Many-Body Localization in Periodically Driven Systems

Pedro Ponte,[1,2] Z. Papić,[1,3,4] François Huveneers,[5] and Dmitry A. Abanin[1,3]
[1]Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
[2]Department of Physics and Astronomy, University of Waterloo, Ontario, N2L 3G1, Canada
[3]Institute for Quantum Computing, Waterloo, Ontario N2L 3G1, Canada
[4]School of Physics and Astronomy, University of Leeds, Leeds, West Yorkshire LS2 9JT, United Kingdom
[5]CEREMADE, Universite Paris-Dauphine, Paris, France
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We consider disordered many-body systems with periodic time-dependent Hamiltonians in one spatial dimension. By studying the properties of the Floquet eigenstates, we identify two distinct phases: (i) a many-body localized (MBL) phase, in which almost all eigenstates have area-law entanglement entropy, and the eigenstate thermalization hypothesis (ETH) is violated, and (ii) a delocalized phase, in which eigenstates have volume-law entanglement and obey the ETH. The MBL phase exhibits logarithmic in time growth of entanglement entropy when the system is initially prepared in a product state, which distinguishes it from the delocalized phase. We propose an effective model of the MBL phase in terms of an extensive number of emergent local integrals of motion, which naturally explains the spectral and dynamical properties of this phase. Numerical data, obtained by exact diagonalization and time-evolving block decimation methods, suggest a direct transition between the two phases.

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Introduction.—The dynamics of closed quantum many-body systems driven out of equilibrium has been the subject of intense investigation over the past decade [1,2]. Many-body systems broadly fall into two classes with distinct dynamical properties: ergodic systems, which reach local thermal equilibrium as a result of the Hamiltonian evolution, and nonergodic ones which fail to thermalize. Thermalization in isolated ergodic systems can be linked to the properties of individual many-body eigenstates that are locally thermal [3–5].

While a complete classification of nonergodic systems remains an open problem, it has recently been established that many-body localization [6–17] provides a robust mechanism of ergodicity breaking in systems with quenched disorder. Many-body localized (MBL) systems are characterized by an extensive number of quasilocal integrals of motion [13,14], which strongly restrict quantum dynamics and prevent energy transport and thermalization. MBL systems have universal dynamical properties, such as the logarithmic-in-time growth of entanglement entropy for initial product states [9,11–15], in contrast to ergodic and Bethe-ansatz-integrable systems where entanglement spreads linearly in time [18–20].

In this Letter, we study the response of MBL systems to periodic driving, which provides a natural experimental probe of both solid-state and cold atoms systems. The ac conductivity of single-particle insulators is conventionally described by the linear-response Mott formula, which predicts a finite absorption at any driving frequency [21]. This suggests that weak periodic driving heats up the system and thermalizes it. On the other hand, a MBL system can be viewed as a collection of localized degrees of freedom with exponentially decaying interactions [13,14]; therefore, one might expect that these degrees of freedom perform nearly independent periodic motion, and a system stabilizes in a highly nonthermal steady state. In what follows, we show that the latter possibility is realized, and MBL persists at weak driving. In contrast, strong driving does delocalize the system, leading to heating and thermalization.

The properties of periodically driven systems with a Hamiltonian $H(t+T) = H(t)$ are determined by the unitary Floquet operator $\hat{F}$, i.e., the evolution operator over one period:

$$\hat{F} = \mathcal{T}\exp\left\{-i\int_0^T H(t)dt\right\}, \qquad (1)$$

where $\mathcal{T}\exp$ denotes a time-ordered exponential. In the eigenstate basis $|\psi_\alpha\rangle$, $\hat{F}$ takes the form $\hat{F} = \sum_{\alpha=1}^{\mathcal{D}} e^{-i\theta_\alpha}|\psi_\alpha\rangle\langle\psi_\alpha|$, where $\mathcal{D}$ is the Hilbert space dimension, and the quasienergies θ_α can be chosen to lie in the interval $[0;2\pi)$. One can introduce an effective Floquet Hamiltonian H_F as $\hat{F} = e^{-iH_F}$, with eigenstates $|\psi_\alpha\rangle$ and eigenvalues $\theta_\alpha + 2\pi n_\alpha$, with n_α integer.

