Universal Adiabatic Quantum Computation via the Space-Time Circuit-to-Hamiltonian Construction

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We show how to perform universal adiabatic quantum computation using a Hamiltonian which describes a set of particles with local interactions on a two-dimensional grid. A single parameter in the Hamiltonian is adiabatically changed as a function of time to simulate the quantum circuit. We bound the eigenvalue gap above the unique ground state by mapping our model onto the ferromagnetic XXZ chain with kink boundary conditions; the gap of this spin chain was computed exactly by Koma and Nachtergaele using its q-deformed version of SU(2) symmetry. We also discuss a related time-independent Hamiltonian which was shown by Janzing to be capable of universal computation. We observe that in the limit of large system size, the time evolution is equivalent to the exactly solvable quantum walk on Young's lattice.

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Adiabatic quantum computation [1] is a computational model where one gradually converts a (efficiently preparable) ground state of a simple Hamiltonian into a (computationally useful) ground state of another Hamiltonian using adiabatic evolution with a slowly changing Hamiltonian.

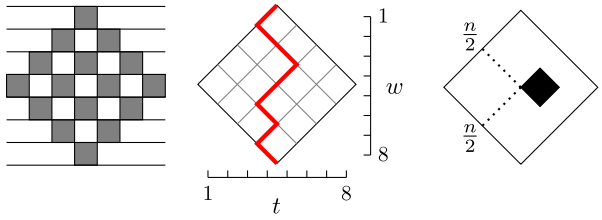
This model was shown to be equivalent to the standard quantum circuit model [2] through the use of the Feynman-Kitaev circuit-to-Hamiltonian construction [3,4]. Although the class of universal Hamiltonians originally considered (nearest neighbor interactions between six-dimensional particles in two dimensions) is not practically viable, perturbation gadget techniques [5,6] were later used to massage it into simpler forms [7,8]. However, these techniques have the disadvantage of requiring impractically high variability in the coupling strengths which appear in the Hamiltonian (see, e.g., the analysis in Ref. [9]). Given this state of affairs, it is of interest to consider how to construct a universal adiabatic quantum computer with a physically plausible Hamiltonian without using perturbative gadgets.

An alternative type of circuit-to-Hamiltonian mapping which is conceptually distinct from the Feynman-Kitaev construction has been used by some authors [10–16]. In these works a quantum circuit is mapped to a Hamiltonian which acts on a Hilbert space with computational and "local" clock degrees of freedom associated with every qubit in the circuit. This idea was first explored by Margolus in 1989 [10], just four years after Feynman's celebrated paper on Hamiltonian computation [3]. Margolus showed how to simulate a one-dimensional cellular automaton by Schrödinger time evolution with a time-independent Hamiltonian. More recently, Janzing [11] presented a scheme for universal computation with a time-independent Hamiltonian. In Ref. [14] it was claimed that an approach along these lines can be used to perform universal adiabatic quantum computation; unfortunately, the analysis presented by Mizel *et al.* does not establish the claimed results. The local clock idea was developed further in the recent "space-time circuit-to-Hamiltonian construction" and was used to prove that approximating the ground energy of a certain class of interacting particle systems is QMA-complete [16].

Our main result is a new method which achieves efficient universal adiabatic quantum computation using the space-time circuit-to-Hamiltonian construction. The Hamiltonian we use describes a system of interacting particles which live on the edges of a two dimensional grid. To prove that the resulting algorithm is efficient we use a mapping from our Hamiltonian to the ferromagnetic XXZ model with kink boundary conditions [17]. Our work can be viewed as a carefully tuned adaptation of the proposal from Ref. [11] to the quantum adiabatic setting. In the final part of this work, we turn our attention to Janzing's proposal for computation with a time-independent Hamiltonian and we present a new analysis based on the quantum walk on Young's lattice.

