Hyperuniformity of Critical Absorbing States

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The properties of the absorbing states of nonequilibrium models belonging to the conserved directed percolation universality class are studied. We find that, at the critical point, the absorbing states are hyperuniform, exhibiting anomalously small density fluctuations. The exponent characterizing the fluctuations is measured numerically, a scaling relation to other known exponents is suggested, and a new correlation length relating to this ordering is proposed. These results may have relevance to photonic band-gap materials.

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Dynamical systems with absorbing states [1–3] are intrinsically out of equilibrium, as they necessarily violate detailed balance—by definition, once they arrive at an absorbing state, they can never get out. There are many such systems, and they have an extensive literature [3]. Recently [4,5], an experimental system of non-Brownian colloids displaying a transition to an absorbing state has been studied, and a model, called "random organization" [5], has been proposed as a description of the experimental phenomena. Additionally, a similar transition has been observed in a dense glassy system of Brownian colloids under shear [6].

Absorbing states satisfy some model-dependent condition, the specifics of which do not appear to change the behavior qualitatively. The control-parameter space is divided into two regions—one where absorbing states are achieved for any typical initial state, the other where absorbing states, although possible, are not attained, resulting in a nonequilibrium steady state with a nonzero average number of "active" particles which violate the stipulated condition. These regions are separated by a critical line, and the transition between the two phases displays characteristics of a continuous phase transition, with a diverging correlation length. This phenomenology is common to all absorbing state models.

In this Letter, we shall study the "order" that develops in several absorbing state models as the system approaches the critical point from within the absorbing phase. Remarkably, even though the initial states are random, the dynamics are random, and there is an enormous number of random absorbing states, those absorbing states actually attained by the system are special, with greatly diminished density fluctuations. As we approach criticality, these fluctuations scale differently from the usual random system—they are, in fact, hyperuniform [7,8]. Hyperuniform systems have recently attracted interest due to their relation to photonic band-gap materials [9,10]. Thus, the setups in [4–6] may provide an efficient method for creating such materials in bulk.

For a system in d dimensions, we characterize the density fluctuations in a region of volume $V = \ell^d$ by

$$\sigma^2(\ell) \equiv \langle \rho^2(\ell) \rangle - \langle \rho(\ell) \rangle^2 \propto \ell^{-\lambda}, \tag{1}$$

where $\rho(\ell)$ is the number of particles in the region divided by V. The average is taken over many different volumes of the same system. Random systems, such as a Poisson process, have the standard "$\sqrt{N}$" fluctuations; this corresponds to $\lambda = d$. When $\lambda > d$, the particles are distributed more uniformly; such distributions are termed hyperuniform [7,11]. We find that, for the models we study, $\lambda > d$ for critical absorbing states. In all cases, the initial configuration is random with no correlations [$\sigma^2(\ell) \propto \ell^{-d}$]; hence, any anomalous value of the exponent λ is due to the dynamics. This behavior should be contrasted with equilibrium models, like the Ising model, whose fluctuations are enhanced at the critical point, with $\lambda < d$ [12].

If the system is not exactly critical, we find a crossover from hyperuniform behavior at short scales to random behavior at large scales; this enables the definition of a static correlation length $\xi_\times$ which diverges as the critical line is approached. This behavior is common to all the models we examined, with exponents that appear to be universal. We note that this correlation length is different in character and origin from that typically studied for such systems, which is defined in the active phase.

The systems we shall study are representatives of the conserved directed percolation ("Manna") universality class [2,3]. In all cases, the particles in a given configuration are distinguished into "active" and "inactive" particles. At each time step, the active particles, if there are any, are displaced randomly with a bounded distribution. Particle displacements are simultaneous, and the dynamics continue until there are no more active particles (this is the absorbing phase) or until a steady state sets in (with a finite active density). Although it is not essential, we employ

periodic boundary conditions and all the data we present are averaged of 50–100 realizations.