We consider a generic class of periodically driven 1D models with quenched disorder, and find that, as a function of driving strength, two distinct phases are realized. At weak driving, the system remains in the MBL phase, characterized by the Poisson statistics of quasienergy levels. The Floquet eigenstates the obey area law for entanglement entropy [22] (that is, entanglement entropy of half the system remains smaller than a constant as system size $L \to \infty$), similar to the ground states in gapped systems. Further, the eigenstates with similar quasienergies typically have different local properties; thus, the eigenstate thermalization hypothesis (ETH) [3–5] breaks down. At some critical driving strength, the system undergoes a transition into a delocalized (ergodic) phase. Here the Floquet eigenstates have an extensive, volume-law entanglement; the quasienergy levels repel, and their statistics is described by the circular orthogonal ensemble (COE). The ETH holds in this phase, and the Floquet eigenstates have identical local properties, described by the infinite-temperature Gibbs ensemble.

The two phases can furthermore be distinguished by their dynamical properties, e.g., the time evolution of the system prepared in a product state, which can be efficiently simulated numerically. In the MBL phase, the states retain the local memory of the initial state, and local observables at long times are correlated with their initial values. This behavior reflects the presence of emergent local integrals of motion [13,14], which we explicitly construct following Ref. [23] (see also Ref. [24]). In contrast, in the delocalized phase, local observables relax to their "equilibrium" values at long times, which are given by the infinite-temperature Gibbs ensemble.

Our results complement previous works [25–28], which studied driven translationally invariant systems, as well as Ref. [29], where the behavior of disordered many-body systems under local driving was studied.

Model.—Our system is a 1D spin 1/2 chain with open boundary conditions. Following Refs. [25,26], we consider a driving protocol in which the system's Hamiltonian is periodically switched between two operators, H_0 and H_1, both of which are sums of local terms. An example of a disordered Hamiltonian H_0, which describes an MBL phase and acts for time T_0, is

$$H_0 = \sum_i h_i \sigma_i^z + J_z \sigma_i^z \sigma_{i+1}^z, \qquad (2)$$

where random fields h_i are uniformly distributed in the interval $[-W; W]$. The eigenstates of H_0 are product states. As a delocalizing Hamiltonian H_1 we choose

$$H_1 = J_x \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y, \qquad (3)$$

which acts for time T_1 such that the driving period is $T = T_0 + T_1$. The Floquet operator is given by

$$\hat{F} = e^{-iH_0T_0} e^{-iH_1T_1}. \qquad (4)$$

The protocol describes a MBL system periodically "kicked" with a delocalizing perturbation H_1. An important difference compared to the periodically kicked rotor model [30–36] is that in our model the one-body Hilbert space has a finite dimension. We fix $J_x = J_z = 1/4$, $T_0 = 1$, $W = 2.5$, and tune the strength of the kick, T_1, observing a transition at critical T_1^* between the MBL phase (small $T_1 < T_1^*$) and the ergodic phase ($T_1 > T_1^*$) [37].

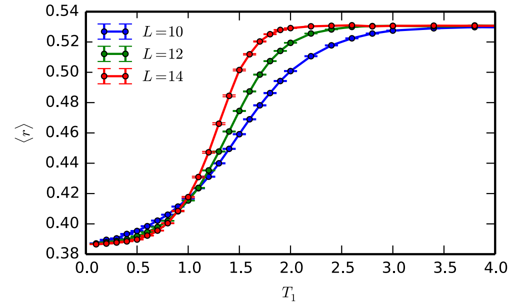

FIG. 1 (color online). Disorder-averaged level statistics parameter $\langle r \rangle$ as a function of the "kick" strength T_1. At small values of T_1, $\langle r \rangle \approx 0.386$, indicating Poisson statistics of quasienergy levels (no level repulsion). At larger T_1 the system undergoes a transition into a delocalized phase with $\langle r \rangle \approx 0.53$, consistent with the COE [27]. Data are for system sizes $L = 10, 12, 14$, and averaging is performed over 1000 disorder realizations.

