Topological Index for Periodically Driven Time-Reversal Invariant 2D Systems

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We define a new $\mathbf{Z}_2$-valued index to characterize the topological properties of periodically driven two dimensional crystals when the time-reversal symmetry is enforced. This index is associated with a spectral gap of the evolution operator over one period of time. When two such gaps are present, the Kane-Mele index of the eigenstates with eigenvalues between the gaps is recovered as the difference of the gap indices. This leads to an expression for the Kane-Mele invariant in terms of the Wess-Zumino amplitude. We illustrate the relation of the new index to the edge states in finite geometries by numerically solving an explicit model on the square lattice that is periodically driven in a time-reversal invariant way.

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Introduction.—The recent discovery of the quantum spin Hall effect [1–4] renewed interest in topological insulating phases which were first encountered in the beginning of the 1980s with the discovery of the quantum Hall effect (QHE) [5–8]. In his seminal paper [9] of 1981, Laughlin related the quantized Hall conductance to a quantum pump adiabatically driven by the magnetic flux. As shown by Thouless [10], such pumps drive through the insulator an integer number of charges whose origin is topological. In deep analogy, several works interpreted topological insulators and their robust boundary states in terms of quantum adiabatic pumps [11–13]. Interestingly, quantum crystals can exhibit original topological features when periodically driven *beyond* the adiabatic regime. While such modulation was first proposed to trigger a topological phase transition [14–16], it can also yield specific topological properties which cannot be understood within the usual framework of topological band theory [17,18]. The search for these so-called *Floquet* topological states quickly became a very active field and has recently stimulated numerous experimental works. A realization of such phases in condensed matter is quite challenging [19,20]. Several alternative artificial systems have been proposed to simulate and probe analogous phases, such as lattices of photonic resonators periodically driven by electro-optic modulators [21], ring resonator lattices [22], or more recently, photons coupled to excitons in semiconductors [23]. Signatures of topological Floquet states have already been revealed in one-dimensional quantum walks with photons [24], as well as in 2D waveguide lattices [25]. Shaken trapped cold atoms were also proposed as a good candidate [26–28] and nontrivial topological phases were recently observed there [29].

Remarkably, although dissipation is inherent to driven systems, signatures of the topological properties observed in these experiments can be captured by Hermitian or unitary operators. In particular, topological properties of 2D periodically driven systems with no additional symmetry are well characterized by the invariant proposed by Rudner *et al.* [18] generalizing the description of static band insulators in terms of first Chern numbers. For equilibrium phases in the presence of symmetries, indices different from Chern numbers are required to describe topological properties. An analogous treatment for periodically driven systems is still lacking. Different works addressed recently this question in 1D systems [30–32]. Of great importance is, however, the case of 2D periodically driven time-reversal invariant (TRI) fermionic systems. In this Letter we introduce a novel topological index that generalizes the Kane-Mele (KM) invariant for the equilibrium quantum spin Hall phases [2] in the spirit of the invariant of [18] for unconstrained periodically driven systems.

We define a $\mathbf{Z}_2$-valued quantity $K_\epsilon[U]$ that depends on the (quasienergy) spectral gap ϵ of the evolution operator over one period of time $U(T)$ in such a way that its difference for two distinct quasienergy gaps ϵ and ϵ' satisfies the relation

$$\mathrm{KM}(\mathcal{E}_{\epsilon,\epsilon'}) = K_{\epsilon'}[U] - K_\epsilon[U], \tag{1}$$

where $\mathrm{KM}(\mathcal{E}_{\epsilon,\epsilon'})$ is the Kane-Mele invariant of the vector bundle $\mathcal{E}_{\epsilon,\epsilon'}$ of eigenstates of $U(T)$ associated with the quasienergy band between the two gaps. Moreover, we find that this index builds unexpected bridges between the Kane-Mele invariant and the 2D Wess-Zumino action functional, providing a new expression for the static Kane-Mele index that brings the field theory toolbox to its analysis.

In the following, we first review the construction of the index W of [18] for the case with no symmetry. Then, we define the new invariant for TRI driven crystals. We finally illustrate some topological properties of the TRI periodically driven systems on a simple lattice model.

