Quantum Critical Spin-2 Chain with Emergent SU(3) Symmetry

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We study the quantum critical phase of an SU(2) symmetric spin-2 chain obtained from spin-2 bosons in a one-dimensional lattice. We obtain the scaling of the finite-size energies and entanglement entropy by exact diagonalization and density-matrix renormalization group methods. From the numerical results of the energy spectra, central charge, and scaling dimension we identify the conformal field theory describing the whole critical phase to be the $\mathrm{SU}(3)_1$ Wess-Zumino-Witten model. We find that, while the Hamiltonian is only SU(2) invariant, in this critical phase there is an emergent SU(3) symmetry in the thermodynamic limit.

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Cold atomic gases in optical lattices have become an ideal framework for studying quantum many-body systems in recent years [1]. In particular, various schemes have been proposed to study quantum magnetism [2]. For spin $1/2$ systems, simulation of the Ising model has been realized using bosons in a tilted optical lattice [3]. It has also been proposed that the spin $1/2$ XYZ Heisenberg model can be realized using p-orbital bosons [4]. This rapid progress in cold atomic physics results in a considerable renewal of interest to study models with higher spins or higher symmetries, especially for models that are potentially realizable by cold atomic systems. A natural direction is to study spinor bosons and their novel phases. For example, it has been proposed that the spin-1 bilinear biquadractic (BB) model can be engineered using spin-1 cold bosons in an optical lattice [5,6]. Furthermore, the phase diagram of spin-1 bosons in a one-dimensional (1D) lattice has been studied numerically and compared to the spin-1 BB model [7]. Since spin-2 bosons are available and have been experimentally studied [8–11], it is of great interest to explore the phases realizable by spin-2 bosons. On the other hand, it has also been pointed out that fermions with hyperfine spin $F = 3/2$ can be used to realize models with SO(5) symmetry [12], or to realize an SU(3) spin chain by effectively suppressing the occupation of one of the middle states [13]. Possibilities to realize higher SU(N) symmetry have also been proposed [14,15]. Along this line, spin dynamics and correlation have been studied experimentally using cold fermions with effective spin ranging from $1/2$ to $9/2$ [16,17]. Another interesting problem is to explore symmetries that emerge in the low energy limit of the models. Indeed, different aspects of emergent symmetries have been discussed widely in the recent literature. Examples include SO(5) and SO(8) symmetries in high temperature superconductors and two-leg ladders [18,19], E_8 symmetry in Ising spin chains under a critical transverse field [20], emergent modular and translational symmetries for quantum Hall states [21] and fractional Chern insulators [22], and supersymmetry at sample boundaries of topological phases [23] or at critical or multicritical points separating different phases [19], especially for confinement-deconfinement or non-Landau phase transitions [24].

Recently we studied the phase diagram of spin-2 bosons in a 1D optical lattice with one particle per site and identified three possible phases for a finite system: ferromagnetic, dimerized, and trimerized phases [25]. Within the trimerized phase, if the system size is a multiple of 3, the ground state is a spin singlet with a finite-size gap and broken lattice symmetry. It was also shown that in the thermodynamic limit the system became gapless with unbroken lattice symmetry. The nature of this extended critical phase was, however, not fully determined. In this Letter we investigate the extended critical phase of the spin-2 boson in a 1D lattice. In particular, we identify the conformal field theory (CFT) describing the low energy physics of the whole critical phase. By using multiple diagnostic tools we show that in the thermodynamic limit the low energy physics of this critical phase is described by the $\mathrm{SU}(3)_1$ Wess-Zumino-Witten (WZW) model. This is the main result of this work.

We begin with the Hamiltonian, which is obtained from spin-2 bosons in a 1D optical lattice with one particle per site in the limit of $t/U_S \ll 1$. Here, t is the hopping between nearest neighbors and U_S is the Hubbard repulsion for two particles with spin S on the same site. Within second order perturbation theory the effective Hamiltonian reads

$$H = \sum_i H_{i,i+1} = \sum_i \epsilon_o P_{0,i,i+1} + \epsilon_2 P_{2,i,i+1} + \epsilon_4 P_{4,i,i+1}, \tag{1}$$