Properties of Floquet eigenstates.—We first explore the properties of the Floquet eigenstates using exact diagonalization (ED). By computing the consecutive quasienergy gaps $\delta_\alpha = \theta_{\alpha+1} - \theta_\alpha$, we characterize the level statistics by their ratio $r = \min(\delta_\alpha, \delta_{\alpha+1})/\max(\delta_\alpha, \delta_{\alpha+1})$ [8,27]. The averaged value of r serves as a probe of ergodicity breaking: it allows one to distinguish between the Poisson and Wigner-Dyson level statistics. In Fig. 1 we show $\langle r \rangle$ averaged over all quasienergy spacings and over 1000 disorder realizations, for several system sizes. At small kick period T_1, $\langle r \rangle$ becomes increasingly close to the Poisson-statistics value $\langle r \rangle_{\text{POI}} \approx 0.386$ as the system size is increased. This indicates the absence of level repulsion and suggests that ergodicity is broken at small T_1 and the system is in the MBL phase. At large T_1, parameter $\langle r \rangle$ is approximately equal to 0.53, which is close to the COE value, $\langle r \rangle_{\text{COE}} \approx 0.527$ [27,38]. This suggests that at large T_1 the system delocalizes. The $\langle r \rangle$ curves for different system sizes cross at $T_1^* \approx 0.9$, suggesting a phase transition between MBL and ergodic phases in the thermodynamic limit. A drift of the crossing point towards smaller T_1 is observed, similar to the time-independent case [10]. This behavior should be contrasted with periodically kicked ergodic spin chains [27], where the ergodicity breaking regime was found at a fixed L and sufficiently small L-dependent T_0, T_1; in this case, ergodicity was found to be restored in the thermodynamic limit, $L \to \infty$.

To further distinguish the two phases, we study the entanglement properties of the Floquet eigenstates. Figure 2 shows disorder- and ensemble-averaged von Neumann entropy $\langle S \rangle$ of the Floquet eigenstates, for the symmetric bipartition, plotted as a function of T_1. The markedly different scaling of $\langle S \rangle$ at small and large values of T_1 lends further support to the existence of two phases. At $T_1 \lesssim T_1^*$, $\langle S \rangle$ is much smaller than the value expected for

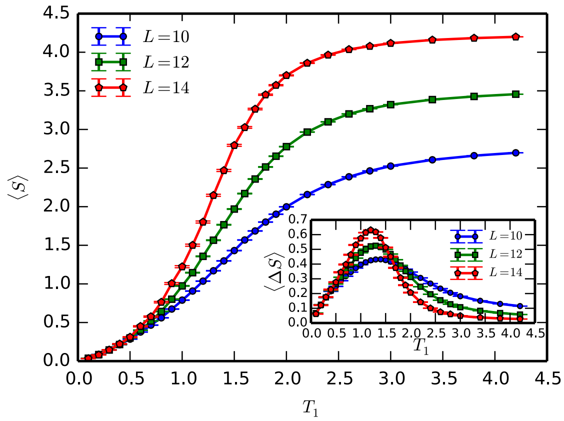

FIG. 2 (color online). Averaged entanglement entropy $\langle S \rangle$ and its fluctuations $\langle \Delta S \rangle$ (inset) as a function of T_1. The scaling of entropy and its fluctuations with system size L are consistent with the existence of a MBL and a delocalized phase for small and large T_1, respectively.