Universal adiabatic quantum computation.—We consider the universal circuit family used in Ref. [11] and depicted in Fig. 1(a), i.e., $2n$-qubit circuits which can be schematically drawn as a rotated $n \times n$ grid [shown in Fig. 1(b)] where each plaquette p on the grid corresponds to a two-qubit gate U_p. For technical reasons we further restrict the circuit so that many of the gates are fixed to be the identity; in particular, we set $k = \sqrt{n}/16$ and select the rotated $k \times k$ subgrid with its left corner in the center of the original lattice as the "interaction region"; see Fig. 1(c). In this interaction region the gates U_p are unrestricted, elsewhere they are identity gates.

We map such a circuit to a Hamiltonian $H(\lambda)$ which depends on a single parameter $\lambda \in [0, 1]$. We will



(a) (b) (c)

FIG. 1 (color online). A quantum circuit of the form shown in (a) (each gray square is a two qubit gate) is mapped to a Hamiltonian which describes a system of interacting particles that live on the edges of the rotated grid shown in (b). In the ground state, the edges occupied by particles form a connected string, as illustrated by the thick (red) line. (c) Many of the gates are fixed to be the identity; the gates which are unrestricted correspond to plaquettes within a $k \times k$ subgrid, the "interaction region," with the left corner in the center of the grid (shown in black).

demonstrate that (a) $H(\lambda)$ has a unique ground state for all $\lambda \in [0,1]$, (b) the ground state of $H(0)$ can be efficiently prepared, (c) the output of the quantum circuit is obtained with sufficiently high probability by performing a simple measurement in the ground state of $H(1)$, and (d) the eigenvalue gap above the ground energy of $H(\lambda)$ is lower bounded as $1/\mathrm{poly}(n)$ for all $\lambda \in [0,1]$. These properties allow us to efficiently simulate the given quantum circuit using the quantum adiabatic algorithm with interpolating Hamiltonian $H(\lambda)$.

We consider a multiparticle Fock space where the particles live on the edges of the rotated $n \times n$ grid, and each particle has a two-dimensional internal degree of freedom that encodes a qubit. For an edge with midpoint that intersects horizontal and vertical coordinates (t, w) (as shown in Fig. 1(b), these are unrotated coordinates) we define an operator $a_{t,x}[w]$ which annihilates a particle on that edge with internal state $x \in \{0,1\}$, and a number operator $n_{t,x}[w] = a^\dagger_{t,x}[w] a_{t,x}[w]$ which counts the number of particles in this state. $H(\lambda)$ is defined using these operators and, as we will see, it conserves the total number of particles on each horizontal line w. We restrict our attention to the sector where there is exactly one particle for each $w \in \{1, \ldots, 2n\}$; for the rest of this Letter we work in this finite-dimensional Hilbert space. The coordinate t can be viewed as a local time variable (local, since different particles may be located on edges with different values of t). For our purposes it is irrelevant whether the particles are fermions, bosons or distinguishable particles, since each particle never strays from its horizontal line of edges.

For a gate U_p with plaquette p bordered by edges $(t,w), (t+1,w), (t,w+1), (t+1,w+1)$, we define

$$H^p_{\mathrm{prop}} = -\sum_{\alpha,\beta,\gamma,\delta} (\langle \beta,\delta | U_p | \alpha,\gamma \rangle a^\dagger_{t+1,\beta}[w] a_{t,\alpha}[w] \times a^\dagger_{t+1,\delta}[w+1] a_{t,\gamma}[w+1]) + \mathrm{H.c.},$$

which allows nearest-neighbor particles to hop together. When the particles are both located before (or after) the plaquette, H^p_{prop} can map them onto being both located after (or before) it, while their internal qubit degrees of freedom are changed according to U_p (or $U_p^\dagger$). For each $\lambda \in [0,1]$ we define a positive semidefinite operator

$$H^p_{\mathrm{gate}}(\lambda) = \mathbf{n}_t[w]\mathbf{n}_t[w+1] + \mathbf{n}_{t+1}[w]\mathbf{n}_{t+1}[w+1] + \lambda H^p_{\mathrm{prop}},$$