The various models differ in the criteria defining which of the particles are active and what sort of random displacements will be performed. Whatever their details, the phase transitions are characterized by a set of universal critical exponents which depend only on the dimensionality. The models we consider in this Letter are (1) conserved lattice gas (CLG) (2D and 3D) [13]: Particles are placed randomly on a d-dimensional cubic lattice of volume L^d with density ρ, such that each lattice site is occupied by, at most, one particle. Particles with adjacent neighbors are considered active. At each time step, each active particle moves to a random empty neighboring site. If several active particles attempt to move to the same empty site, only one of these moves is allowed; the other particles remain where they were. (2) Manna model (1D) [2]: Particles are distributed randomly on a one-dimensional lattice of length L with density ρ, allowing multiple occupancy. If there are more than two particles on a site, they are considered active, and each of them independently moves randomly to the neighboring site to the left or the right. (3) Random organization (1D): N particles of unit diameter are distributed at random positions in a one dimensional segment of length L. Overlapping particles are considered active, and at every time step, each is given a random displacement, distributed uniformly in $[-\epsilon, \epsilon]$. In this Letter, we have taken $\epsilon = 0.25$. (4) Random organization (2D) [5]: Particles with diameter d are distributed uniformly in an $L \times L$ square, with a covering fraction $\phi = (N/L^2)(\pi d^2/4)$. The system is then sheared in the x direction with strain amplitude γ. If, in this process, two particles collide, they are deemed active [14] and are given a random displacement whose magnitude is drawn uniformly from the range $[0, d/2]$, with a uniformly chosen angle. γ and ϕ are the control parameters of the model, and the critical value of γ depends on ϕ.

We are particularly interested in the spatial distribution of particles, and we measure the variance of the density $\sigma^2(\ell)$ in a region of volume $\sim \ell^d$ to see how it scales with ℓ, as in Eq. (1). The asymptotic scaling is related to large scale correlations and is not sensitive to short range fluctuations. Typically, when $\lambda \neq d$ or $d+1$, long range correlations are present in the system. To see this, consider the variance of the number of particles in a region which, for simplicity, we take to be a hypersphere

$$\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle + \int d^d r_1 \int d^d r_2 h(r_1, r_2), \qquad (2)$$

where $h(r_1, r_2) = \langle \rho(r_1)\rho(r_2) \rangle - \langle \rho(r_1) \rangle \langle \rho(r_2) \rangle$ is the two-point correlation function, and the integration is over a hypersphere. If h is translationally invariant, $h(r_1, r_2) = h(r_1 - r_2)$, and we get

$$\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = 1 + \frac{1}{\rho} \int d^d r h(r) \alpha\left(\frac{r}{\ell}\right), \qquad (3)$$

where $\alpha(r/\ell)$, a dimension-dependent function which takes account of the domain of integration, lies between 0 and 1 for $0 < (r/\ell) < 2$, and vanishes, otherwise [7]. For a hyperuniform system, the left hand side of Eq. (3) must vanish as $\ell \to \infty$, so we must have $\int d^d r h(r) = -\rho$ [15].

As discussed in Ref. [7], long-range negative correlations typically characterize hyperuniform systems. This is most easily shown in 1D, where the function $\alpha(r/\ell) = 1 - (r/2\ell)$ [16]. In this case, assuming the scaling form of Eqs. (1) and (3) becomes

$$\ell^{2-\lambda} \sim \ell + \frac{1}{\rho} \int_0^{2\ell} dr h(r) \left(\ell - \frac{r}{2}\right). \qquad (4)$$

Differentiating twice with respect to ℓ yields $h(r) \sim -r^{-\lambda}$ [17].

Power law correlations in continuous phase transitions often imply a diverging length scale. In absorbing-state systems, this is manifested in a spreading length which diverges as the critical point is approached. This length is the distance that activity ensues following a perturbation, and is discussed in [1–3]. We shall presently discuss another length which diverges at the transition.

We now discuss the numerical results for the different models, starting with the 2D CLG. In Fig. 1, we plot $\sigma^2(\ell)$ vs ℓ for several densities below the critical density $\rho_c \simeq 0.2391$. For $\rho < \rho_c$, we find, at short distances, that $\sigma^2(\ell)$ scales as $\ell^{-\lambda}$ with $\lambda = 2.45 \pm 0.03$, going over, at larger distances, to a ℓ^{-2} decay. This enables us to define a crossover length $\xi_\times$ on the absorbing side of the transition. As ρ approaches ρ_c, $\xi_\times$ grows, until finally at ρ_c, it diverges, and the second scaling region disappears.