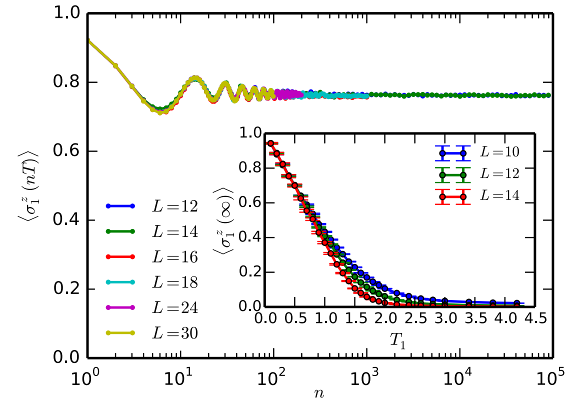

FIG. 3 (color online). Dynamical properties: Decay of magnetization at a given site $I = 1$ for the Néel initial configuration. Inset: Long-time magnetization remains nonzero in the MBL phase as the system size is increased. In the delocalized phase, magnetization decays to zero at long times. Averaging was performed over 6000 disorder realizations.

random vectors in the Hilbert space, $S_{\mathrm{Th}} \approx L/2 \ln 2$ [39], which signals ergodicity breaking. Moreover, at $T_1 \lesssim 0.6$ the entanglement entropy grows very weakly with system size, consistent with area-law in 1D. On the contrary, at large $T_1 > T_1^*$, $\langle S \rangle$ approaches S_{Th}, indicating that almost all eigenstates are essentially random vectors in the Hilbert space, as expected in the ergodic phase.

It is also instructive to study the fluctuations of entanglement entropy, as they have been shown to provide a useful probe of the MBL-delocalization transition in time-independent models [40]. The disorder-averaged fluctuations of S, defined as $\Delta S = \sqrt{\langle (S - \langle S \rangle)^2 \rangle}$ are expected to be small deep in the delocalized phase, as well as in the MBL phase: in the former case, almost all eigenstates are highly entangled, with $S \approx S_{\mathrm{Th}}$, with small fluctuations around this value, while in the latter case, S obeys the area law and is therefore small, as are its state-to-state fluctuations. In contrast, at the transition S has a broad distribution [13,40], and therefore its fluctuations are maximal. Thus, the localization-delocalization transition can be detected by the location of the peak in ΔS. Figure 2(inset) shows ΔS as a function of T_1. Entanglement fluctuations ΔS exhibit a maximum at $T_1 \approx 1.1$ that roughly agrees with the T_1^* value found from analyzing level statistics; further, we observe a slight drift of the maximum with the system size, similar to the previous study of the static case [40]. We attribute the difference between the position of the maximum in ΔS and value T_1 determined from the level statistics, to the finite-size effects. We have also directly tested the ETH and its violation in the MBL phase in the Floquet eigenstates, finding behavior consistent with the existence of two phases [41] (see also Ref. [42], where the ETH for driven ergodic systems was tested).

Dynamics.—We next study the dynamical properties of the model [Eqs. (2) and (3)]. We consider a standard quantum quench protocol: the system is initially prepared in the Néel (product) state $|\psi_0\rangle$ of spins $\sigma_i^z = \pm 1$ at $t = 0$, and this state is evolved under the Hamiltonian (2), (3) at $t > 0$. This protocol is particularly easy to simulate using Krylov subspace projection methods [43] or the time-evolving block decimation (TEBD) [44] method, both of which allow us to access larger systems beyond ED due to the sufficiently slow growth of entanglement in the MBL phase. For the TEBD algorithm we use a second order Trotter decomposition with time step $\Delta t = 0.1$. The growth of the bond dimension is controlled by requiring the neglected weight to be less than 10^{-7} at each Schmidt decomposition.

Local observables: We first focus on the evolution of local observables, and compute the expectation value of the spin on a given site I, $\sigma_I^z(t)$, and its long-time limit $\langle \sigma_I^z(\infty) \rangle$ [23,45,46]. Figure 3 illustrates the time evolution $\sigma_I^z(t)$ for the Néel initial state $|\psi_0\rangle$ and site $I = 1$, and for system sizes ranging from $L = 10$–14 (obtained via ED), $L = 16, 18$ obtained using Krylov subspace projection, and $L = 24, 30$ obtained using TEBD. We find that the on-site magnetization remains finite at very long times even for the largest systems without any visible finite-size effects. This indicates that the MBL phase remains stable in the thermodynamic limit.