where $\epsilon_S = -4t^2/U_S$. Here, i is the site index, and $P_{S,i,i+1}$ denotes the projection operator for sites i and $i+1$ onto a state with total spin S. We focus on the regime with $U_S > 0$ (hence $\epsilon_S < 0$) to ensure stability for one particle per site. We use the non-Abelian density matrix renormalization group (DMRG) method that preserves the SU(2) symmetry. This not only increases the accuracy and but also allows us to target any specific total SU(2) spin sector. Within the DMRG, however, it is more convenient to explicitly express $H_{i,i+1}$ in terms of spin-2 operators $\mathbf{S}_i$, resulting in $H_{i,i+1} = \sum_{n=1}^{4} \alpha_n(\epsilon_0, \epsilon_2, \epsilon_4)(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^n$, where the expressions of α_n can be found in Ref. [25]. While the mean-field and exact phase diagrams have been studied [25,26], the nature of the critical phase is not known. In the following we will use the finite-size scaling of the energies and the entanglement entropy to identify the CFT. Since the whole critical phase is described by a unique CFT, it suffices to use one particular parameter set for the determination. Throughout this Letter, we will use $\vec{\epsilon} \equiv (\epsilon_0, \epsilon_2, \epsilon_4) = (0, -1, 0)$. This sets the system deep in the trimerized phase and far from the boundary of the ferromagnetic and dimerized phases.

We start with the exact diagonalization (ED) to obtain the low energy spectrum of small size systems with periodic boundary conditions. In Fig. 1 we show the excitation spectrum for $L = 15$. We set the ground state energy to be zero and use the energy of the $S = 2$ state at $k = 2\pi/L$ as the energy unit (for the reason described later in the Letter). We find that the ground state has $S = 0$ when the system size is a multiple of 3. Furthermore, we observe that there is a period-3 structure and soft modes develop at $k = \pm 2\pi/3(\mathrm{mod}\, 2\pi)$, with a cluster of low energy states

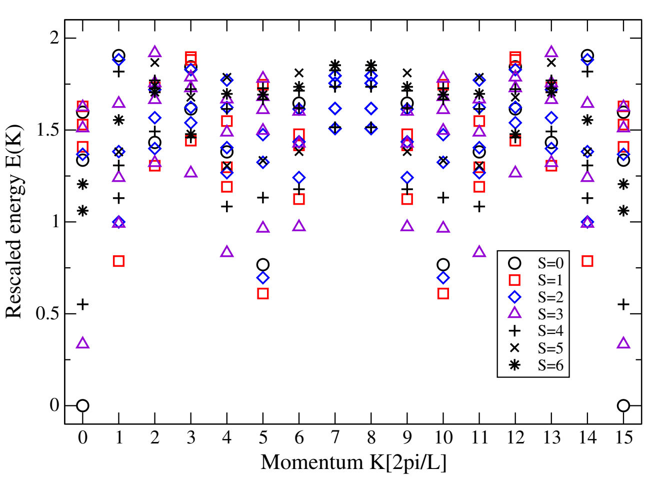

FIG. 1 (color online). Rescaled energy spectrum obtained from exact diagonalization with $L = 15$ and $\vec{\epsilon} = (0, -1, 0)$.

with $S = 0, 1, 2$. The period-3 structure and the fact that a gapped trimerized state is formed for a finite-size chain strongly suggests that the critical theory has approximate SU(3) symmetry at finite sizes. The observation above allows us to rule out some models as the potential CFT of the critical phase. Since our Hamiltonian is SU(2) symmetric, the corresponding CFT must contain SU(2) as a subgroup. $\mathrm{SU}_k(2)$ models are natural candidates; however, their spectrum would have minima at $k = \pi$ rather than $\pm 2\pi/3$, and the spin correlations have period 2 rather than period 3. Consequently, one can rule out $\mathrm{SU}_k(2)$ models as the associated CFT. On the other hand, the low energy spectrum is compatible with the DMRG calculation results for the SU(3) Heisenberg model in Ref. [27] and the ED results of the spin-1 BB model in the critical period-3 phase [28]. This makes the $\mathrm{SU}(3)_1$ WZW model with central charge $c = 2$ an appealing candidate.