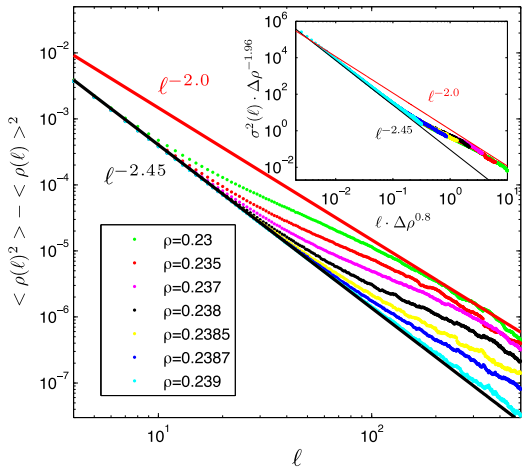

FIG. 1 (color online). Density fluctuations in an $\ell \times \ell$ box for the 2D CLG for some different densities. Data collapse is shown in the inset. Here, $L = 1000$ and $\rho_c \simeq 0.2391$.

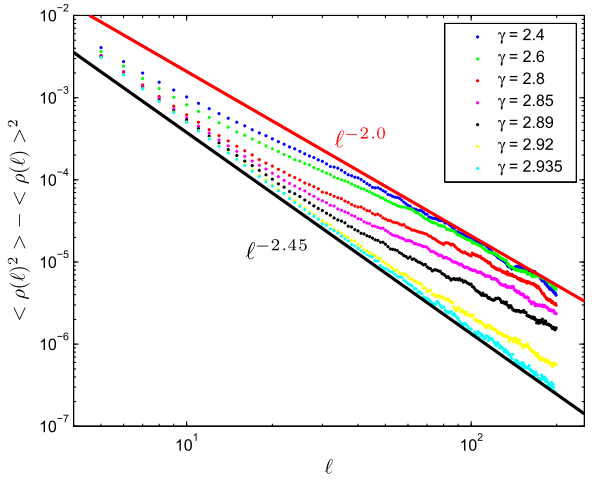

FIG. 2 (color online). Mean square density fluctuations for the 2D random organization model at various values of strain $\gamma < \gamma_c \simeq 2.935$, with $\phi = 0.2$ and $L = 400$.

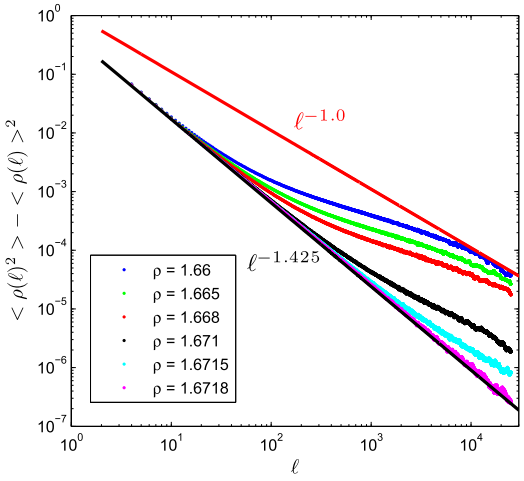

FIG. 3 (color online). Density fluctuations in the 1D Manna model as a function of length ℓ. Here, $L = 100\,000$ and $\rho < \rho_c \simeq 1.6718$.

To check if $\xi_\times$ has the same dependence as the active site-active site correlation length [2] which diverges as $\xi_{AA} \propto |\rho - \rho_c|^{-\nu_\perp} \equiv |\Delta\rho|^{-\nu_\perp}$, we rescale ℓ by $|\Delta\rho|^{-\nu_\perp}$, where $\nu_\perp \simeq 0.8$ in 2D [1]. A true data collapse is not quite possible since there are two scaling regimes. We choose to match the first scaling regime, so we rescale the y axis by $\Delta\rho^{-\nu_\perp\lambda}$. As seen in the inset to Fig. 1, the data collapses rather well, suggesting that $\xi_\times$ scales in the same way as the active-active correlation length [1], despite the fact that the former is defined in the absorbing phase, and the latter in the active phase.

To relate to the experiment of Ref. [4], and to test the universality of the exponent λ, we simulate the 2D random organization model introduced in [5]. This model is intrinsically anisotropic, due to the directionality of the shear. As in the experiment, we set the covering fraction ϕ and vary the strain γ. In Fig. 2, $\sigma^2(\ell)$ is shown for $\phi = 0.2$ and various values of the strain. Phenomenology similar to that of the CLG is seen, with the same value of the exponent λ, suggesting that, indeed, λ is universal despite the anisotropy of the system. This is in agreement with Ref. [18] which asserts that anisotropy should not change the universality class of the system.