The long-time average $\langle \sigma_I^z(\infty) \rangle$ can be expressed in terms of the Floquet eigenstates as $\langle \sigma_I^z(\infty) \rangle = \lim_{t\to\infty} (1/t) \int_0^t \langle \psi_0 | \sigma_I^z(t') | \psi_o \rangle dt'$, which in terms of the eigenstates $|\psi_\alpha\rangle$ reads $\sum_\alpha \langle \psi_\alpha | \sigma_I^z | \psi_\alpha \rangle |\langle \psi_0 | \psi_\alpha \rangle|^2$. The long-time value $\langle \sigma_I^z(\infty) \rangle$, calculated using ED and averaged over 6000 disorder realizations, is illustrated in Fig. 3 (inset). This quantity behaves differently in the two phases: at $T \lesssim T_1^*$, $\langle \sigma_I^z(\infty) \rangle$ is positive and weakly dependent on the system size, which shows that in the MBL phase the local memory of the initial state is retained. Deep in the ergodic phase, at $T_1 \gg T_1^*$, $\langle \sigma_I^z(\infty) \rangle \to 0$, reflecting the decay of

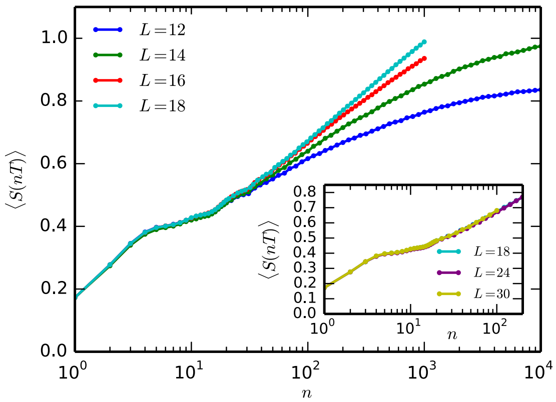

FIG. 4 (color online). Disorder-averaged entanglement entropy following a quantum quench, for the Néel initial state. Data for system sizes $L = 12, 14$ were obtained by ED, for $L = 16, 18$ using Krylov subspace projection, and $L = 24, 30$ using TEBD. Averaging is performed over 6000 disorder realizations.

the initial magnetization and therefore a loss of the memory of the initial state.

Entanglement growth: Finally, we explored the spreading of entanglement following a quantum quench, known to be a sensitive probe of many-body localization: in the MBL phase, entanglement grows logarithmically in time [9,11–14], while in the ergodic phase, as well as in Bethe-ansatz-integrable systems, it grows linearly in time [18–20]. The disorder-averaged entanglement entropy as a function of time, calculated for fixed $T_1 = 0.4$ and the symmetric bipartition, is shown in Fig. 4. Averaging was performed over 6000 disorder realizations. Entanglement initially rises from zero, followed by a plateau and a logarithmic growth for several decades in time, $\langle S(t)\rangle \propto \ln(t)$. This behavior is qualitatively similar to that found in the MBL phase in systems with time-independent Hamiltonians [11–14], which gives further support for the existence of the MBL phase in driven systems with strong disorder.