Invariant for 2D periodically driven systems.—The Hamiltonian H describing a system on a translation invariant lattice may be block diagonalized by the Fourier transform. This produces the Bloch Hamiltonians $H(k)$ acting on $\boldsymbol{C}^N$ for k in the Brillouin torus (BZ), where N is the (finite) number of internal degrees of freedom. It is always possible to assure that $H(k) = H(k+G)$ for G in the reciprocal lattice. A periodically driven system may be described by a time periodic Hamiltonian which corresponds to Bloch Hamiltonians periodic both in quasimomenta and in time, $H(t,k) = H(t,k+G) = H(t+T,k)$.

The time evolution of such systems is described by unitary operators $U(t,k) \in U(N)$ satisfying the equation $i\dot{U}(t,k) = H(t,k)U(t,k)$ with initial condition $U(0,k) = I$. Operators $U(t,k)$ define a smooth mapping from $[0,T] \times \mathrm{BZ}$ to $U(N)$. A natural invariant that characterizes the topological properties of smooth maps between two manifolds is their homotopy class [33], which is, however, trivial for U. This could be different if U were periodic in time defining a map from $S^1 \times \mathrm{BZ}$ to $U(N)$. One may periodize U in a natural way using Floquet theory if unitary operators $U(T,k)$ have a common spectral gap [18].

To do so explicitly, one diagonalizes the unitary operator $U(T,k)$ (this is the essence of the Floquet theory) as

$$U(T,k) = \sum_{j=1}^{N} \lambda_j |\psi_j\rangle\langle\psi_j| \tag{2}$$

and one defines the effective Hamiltonian

$$H_\epsilon^{\mathrm{eff}}(k) = \frac{i}{T}\sum_{j=1}^{N} \ln_{-T\epsilon}(\lambda_j)|\psi_j\rangle\langle\psi_j|, \tag{3}$$

where $\ln_{-T\epsilon}$ is the logarithm with cut at argument $-T\epsilon$, so that $U(T,k) = e^{-i\mathrm{T}H_\epsilon^{\mathrm{eff}}(k)}$. For this effective Hamiltonian to depend smoothly on k, $e^{-iT\epsilon}$ has to lie in an eigenvalue gap of $U(T,k)$ for all k. The quantities ϵ_j such that $\lambda_j = e^{-iT\epsilon_j}$ are called *quasienergies*, so this gap is a quasienergy gap. This allows us to define the periodized versions of $U(t,k)$

$$V_\epsilon(t,k) = U(t,k)e^{itH_\epsilon^{\mathrm{eff}}(k)}, \tag{4}$$

which satisfy $V_\epsilon(0,k) = I = V_\epsilon(T,k)$. The maps V_ϵ, explicitly dependent on the cut $-T\epsilon$, may be considered as defined on the 3-torus $S^1 \times \mathrm{BZ}$. As described in [18], the integer-valued integral [34]

$$W_\epsilon[U] = \frac{1}{24\pi^2}\int_{[0,T]\times \mathrm{BZ}} \mathrm{tr}(V_\epsilon^{-1}dV_\epsilon)^3 \equiv \deg(V_\epsilon), \tag{5}$$

which we shall, somewhat abusively, call the degree of map V_ϵ, defines a topological invariant that may be associated to the gap containing ϵ. It is well defined until the gap closes. The invariants W are thus attached to gaps in the spectrum unlike the first Chern numbers that are attached to spectral bands. Nonetheless, when $U(T,k)$ has two quasienergy gaps $0 \le \epsilon < \epsilon' < 2\pi/T$ then $H_{\epsilon'}^{\mathrm{eff}}(k) - H_\epsilon^{\mathrm{eff}}(k) = (2\pi/T)P_{\epsilon,\epsilon'}(k)$, where $P_{\epsilon,\epsilon'}(k)$ are projectors on states $|\psi_j\rangle$ with eigenvalues λ_j such that $\epsilon < \arg(\lambda_j^{-1})/T < \epsilon'$. The first Chern number of the vector bundle $\mathcal{E}_{\epsilon,\epsilon'}$ on which $P_{\epsilon,\epsilon'}(k)$ projects is then related to indices W by $c_1(\mathcal{E}_{\epsilon,\epsilon'}) = W_{\epsilon'}[U] - W_\epsilon[U]$ [18].