In addition to the two-dimensional systems above, we have also simulated the 1D Manna model and the 1D random organization model of [5], as well as the 3D CLG. We find that, in both one-dimensional models, $\lambda_{1d} \simeq 1.425 \pm 0.025$, as shown in Figs. 3 and 4, while for the 3D CLG, $\lambda_{3d} \simeq 3.24 \pm 0.07$, as discussed below.

The picture that emerges from these results is that wherever the dynamics act, they smooth out density fluctuations in such as way as to become hyperuniform [19]. When the system is in the absorbing phase (say, at low densities), the dynamics take place over regions of size $\xi_\times$, so the system becomes hyperuniform on length scales of this order. As criticality is approached, the dynamics occur over larger and larger scales, and $\xi_\times$ diverges. For this reason, $\xi_\times$ can be regarded as a basic correlation length describing the ordering of the system.

Experimentally, hyperuniformity may be identified by measuring the structure factor $S(k)$ [20], which, for small values of k, scales as $S(k) = Ak^{\lambda-d} + N\delta(k)$

$$S(k) = 1 + \frac{1}{\rho}\int d^d r \langle \rho(r)\rho(0)\rangle e^{-ikr}$$
$$= 1 + \frac{1}{\rho}\int d^d r h(r) e^{-ikr} + N\delta(k). \quad (5)$$

With $h(r) \propto r^{-\lambda}$, the result follows [21]. Disregarding the delta function, $S(k) \to 0$ as $k \to 0$; this is the signature of hyperuniformity [7].

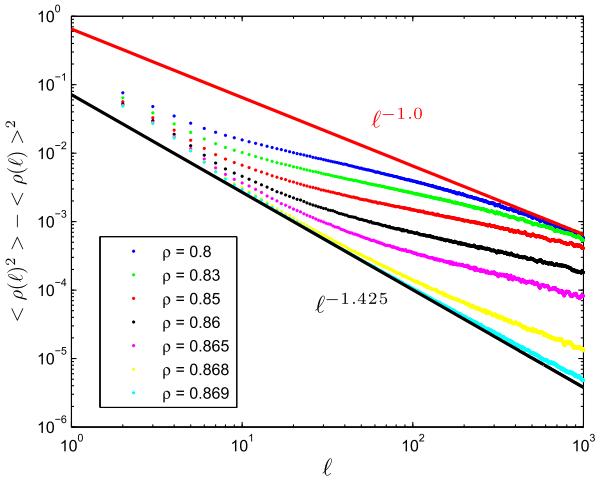

FIG. 4 (color online). Density fluctuations as a function of ℓ for 1D random organization. Here, $L = 10\,000$ and $\rho < \rho_c \simeq 0.8692$.

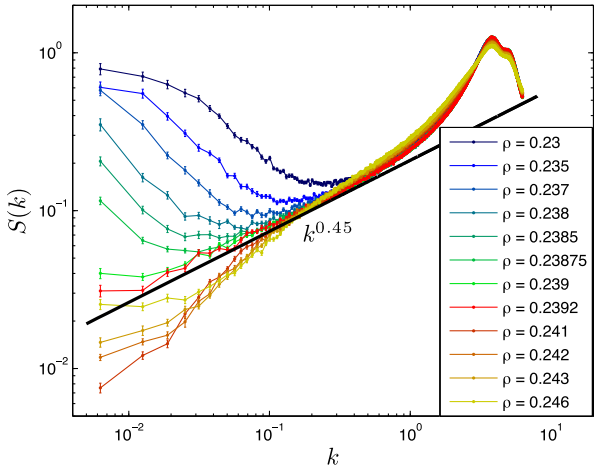

FIG. 5 (color online). The structure factor $S(k)$ at small k for the 2D CLG. Here, $L = 1000$ and $\rho_c \simeq 0.2391$.