where $\mathbf{n}_t[w] = n_{t,0}[w] + n_{t,1}[w]$. The Hamiltonian $H(\lambda)$ is built out of these gate operators as well as an operator H_{string} which ensures that the time variables for different particles remain synchronized. Consider a state where the $2n$ occupied edges of the grid form a connected string with endpoints at the top and bottom [e.g., the red string in Fig. 1(b)]. Such a string can be represented by $2n$ bits $z = z_1 z_2 \ldots z_{2n}$, where $0 = /$ represents an edge going down and to the left and $1 = \backslash$ is an edge going down and to the right, with total Hamming weight $\mathrm{wt}(z) = n$. The subspace of the Hilbert space with this property can be identified [18] with the space

$$S_{\mathrm{string}} = \mathrm{span}\{|x\rangle|z\rangle : x, z \in \{0,1\}^{2n}, \mathrm{wt}(z) = n\} \quad (1)$$

where z describes the string and x represents the $2n$-qubit state encoded in the internal degrees of freedom. It is clear that S_{string} is an invariant subspace for each of the gate operators $H^p_{\mathrm{gate}}(\lambda)$—acting with these operators on a state in S_{string} can move the string forward (or backward) and modify the internal state of the particles, but the string remains connected and fixed at the bottom and top of the grid. $H(\lambda)$ will contain a term H_{string} which penalizes particle configurations which do not correspond to connected strings; this will ensure that the ground state of $H(\lambda)$ is in S_{string}. We define $H_{\mathrm{string}} = \sum_v H^v_{\mathrm{string}}$ as a sum of terms for each vertex in the grid, where, if vertex v has four incident edges labeled $(t,w), (t+1,w), (t,w+1), (t+1,w+1)$, we let

$$H^v_{\mathrm{string}} = \mathbf{n}_t[w] + \mathbf{n}_{t+1}[w] + \mathbf{n}_t[w+1] + \mathbf{n}_{t+1}[w+1] - 2(\mathbf{n}_t[w] + \mathbf{n}_{t+1}[w])(\mathbf{n}_t[w+1] + \mathbf{n}_{t+1}[w+1]). \quad (2)$$

For vertices at the boundaries of the grid which have degree <4, this definition is modified so that it only includes operators for the edges which are present. Note that $H_{\mathrm{string}} \geq 0$ in the Hilbert space we are working in (the space with exactly one particle per horizontal line), and its null space is equal to S_{string}. More generally, a particle configuration corresponding to a set of occupied edges which form L string segments which are disconnected from

one another has energy $2L-2$, the number of "loose ends." In particular, the smallest nonzero eigenvalue of H_{string} is 2.

We are now ready to define the Hamiltonian $H(\lambda)$. For $\lambda \in [0,1]$ we let

$$H_{\text{circuit}}(\lambda) = \sum_p H^p_{\text{gate}}(\lambda) + \sqrt{1-\lambda^2} H_{\text{init}},$$
$$H(\lambda) = H_{\text{string}} + H_{\text{circuit}}(\lambda) + H_{\text{input}},$$

where $H_{\text{init}} = \mathbf{n}_{n+1}[w=1] + \mathbf{n}_{n+1}[w=2n]$ is chosen so that in the ground state of $H(0)$ all particles are located at the left boundary of the grid, and $H_{\text{input}} = \sum_{w=1}^{2n}\sum_{t\le n} n_{t,1}[w]$ ensures that the internal state of each particle is correctly initialized to $|0\rangle$ when the particle is on the left-hand side of the grid. We now investigate the ground space of $H(\lambda)$.

To begin, observe that H_{string} commutes with each of the plaquette operators H^p_{prop} [19] and also with each of the number operators $n_{t,z}[w]$. Thus $[H_{\text{string}}, H(\lambda)] = 0$. As noted above, the ground energy of H_{string} is zero and its first excited energy is 2. In the following we show that the smallest eigenvalue of $H(\lambda)$ within the space S_{string} is $\sqrt{1-\lambda^2}$. Since $\sqrt{1-\lambda^2} < 2$ this establishes that the corresponding eigenvector of $H(\lambda)$ is the ground state.