Index K for TRI periodically driven systems.—In static systems, time-reversal invariance is described at the level of Hamiltonian as $\Theta H \Theta^{-1} = H$ with the time-reversal operator $\Theta = e^{-i\pi S_y/\hbar}C$, where S is the spin operator and C represents the complex conjugation. More generally, a system is time-reversal invariant when $\Theta H(t)\Theta^{-1} = H(-t)$, or, equivalently, $\Theta U(t)\Theta^{-1} = U(-t)$, up to a choice of the origin of time. For a family of Bloch Hamiltonians, this property translates into $\Theta H(t,k)\Theta^{-1} = H(-t,-k)$, i.e., $\Theta U(t,k)\Theta^{-1} = U(-t,-k)$ for the evolution operators.

The first key observation is that, in a TRI periodically driven system, Kramers pairs $(\psi_j, \Theta\psi_j)$ of eigenstates of $U(T,k)$ and $U(T,-k)$, respectively, "evolve in time" in opposite directions: $\Theta U(t,k)\psi_j(k) = U(-t,-k)\Theta\psi_j(k)$ and similarly $\Theta V_\epsilon(t,k)\psi_j(k) = V_\epsilon(T-t,-k)\Theta\psi_j(k)$, as illustrated in Fig. 1(a). This property implies that $W_\epsilon[U] = 0$ because the contributions from two Kramers partners cancel. Consequently, also the first Chern numbers of quasienergy bands vanish in TRI periodically driven systems, as for the energy bands in the static case. To circumvent this cancellation, we shall keep the contribution only of one member of each Kramers pair by restricting the time evolution to times between $t = 0$ and $t = T/2$ [see Fig. 1(a)] in the same spirit than the construction by Moore and Balents in static systems [35].

The second key observation is that TR relates states at time $t = T/2$ to states at the same time. We may thus

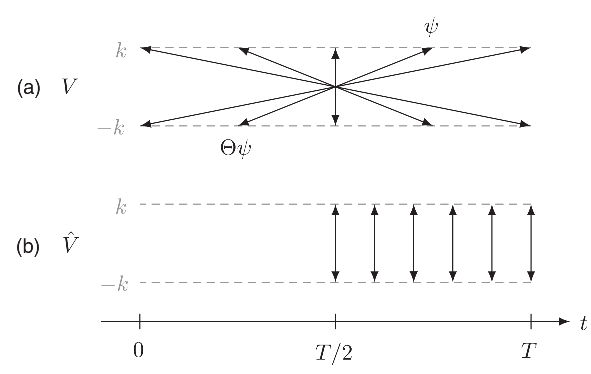

FIG. 1. Sketch of the periodized evolution over period T. (a) Periodized evolution $V(t,k)$. Time-reversal relates pairs $(\psi, \Theta\psi)$ of states at (t,k) and $(-t,-k)$, as shown by black arrows. (b) Contracted half-evolution $\hat{V}(t,k)$. The second half of the initial evolution was discarded and replaced by a contraction respecting an equal-time constraint (6b) depicted as black arrows.

directly compare the evolved Kramers partners at $t = T/2$. Then it is possible to deform the map $k \mapsto V_\epsilon(T/2, k)$ to the identity through a contraction of the Kramers pairs present at $t = T/2$. This will result in a periodic map containing the information on the topological winding of Kramers pairs during the first half-period; see Fig. 1(b).

More concretely, we consider a smooth map $\hat{V}_\epsilon$ from $[0, T] \times \mathrm{BZ}$ to $U(N)$ such that

$$\hat{V}_\epsilon(t, k) = V_\epsilon(t, k) \quad \text{for } 0 \leq t \leq T/2 \tag{6a}$$

and

$$\Theta \hat{V}_\epsilon(t, k) \Theta^{-1} = \hat{V}_\epsilon(t, -k) \quad \text{for } T/2 \leq t \leq T, \tag{6b}$$

with $\hat{V}_\epsilon(T, k) = I = \hat{V}_\epsilon(0, k)$. The $\mathbf{Z}_2$-valued index K is defined by the relation

$$K_\epsilon[U] = \deg(\hat{V}_\epsilon) \bmod 2 \tag{7}$$

(remember that V_ϵ is defined from U). This quantity is well defined. The existence of contractions (6b), the independence of $K_\epsilon[U]$ upon their choice, and the link with bundle Kane-Mele invariants (1) will be discussed elsewhere (see also Supplemental Material [36]). In a semi-infinite system, $K_\epsilon[U]$ should give the parity of the number of Kramers pairs of edge states that lie in the corresponding bulk quasienergy gap.