For the $\mathrm{SU}(3)_1$ WZW model the soft modes at $k = \pm 2\pi/3$ should have a degeneracy that matches the dimension of the SU(3) representation. Here, because the bare Hamiltonian has SU(2) but not SU(3) symmetry, for a finite-size system it is natural that the energies at $k = \pm 2\pi/3$ will split according to the SU(2) spin as observed in Fig. 1. However, the $S = 3, 4$ states at $k = 0$ have lower energies than the states at $k = \pm 2\pi/3$. These low energy states are not expected if the critical theory is the $\mathrm{SU}(3)_1$ WZW model. We shall argue that the presence of these states is due to the proximity to the ferromagnetic phase and the lack of SU(3) symmetry in the Hamiltonian. We shall provide more details on this point below. Since the SU(3) symmetry is only emergent, it is then not surprising that non-CFT behavior is observed for small sizes. However, we expect that for sufficiently large sizes, their rescaled energies will move up and the low energy spectrum will become fully consistent with the CFT. In the following we shall use the DMRG method to calculate the energies of the states that are consistent with the $\mathrm{SU}(3)_1$ WZW model. The finite-size scaling of these energies then are used to identify the CFT. While we can no longer specify the momenta of the excited states in the DMRG, we can target a particular SU(2) spin sector to obtain the low energy states needed.

Before studying the excited states, we first use the finite-size scaling of the ground state energy to estimate the central charge c. According to the CFT, for a system with size L, the ground state energy $E_0(L)$ should scale as [35,36]

$$\frac{E_0(L)}{L} = \epsilon_\infty - \frac{\pi}{6L^2} c v, \tag{2}$$

where ϵ_∞ is the ground state energy per site in the thermodynamic limit and v is the spin-wave velocity. In Fig. 2(a) we show the finite-size scaling of the ground state energy $E_g(L)$ with $L = 12$–30, from which we find $(\pi/6)cv = 1.3968$. To find c we need to estimate the value

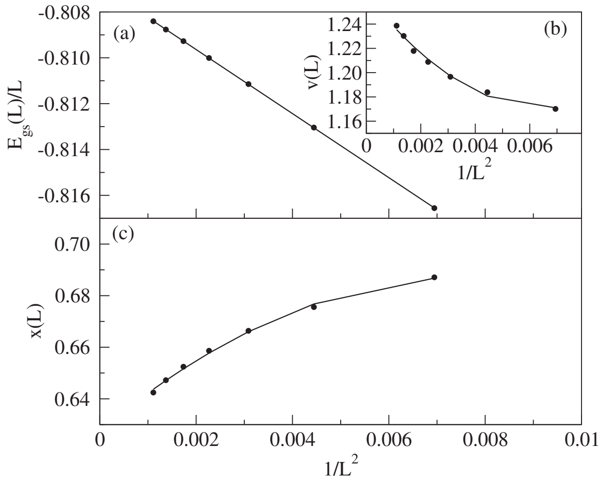

FIG. 2. (a) Finite-size scaling of the ground state energy $E_g(L)$. (b) Finite-size scaling of the spin-wave velocity $v(L)$. (c) Finite-size scaling of the scaling dimension $x(L)$, which is obtained from excited state energies by applying the sum rule as described in the text.

of v, which is determined by the energy of the state at $k = 2\pi/L$. In order to decide which state at $k = 2\pi/L$ should be used, we resort to the excitation spectrum of the spin-1 BB model. For the spin-1 BB model at the SU(3) point, we find that at $k = 2\pi/L$ the $S = 1, 2$ states are degenerate, indicating that the spin-wave excitation belongs to the **8** or $\bar{\mathbf{8}}$ representation of the SU(3) group [28]. In contrast, for our spin-2 model the $S = 1, 2$ states at $k = 2\pi/L$ are split due to the absence of SU(3) symmetry for the Hamiltonian. In this work we use the $S = 2$ state at $k = 2\pi/L$ to define the spin-wave velocity (alternative choices only make minor differences, see Ref. [28]) and set the energy scale in Fig. 1. It corresponds to the first excited state in the $S = 2$ sector and its energy can be obtained accurately by the non-Abelian DMRG method. In Fig. 2(b) we show the L-dependent velocity $v(L)$ as a function of L. By extrapolation of $v(L)$ as $v + a/L^2 + b/L^4$ we find $v = 1.2643$. Combined with the value of cv above we find that $c = 2.11$, which is very close to the expected value of $c = 2$ for the $\mathrm{SU}(3)_1$ WZW model [28].