Let us consider the small k behavior of $S(k)$ for the 2D CLG (other models are similar). To do this, we first evaluate $S(k)$ numerically at values of $\vec{k} = (n, m)(2\pi/L)$, with n, m integers [22], and then perform an angular average for a given $|k| = (2\pi n/L)$ using all the wave vectors in the range $(2\pi n/L) \leq |k| < [2\pi(n+1)/L]$. As seen in Fig. 5, at the critical point, $S(k) \sim |k|^{\lambda-d}$ for small k, with $\lambda \simeq 2.45$. When $\rho < \rho_c$, $S(k)$ is proportional to $|k|^{\lambda-d}$ for intermediate values of k, with $S(k) \to$ const as $k \to 0$ [23].

Surprising behavior is observed for $\rho > \rho_c$ in the active region. Here, we measure $S(k)$ from snapshots of the system, since it includes active particles and is dynamic. Slightly above the critical density, $S(k)$ appears to drop faster than $|k|^{\lambda-d}$ for small values of k, as shown in Fig. 5: fluctuations are reduced on large scales. We believe that this behavior must get cut off at large enough length scales when off criticality, with $\lim_{k\to 0} S(k) > 0$ indicating a finite (perhaps large) correlation length.

Last, we observe a relation between the critical exponent λ to a known exponent of the Manna universality class. First, we note that at criticality, the active-active correlation function [1] (measured in the active phase, of course) goes as $c(r) \sim r^{2-d-\eta_\perp}$, where $\eta_\perp$ is an exponent whose values are given in Table I. As noted above, the density-density correlation function at criticality scales as $h(r) \sim r^{-\lambda}$. As shown in Table I, our data is well fit in all dimensions by the form

$$\lambda = d + 2 - \eta_\perp, \tag{6}$$

which is different from the active-active correlation exponent, though both involve the exponent $\eta_\perp$.

TABLE I. The measured values of λ and the proposed scaling relation. Values for $\eta_\perp$ are from Ref. [3]. The exponent λ and its error were measured in 1D for the Manna model while in 2D and 3D for the CLG model.

d	λ	$\eta_\perp$	$d + 2 - \eta_\perp$
1	1.425 ± 0.025	1.592 ± 0.040	1.4076 ± 0.040
2	2.45 ± 0.03	1.541 ± 0.025	2.4588 ± 0.025
3	3.24 ± 0.07	1.744 ± 0.029	3.2557 ± 0.029

We conclude by comparing this behavior to that of an equilibrium system. In equilibrium, particle number fluctuations are related to the isothermal compressibility $\kappa_T \equiv -(1/V)(\partial V/\partial P)_T$ by $\langle N^2\rangle - \langle N\rangle^2 = \langle N\rangle \rho k_B T \kappa_T$. If κ_T is finite and nonzero, the particle number variance is extensive, yielding $\lambda = d$. This result is expected for finite ranged interactions since there is a finite energy cost to move a particle to some random location, so that the density of displaced particles is proportional to the system volume. Hyperuniformity, therefore, would not be manifested in equilibrium states of systems with finite-range interactions at finite temperature.

Why then, do the absorbing state models herein considered exhibit hyperuniformity? In the models we study, there is an essential irreversibility—where the density is below critical, the system does not evolve. However, active regions are of higher density, and the dynamics tend to push particles away to neighboring regions of lower density. For sparsely populated systems, this process proceeds over a finite distance (the correlation length), but as the density increases, the redistribution takes place over ever larger distances.

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[11] We note two limiting cases: (a) Poisson distributed particles: $\sigma^2(\ell) \sim \ell^{-d}$, and (b) particles in a perfect lattice, which typically have $\sigma^2(\ell) \sim \ell^{-(d+1)}$. The systems we study lie between these two cases.
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[14] This is equivalent to surrounding each particle by an envelope of a particular shape; the particles are active if the envelopes overlap.
[15] See Ref. [7] for a detailed discussion about the behavior of correlation functions in the context of equilibrium and hyperuniform systems.
[16] Explicit forms for α are given in Ref. [7] for several dimensions.
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[19] This crossover is clearly seen for the CLG and Manna models. For random organization, the finite size of the particle (or, equivalently, the applied strain) obscures this at very small scales. It becomes more evident, however, close to the critical point, as seen in Figs. 2 and 4.
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[22] Note that this is equivalent to putting the system under periodic boundary conditions, which removes the finite size effects near $k = 0$.
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.114.110602 for graphs on system size dependence of $S(k)$ and for $S(k)$ for the 1D Manna model.