entropy of the non-Schwarzschild black hole of a given mass is always less than the entropy of the Schwarzschild black hole of the same mass. The two entropies approach each other asymptotically as r_0 approaches $r_0^{\rm min}$. This can be seen in the left-hand plot in Fig. 3. It is also of interest to look at the free energy $F = M - TS$ as a function of temperature. This is shown in the right-hand plot in Fig. 3. It can be seen that the free energy is always larger for the non-Schwarzschild black hole at a given temperature, with the two curves again meeting at the lower limit when $r_0 = r_0^{\rm min}$.

In this Letter, we have used black holes to probe some of the consequences of interpreting the action (1) as a complete classical action in its own right. We have seen that there exists a second branch of static, spherically symmetric black holes, over and above the Schwarzschild solutions. These are not Ricci flat, although they do have a vanishing Ricci scalar. Restoring the factors of α and γ that we fixed in our numerical simulations, the second branch of black holes has masses, which can become negative, bounded approximately by $M \leq 0.438\sqrt{2\alpha\gamma}$. Thus, in a regime where α is small, which one might hope would correspond to a small correction to Einstein gravity, the second branch of black holes will be tiny, and will actually have very large curvature near the horizon, thus tending to invalidate the requirement that the curvature squared should be small. The fact that their mass can be negative, violating the usual positive-mass theorem of standard Einstein gravity, indicates that the ghostlike nature of the quadratically corrected action is becoming dominant in this regime; one may view the negative-mass black holes as condensates dominated by the contribution of the ghostlike massive spin-2 modes. It could be viewed as a satisfactory outcome of our investigation that the only indications of the existence of black holes with potentially pathological properties in the quadratic-curvature theories occur in a regime where yet higher-order corrections, as in string theory, are going to be important also. It would be interesting to obtain analytical proofs of the existence of the numerical solutions we have found. Although this could be challenging, it might perhaps be easier to obtain restricted no-hair theorems that confirm the apparent absence of non-Schwarzschild black holes outside the parameter range where we have found them.

Naturally, the fourth-order equations of motion (2) have wider classes of solutions than the black-hole solutions with horizons that we have considered here. These were initially investigated in Ref. [4] and will be given a more detailed analysis, along with a more extensive analysis of the black hole solutions, in Ref. [11].

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