To further support the CFT to be the $\mathrm{SU}(3)_1$ WZW model we estimate the scaling dimensions x_i, which are related to the scaling of the excited state energies $E_i(L)$ as

$$\frac{E_i(L) - E_0(L)}{L} = \frac{2\pi v}{L^2}\left(x_i + \frac{d_i}{\ln L}\right), \tag{3}$$

where $x_i = h_L + h_R$ and d_i is the coefficient of the logarithmic correction due to the marginal operator. Here, $h_L = h_L^0 + m_L$ and $h_R = h_R^0 + m_R$, where h_L^0, h_R^0 correspond to the holomorphic and antiholomorphic conformal weights of the primary fields, and m_L and m_R are non-negative integers describing descendant fields. When the system size is a multiple of 3, the lowest energy states at $k = \pm 2\pi/3$ are expected to belong to the representation $\mathbf{3} \times \bar{\mathbf{3}}$ with $h_L = h_R = 1/3$ [27]. Since the Hamiltonian has SU(2) symmetry but not SU(3) symmetry, the excited state energies split according to their SU(2) spin S. Furthermore, states with different S will pick up a different logarithmic correction d_S. Fortunately, they can be removed by using the sum rule $\sum_S (2S+1) d_S = 0$ [37]. By using the appropriate average of the excited states, one can define an L-dependent scaling dimension $x(L)$ [28]. In Fig. 2(c) we show $x(L)$ as a function of L. By extrapolating $x(L)$ as $x + a/L^2 + b/L^4$ we find $x = 0.628$. This agrees well with the expected value $x = 2/3$ of the $\mathrm{SU}(3)_1$ WZW model. The above results on the central charge c and scaling dimension x confirm the CFT to be the $\mathrm{SU}(3)_1$ WZW model. One can argue that another possibility is the $\mathrm{SU}(3)_2$ model. However, it has $c = 3.2$, which makes it unlikely to be the correct CFT. Furthermore, we find that the excitation spectrum of the $\mathrm{SU}(3)_2$ model is not compatible with our ED result. The $\mathrm{SU}(3)_2$ model thus can be safely excluded. In the calculations above we keep $m = 2800$ states of the SU(2) reduced basis. This is equivalent to about 25000 standard DMRG states. We have checked that m is big enough to ensure that the results are in the finite-size scaling regime as distinct from the finite entanglement regime [28].

In the above we obtained the central charge c by considering the ground state energy. In recent years the finite-size scaling of the entanglement entropy (EE) instead has been used intensively to estimate the central charge. Consider a system with periodic boundary conditions. The EE of a subsystem of size l is the von Neumann entropy of the reduced density matrix ρ_l of the subsystem: $S(l) = -\mathrm{Tr}(\rho_l \log \rho_l)$. It is known that for 1D conformal invariant critical system of size L, the EE scales asymptotically as [38]

$$S(l, L) = \frac{c}{3}\log\left[\frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right] + S_0, \tag{4}$$

where c is the central charge of the CFT and S_0 is a nonuniversal constant. Within the DMRG it is straightforward to calculate $S(l, L)$ once the optimized ground state is obtained. The accuracy of the DMRG calculation is controlled by the truncation dimension m and the result is numerically exact in the limit of $m \to \infty$. For a given pair of L and m we fit Eq. (4) to obtain an effective central charge $c(L, m)$. In Fig. 3 we plot $c(L, m)$ as a function of m for several system sizes. To our surprise we observe enormous finite-size and truncation effects. A similar phenomenon is also reported for the critical $S = 1/2$ XXZ chain, but only when the system is extremely close to the ferromagnetic boundary with an emerging ferromagnetic length scale [39]. Here, we have set the system to be far away from the ferromagnetic boundary but a pronounced effect is still observed. In general $c(L, m)$ is a