First consider $H(0)$. Since $\sum_p H^p_{\text{gate}}(0)$ has minimal energy when the string is either 1^n0^n or $z_{\text{init}} = 0^n1^n$, and since H_{init} penalizes configurations where the first edge of the string is \ or the last edge is /, we see that the ground space of $H_{\text{circuit}}(0) + H_{\text{string}}$ (with eigenvalue 1) is spanned by states $|x\rangle|z_{\text{init}}\rangle$. The term H_{input} penalizes all of these states except $|0^{2n}\rangle|z_{\text{init}}\rangle$ which is the unique ground state of $H(0)$, with ground energy 1. Note that our adiabatic quantum computation can be efficiently initialized since this state is easy to prepare.

To understand the ground space of $H(\lambda)$ when $\lambda > 0$, it will be convenient to work with a different basis for the space S_{string} which builds in the details of the quantum circuit. For any configuration of the string $z \in \{0,1\}^{2n}$ with $\text{wt}(z) = n$, let $V(z)$ be the unitary equal to the product of all the two-qubit gates associated with plaquettes which lie to the left of the string. In other words $V(z)$ is the total unitary of the partially completed circuit with boundary described by z. Define basis vectors

$$|x,z\rangle_V = V(z)|x\rangle|z\rangle \qquad x,\ z \in \{0,1\}^{2n},\ \text{wt}(z) = n \tag{3}$$

which span S_{string}. The action of $H_{\text{circuit}}(\lambda)$ in this basis has a nice form: it acts nontrivially only on the string degree of freedom; the two-qubit gates which make up the circuit are "rotated away." Moreover, its action on the string register is equivalent (up to a term proportional to the identity and a multiplicative constant) to the ferromagnetic XXZ chain with kink boundary conditions

$$\begin{aligned} {}_V\langle x',z'|\left(H_{\text{circuit}}(\lambda) - \sqrt{1-\lambda^2} I\right)|x,z\rangle_V \\ = 2\delta_{x',x}\langle z'|H_{XXZ}(\lambda)|z\rangle \end{aligned} \tag{4}$$

where [17] (writing X, Y, Z for the Pauli operators)

$$\begin{aligned} H_{XXZ}(\lambda) &= \frac{1}{4}\sqrt{1-\lambda^2}(Z_{2n} - Z_1) \\ &\quad - \frac{1}{4}\sum_{w=1}^{2n-1}\left[(Z_wZ_{w+1} - I) + \lambda(X_wX_{w+1} + Y_wY_{w+1})\right] \\ &= \sum_{w=1}^{2n-1} |\Psi_{q(\lambda)}\rangle\langle\Psi_{q(\lambda)}|_{w,w+1}, \\ \lambda &= \frac{2}{q(\lambda) + q(\lambda)^{-1}}, \end{aligned} \tag{5}$$

where $0 \le q(\lambda) \le 1$ and the q-deformed singlet equals $|\Psi_q\rangle = (1/\sqrt{q^2+1})(|10\rangle - q|01\rangle)$. This spin chain can be viewed as a q analogue of the ferromagnetic Heisenberg chain; it has a remarkable $\text{SU}_q(2)$ quantum group symmetry which is a deformation of the SU(2) symmetry of the Heisenberg ferromagnet. Its spectral gap, ground space [17], and excitations are known [17,20]. In the Supplemental Material [21] we derive an expression for the zero energy ground state of $H_{XXZ}(\lambda)$ in the sector with Hamming weight n. Using this expression and Eq. (4) we immediately obtain a spanning basis for the $\sqrt{1-\lambda^2}$ energy ground space of $H_{\text{string}} + H_{\text{circuit}}(\lambda)$, given by (up to normalization)

$$|\Phi_\lambda(x)\rangle = \sum_{z:\ \text{wt}(z)=n} q(\lambda)^{-A(z)}|x,z\rangle_V \qquad x \in \{0,1\}^{2n}, \tag{6}$$

where $A(z) = \sum_{j=1}^{2n} jz_j - [n(n+1)/2]$ is the area of the grid which lies to the right of the string (and z_j is the jth bit of z). We see that when $\lambda < 1$ the associated probability distribution over strings favors the left-hand side of the grid; the most likely string is $z_{\text{init}} = 0^n1^n$ [with $A(z_{\text{init}}) = n^2$], the least likely is 1^n0^n [with $A(z) = 0$], etc. The term H_{input} penalizes every state [Eq. (6)] except $|\Phi_\lambda(0^{2n})\rangle$, which is the unique ground state of $H(\lambda)$, with energy $\sqrt{1-\lambda^2}$, for $0 < \lambda \le 1$.