Local integrals of motion and effective description of the driven MBL phase.—In order to understand the spectral and dynamical properties of the MBL phase observed in the numerical simulations, we propose an effective model of this phase. Intuitively, a MBL phase is stable because weak driving only induces periodic motion of localized degrees of freedom, but not transitions between them. Thus, we expect that an extensive number of local integrals of motion which is known to exist in static MBL phase [13,14], persists under driving. To define the integrals of motion, we first note that the area-law entanglement of the Floquet eigenstates suggests that they can be obtained from the product states (in the $\sigma_i^z = \pm 1$ basis) by a quasilocal unitary transformation U which brings the Floquet operator into a diagonal form in that basis: $U\hat{F}U^\dagger = \hat{F}_{\rm diag}$. Since L of the operators σ_i^z commute with $\hat{F}_{\rm diag}$, we can introduce a set of L "pseudospin" operators $\tau_i^z = U^\dagger \sigma_i^z U$. These operators commute with the Floquet operator $[\hat{F}, \tau_i^z] = 0$, as well as with each other $[\tau_i^z, \tau_j^z] = 0$. Operators τ_i^z have eigenvalues ± 1 and therefore satisfy the relation $(\tau_i^z)^2 = 1$; they can be viewed as z components of some "effective" spins. We emphasize that the operators τ_i^z can be introduced for any driven system, but the special property of the MBL phase is that their support is localized near site i, and they affect remote physical degrees of freedom *exponentially weakly*. In terms of τ operators, the operator F takes a simple form, as it can only depend on τ_i^z operators and their products (but not on the τ_i^x, τ_i^y operators). It is convenient to represent $\hat{F}$ as

$$\hat{F} = e^{-iH_{\rm eff}(\{\tau_i^z\})}, \tag{5}$$

where $H_{\rm eff}(\{\tau_i^z\})$ is a real function of operators τ_i^z. (Such a representation takes into account the fact that eigenvalues of $\hat{F}$ have the absolute value one). Further, since $(\tau_i^z)^2 = 1$, $H_{\rm eff}$ can generally be written as

$$H_{\rm eff}(\{\tau_i^z\}) = \sum_i \tilde{h}_i \tau_i^z + \sum_{ij} J_{ij}\tau_i^z\tau_j^z + \sum_{ijk} J_{ijk}\tau_i^z\tau_j^z\tau_k^z + \cdots \tag{6}$$

It is natural to assume that in the MBL phase the couplings J between remote effective spins decay exponentially with distance, similar to the static case [13,14].

The effective model introduced above naturally explains the spectral and dynamical properties of the MBL phase established numerically, e.g., the absence of decay of the on-site magnetization at long times and the logarithmic growth of entanglement, which directly follows from Eqs. (5) and (6) and the exponential decay of interactions between remote effective spins [12–14]. To provide further justification for the effective description [Eqs. (5) and (6)], we have also numerically constructed [41] the local integrals of motion following Ref. [23]. These form an extensive set, although they are not identical to τ_i^z operators.

Discussion.—We have demonstrated the existence of two dynamical regimes in periodically driven MBL systems. We found that weak driving does not destroy MBL systems. In this regime, localized degrees of freedom (effective spins) perform periodic motion, and driving does not induce transitions between distant effective spins, even though they are interacting [13,14]. This indicates that MBL systems do not absorb energy under weak driving, and signals the inapplicability of linear response theory (Mott's formula) in driven MBL systems, which predicts finite absorption and heating. Further, there exists a finite driving threshold above which the system delocalizes, and ergodicity is restored. Our study shows that periodic driving, which is a common tool in both solid-state and cold atoms systems, provides a new experimental probe of MBL systems, and in particular allows one to induce and characterize the MBL-delocalization transition.

Our results indicate that many-body localization does not rely on global conservation laws, and a complete set of local integrals of motion exists even in the absence of energy conservation. This implies that the dynamics of Floquet MBL systems is described by an effective quasi-local *time-independent* Hamiltonian H_{eff}, which is itself many-body localized. This is in sharp contrast to the ergodic phase, where the Floquet Hamiltonian does not have a quasilocal representation [27–29]. An interesting open question is whether the Magnus expansion, which is a high-frequency expansion for the effective Floquet Hamiltonian (see Ref. [47] for a recent overview), converges in the MBL phase.

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Note added.—Recently, we became aware of a related work [48].