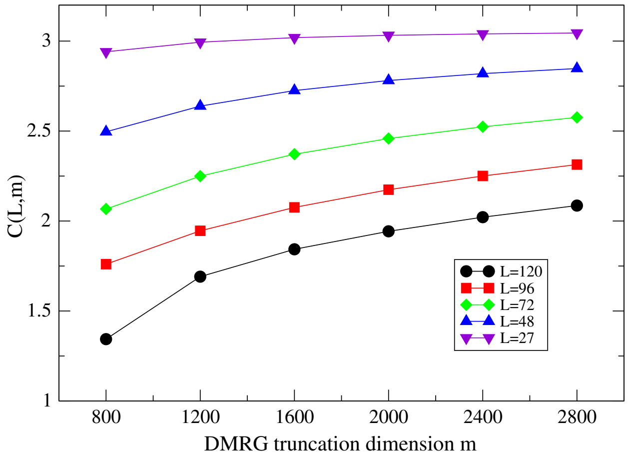

FIG. 3 (color online). $c(L,m)$ as a function of the DMRG truncation number m of SU(2) states for various systems sizes L with $\vec{\epsilon}=(0,-1,0)$. The $m=2800$ states are equivalent to at least about 25000 standard DMRG states.

decreasing function of L but an increasing function of m. The true central charge is obtained in the limit of $c=c(L\to\infty, m\to\infty)$, while $c(L, m\to\infty)$ provides an upper bound for c. Because of the enormous finite-size and truncation effects, we find it difficult to accurately determine the value of c from EE scaling, but certain bounds can be estimated. From the smaller size data where $c(L,m)$ already saturates as m increases, we find strong evidence that $c\lesssim 3$. This again excludes the $\mathrm{SU}(3)_2$ model with $c=3.2$. For the largest $L=120$ used we find that $c(L,2800)>2$, which is consistent with the $\mathrm{SU}_1(3)$ model with $c=2$. For an even larger L it is expected that an even larger $m\gg 2800$ ($m\gg 25000$ standard DMRG states) is needed, which is however beyond the typical size of DMRG calculations.

Some comments are now in order. First, we find that there are similarities between the extended critical phase of the spin-2 model and the extended critical period-3 phase of the spin-1 BB model [37]. For both models there are period-3 structures and gapped trimerized states are formed for finite-size chains. Furthermore, the corresponding critical theory is the $\mathrm{SU}(3)_1$ WZW model in both cases. There are, however, some crucial differences. The spin-1 BB model has an enlarged SU(3) symmetry when the strengths of the bilinear and biquadractic terms are the same, but the spin-2 model is never SU(3) symmetric within the phase space available, except when $\epsilon_{0,2,4}$ are all equal where one has SU(5) symmetry. It is only in the thermodynamic limit that the SU(3) symmetry emerges. Furthermore, the critical phase of the spin-1 BB model is not accessible from spin-1 bosons in the lattice, while for the spin-2 model the critical phase is accessible from spin-2 bosons.

Second, we observe that by using the finite-size scaling of the energies, data from smaller sizes are enough to precisely identify the CFT. In contrast, it is difficult to identify the CFT via the finite-size scaling of the EE with typical computational resources. The physical picture is as follows. Because of the proximity to the ferromagnetic phase, there is a competition between the conformally invariant state and the permutation symmetric state. The two kinds of states have very different EE scaling behavior, leading to enormous finite-size and truncation effects [39]. Analysis based on energy scaling is less sensitive to such a competition and accurate results for central charge and scaling dimensions can be obtained from smaller size data. Our picture is also consistent with the existence of anomalous low energy states beyond the CFT prediction, for example, the $S=3,4$ states at $k=0$ in Fig. 1. They appear because at a small length scale the system looks ferromagnetic. While their energies are lower than the states associated with the primary field at small sizes, it is expected that for larger sizes their rescaled energy will move up to higher energies and become irrelevant. In contrast the rescaled energy of the states at $k=2\pi/3$ will converge to $2/3$ when SU(3) symmetry emerges at larger sizes.

It is also natural to ask what kind of experimental signature can be observed. Because of the nonlocal nature of the EE, it is not easy to measure the EE directly. Recently, there are proposals to measure the related quantities, the Rényi entropies, within the cold atom framework [40]. It has also been shown that the influence of the ferromagnetic state and the value of the effective central charge can be detected and measured via Rényi entropies [41]. It is thus in principle possible to experimentally verify the findings here with cold atom experiments.

In summary, we study the critical phase of the spin-2 model obtained from spin-2 bosons in a 1D lattice. By using multiple approaches we identify the critical theory to be the $\mathrm{SU}(3)_1$ WZW model. The Hamiltonian is never SU(3) symmetric but SU(3) symmetry emerges in the thermodynamic limit.

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