The ground state $|\Phi_{\lambda=1}(0^{2n})\rangle$ of the final Hamiltonian is a uniform superposition over basis vectors $|0^{2n},z\rangle_V$ corresponding to all possible configurations of the string z. To obtain the output of the quantum circuit we measure the locations of the $2k$ particles which lie on horizontal lines that intersect the interaction region. If we find that all of these particles are located on edges to the right of the interaction region then their internal degrees of freedom give the output of the quantum circuit. Since the string is connected, this is guaranteed to occur as long as the nth particle (i.e., the particle on horizontal line $w = n$) is

located on an edge which lies to the right of the interaction region. In the Supplemental Material [21] we show that, with our choice $k = \sqrt{n}/16$, this occurs with probability lower bounded by a positive constant. Finally, we lower bound the eigenvalue gap of $H(\lambda)$.

Theorem 1: The smallest nonzero eigenvalue of $H(\lambda) - \sqrt{1-\lambda^2}I$ is at least $(1/(4n+3))(1-\lambda\cos(\pi/2n))$ for all $\lambda \in [0,1]$.

This $\Omega(n^{-3})$ bound establishes that the adiabatic quantum computation can be performed efficiently. The proof, given in the Supplemental Material [21], uses the known expression for the eigenvalue gap of $H_{XXZ}(\lambda)$ and a Lemma for bounding the smallest nonzero eigenvalue of an operator sum.

In an attempt to improve the success probability of a final measurement, one might consider modifying this scheme so that the ground state of the final Hamiltonian is localized at the right side of the grid. This can be achieved by adding another segment to the adiabatic path: after reaching $H(1)$, replace H_{init} with $H_{\text{endit}} = \mathbf{n}_n[w=1] + \mathbf{n}_n[w=2n]$ and then reduce λ from 1 to 0. With this choice, every state in the ground space of the final Hamiltonian has particle configuration corresponding to the string 1^n0^n on the far right. However, the ground space is degenerate (since $H_{\text{input}}|x, 1^n0^n\rangle_V = 0$ for all computational input states x). Although the error-free Hamiltonian has a symmetry which prevents transitions between the ground state corresponding to the correct input and the other wrong-input states, an imperfect realization could potentially derail the computation.

Universal computation with a time-independent Hamiltonian.—We now discuss a bare-bones version of the related scheme from Ref. [11]. The quantum circuit family is the same as before, except that now the interaction region is chosen to be the first K^2 gates in the circuit with $K = n/4$, i.e., the $K \times K$ subgrid at the far left side of the $n \times n$ grid. The circuit is simulated using Schrödinger time evolution with initial state $|0^{2n}, z_{\text{init}}\rangle_V$ and time-independent Hamiltonian $H_{\text{prop}} = \sum_p H^p_{\text{prop}}$. After evolving for time t, one measures the location of each particle and if one finds them all outside the interaction region then the internal degrees of freedom give the output of the circuit. Janzing's analysis of this scheme is based on an equivalence between H_{prop} and the XY model, which can be diagonalized using a Jordan-Wigner transformation (a unitary mapping to a system of noninteracting fermions in one dimension). In the Supplemental Material [21] we extend one of Janzing's Theorems to prove that the above scheme efficiently simulates a quantum circuit. Specifically we prove that, if the evolution time t is randomly (uniformly) chosen in the interval $[0,T]$ with $T = cn^3$ (for some constant c), the probability to measure all the particles outside the interaction region is at least $\frac{1}{4} + \mathcal{O}(1/\sqrt{n})$.

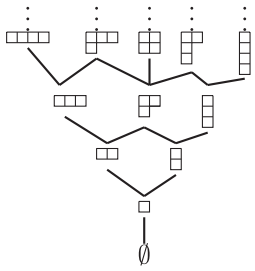


FIG. 2. Young's lattice.