[1] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[2] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[3] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
[4] M. Srednicki, Phys. Rev. E **50**, 888 (1994).
[5] M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).
[6] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. (Amsterdam) **321**, 1126 (2006).
[7] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).
[8] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
[9] M. Znidaric, T. Prosen, and P. Prelovsek, Phys. Rev. B **77**, 064426 (2008).
[10] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[11] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[12] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013).
[13] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **111**, 127201 (2013).
[14] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014).
[15] R. Vosk and E. Altman, Phys. Rev. Lett. **110**, 067204 (2013).
[16] B. Bauer and C. Nayak, J. Stat. Mech. (2013) P09005.
[17] D. Pekker, G. Refael, E. Altman, E. Demler, and V. Oganesyan, Phys. Rev. X **4**, 011052 (2014).
[18] P. Calabrese and J. Cardy, J. Stat. Mech. (2005) P04010.
[19] G. De Chiara, S. Montangero, P. Calabrese, and R. Fazio, J. Stat. Mech. (2006) P03001.
[20] H. Kim and D. A. Huse, Phys. Rev. Lett. **111**, 127205 (2013).
[21] N. Mott, Philos. Mag. **17**, 1259 (1968).
[22] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).
[23] A. Chandran, I. H. Kim, G. Vidal, and D. A. Abanin, Phys. Rev. B **91**, 085425 (2015).
[24] V. Ros, M. Mueller, and A. Scardicchio, Nucl. Phys. **B891**, 420 (2015).
[25] T. Prosen, Phys. Rev. Lett. **80**, 1808 (1998).
[26] L. D'Alessio and A. Polkovnikov, Ann. Phys. (Amsterdam) **333**, 19 (2013).
[27] L. D'Alessio and M. Rigol, Phys. Rev. X **4**, 041048 (2014).
[28] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. E **90**, 012110 (2014).
[29] P. Ponte, A. Chandran, Z. Papić, and D. A. Abanin, Ann. Phys. (Amsterdam) **353**, 196 (2015).
[30] G. Casati and J. Ford, *Stochastic behavior in classical and quantum Hamiltonian systems*, 1st ed. (Springer, Berlin, 1979) Vol. 93.
[31] D. R. Grempel, R. E. Prange, and S. Fishman, Phys. Rev. A **29**, 1639 (1984).
[32] F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, Phys. Rev. Lett. **73**, 2974 (1994).
[33] G. Lemarié, J. Chabé, P. Szriftgiser, J. C. Garreau, B. Grémaud, and D. Delande, Phys. Rev. A **80**, 043626 (2009).
[34] Fritz Haake, *Quantum signatures of chaos*, 2nd ed. (Springer-Verlag, Berlin, 1991).
[35] F. M. Izrailev and D. L. Shepelyanskii, Theor. Math. Phys. **43**, 553 (1980).
[36] T. Dittrich and R. Graham, Ann. Phys. (N.Y.) **200**, 363 (1990).
[37] We note that the model [Eqs. (2) and (3)] always has one conserved quantity, the z projection of the total spin, $S_z = \sum_{i=1}^{N} \sigma_i^z$. However, we have checked that this global conservation law is not essential for the existence of MBL systems and ergodic phases, by studying other models where S_z is not conserved.
[38] We have also checked that the full distribution of parameter r is described by the COE and Poisson distribution in the two regimes.
[39] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993).
[40] J. A. Kjäll, J. H. Bardarson, and F. Pollmann, Phys. Rev. Lett. **113**, 107204 (2014).
[41] P. Ponte *et al.*, arXiv:1410.8518 [Phys. Rev. Lett. (to be published)].
[42] H. Kim, T. N. Ikeda, and D. A. Huse, Phys. Rev. E **90**, 052105 (2014).
[43] R. B. Sidje, ACM Trans. Math. Softw. **24**, 130 (1998).
[44] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[45] R. Vasseur, S. A. Parameswaran, and J. E. Moore, arXiv:1407.4476.
[46] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. B **90**, 174302 (2014).
[47] M. Bukov, L. D'Alessio, and A. Polkovnikov, arXiv:1407.4803.
[48] A. Lazarides, A. Das, and R. Moessner, arXiv:1410.3455.