Here we focus on the limit $n \to \infty$ and directly analyze the time evolution in the given basis without using a Jordan-Wigner transformation. In this way we obtain a detailed picture of the dynamics of the string. To begin, note that a string is associated with a Young diagram (or, equivalently, an integer partition) obtained by rotating the portion of the grid which lies to the left of the string by 45 degrees. In the limit $n \to \infty$, the set of string configurations is in one-to-one correspondence with the set of Young diagrams. In the basis [Eq. (3)], H_{prop} acts nontrivially only on the string degree of freedom and it acts on this space as $-H_{\mathbb{Y}}$, where $H_{\mathbb{Y}}$ is the adjacency matrix of Young's lattice, shown in Fig. 2. In this infinite graph two Young diagrams are connected by an edge if they differ by one box. The dynamics of our system is given by the quantum walk on Young's lattice starting from a very special initial state: the empty partition Ø. This quantum walk can be solved exactly [26]; the solution is

$$e^{iH_{\mathbb{Y}}t}|\emptyset\rangle = e^{-(t^2/2)} \sum_{m=0}^{\infty} \frac{(it)^m}{\sqrt{m!}} |\phi_m\rangle, \tag{7}$$

where the normalized state $|\phi_m\rangle = (1/\sqrt{m!})\sum_{\sigma\vdash m} d_\sigma |\sigma\rangle$, $\sigma\vdash m$ indicates that σ is a partition of m, and d_σ is the dimension of the irreducible representation of the symmetric group S_m associated with σ (given by the hook-length formula). For completeness, in the Supplemental Material [21] we review the derivation of Eq. (7).

We see that the quantum walk takes place in a tiny subspace of the full Hilbert space spanned by $\{|\phi_m\rangle : m \geq 0\}$. The probability distribution over partitions σ as a function of time is given by $p(\sigma,t) = (m!)^{-2} e^{-t^2} t^{2m} d_\sigma^2$ (where $\sigma\vdash m$) which is a *Poissonized Plancherel measure* [27]. The marginal distribution of m is Poisson with mean and variance $\mathbb{E}[m] = \text{Var}(m) = t^2$. In our case m represents the area to the left of the string (i.e., the number of gates that have been applied) and this shows that, roughly speaking, this area increases quadratically. For large times the random variable m is peaked about its mean in the sense that $\sqrt{\text{Var}(m)}/\mathbb{E}[m]$ is small. The conditional distribution over partitions $\lambda\vdash m$ for fixed m is the widely studied Plancherel measure $\rho_m(\sigma) = d_\sigma^2/m!$, which is known to exhibit a limiting behavior [28]. Imagine

sampling a partition from ρ_m, drawing it in the x-y plane and then rescaling both axes by $1/\sqrt{m}$. As $m \to \infty$, the resulting picture approaches a fixed shape with probability $\to 1$ [27,28] (we include a plot of this shape in the Supplemental Material [21]). Roughly speaking, for large times we envision the string as a wave front which moves with constant speed and with scaled shape described by this limit theorem.

Finally, note that although it was convenient to consider the limit $n \to \infty$, we expect this analysis to be approximately valid for finite n when t is small enough so that Eq. (7) is supported almost entirely on partitions contained in the left-hand side of the rotated $n \times n$ grid.

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[18] This identification is just a matter of notational convenience and we use it throughout the Letter. It is formally defined in the following straightforward way. The string z specifies a set of occupied edges; for each $w = 1, \ldots, 2n$ let (t_w, w) be the occupied edge on the wth horizontal line. Then we identify the state $(\Pi_{w=1}^{2n} a^\dagger_{t_w,x_w}[w])|\mathrm{vac}\rangle$, where $|\mathrm{vac}\rangle$ is the vacuum state with no particles, with $|x\rangle|z\rangle$.
[19] To see why $[H_{\mathrm{string}}, H^p_{\mathrm{prop}}] = 0$, recall that the eigenvectors of H_{string} with eigenvalue $2L-2$ correspond to particle configurations with L disconnected string segments, and note that applying H^p_{prop} to such a state does not change the number of string segments.
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