Result (1) greatly simplifies when spin is conserved. In this case, the evolution operator U is block diagonal in the $(\uparrow, \downarrow)$ basis and so is V_ϵ. The two blocks are related by time reversal and the K index can be related to the W index of one of the spin blocks, namely,

$$K_\epsilon\left[\begin{pmatrix} U_\uparrow & 0 \\ 0 & U_\downarrow \end{pmatrix}\right] = \frac{W_\epsilon[U_\uparrow] - W_\epsilon[U_\downarrow]}{2} \bmod 2, \tag{8}$$

where $W_\epsilon[U_\uparrow] = W_\epsilon[\Theta U_\downarrow \Theta^{-1}] = -W_\epsilon[U_\downarrow]$. This expression is reminiscent of the "spin Chern number" [37]. Indeed, when considering the difference between indices at two quasienergy gaps, the usual spin Chern number is recovered.

Relation to the Wess-Zumino amplitude.—The difference $K_{\epsilon'}[U] - K_\epsilon[U]$ is equal to the degree taken modulo 2 of the map $\hat{V}_{\epsilon,\epsilon'}$ constructed as in (6) from $V_{\epsilon,\epsilon'} = e^{-2\pi i t P_{\epsilon,\epsilon'}(k)/T}$. Because of the relation $\Theta V_{\epsilon,\epsilon'}(t, k) \Theta^{-1} = V_{\epsilon,\epsilon'}(t, -k)^{-1}$, the first half-period of time does not contribute to the integral for $\deg(\hat{V}_{\epsilon,\epsilon'})$. The latter involves then only the contribution of the contraction (6b) of $V_{\epsilon,\epsilon'}(T/2)$. Up to a factor, this contribution coincides with the Wess-Zumino (WZ) action [38,39] of the $U(N)$-valued field $V_{\epsilon,\epsilon'}(T/2, k) = I - 2P_{\epsilon,\epsilon'}(k)$ defined on BZ:

$$\deg(\hat{V}_{\epsilon,\epsilon'}) = -\frac{1}{2\pi} S_{\mathrm{WZ}}[V_{\epsilon,\epsilon'}(T/2)]. \tag{9}$$

The action S_{WZ} is normally determined modulo 2π but the condition (6b) for $\hat{V}_{\epsilon,\epsilon'}$ makes it well-defined modulo 4π. We infer then from (1) the identity

$$(-1)^{\mathrm{KM}(\mathcal{E}_{\epsilon,\epsilon'})} = e^{(i/2) S_{\mathrm{WZ}}[V_{\epsilon,\epsilon'}(T/2)]}, \tag{10}$$

relating the Kane-Mele invariant to the square root of the Wess-Zumino amplitude. This identity holds also in the static TRI case, providing a new expression for the 2D Kane-Mele invariant of the valence sub-bundle of states.

Time-dependent lattice model.—We shall now illustrate some of the basic properties of TRI periodic evolution operators characterized by the K index with a simple time-dependent lattice model. A natural starting point is to define for spin $\uparrow$ states a periodic evolution possessing a nonvanishing invariant W_ϵ inside a spectral gap, and to deduce by time-reversal symmetry the evolution of spin $\downarrow$ states with the opposite W_ϵ. Moreover, to verify that for a finite sample the index K_ϵ gives the parity of the number of pairs of edge states in the corresponding gap independently of TRI gap-preserving perturbations, we add TRI spin flipping couplings to the model. For simplicity we define a time-step periodic dynamics such that, during each step α of the evolution, the corresponding Hamiltonian H_α is constant in time. One cycle of period T is split into four steps $(\alpha - 1)(T/4) \leq t < \alpha(T/4)$ so that the evolution operator $U(T) = U_4 U_3 U_2 U_1$, where $U_\alpha = \exp(-\mathrm{i} H_\alpha T/4)$. As the initial dynamics of spin $\uparrow$, we consider a quantum analog of walks along classical cyclotron orbits on a square lattice [17,18]. We distinguish two sublattices A and B and define the corresponding Hamiltonian for spin $\uparrow$ for the first step $\alpha = 1$ as $H_1^{\uparrow\uparrow} = J T_{+x}^{A \to B} + \mathrm{H.c.}$, where $T_{+x}^{A \to B}$ is the translation operator by one horizontal lattice spacing from sublattice A to B; see Fig. 2. Hamiltonians $H_\alpha^{\uparrow\uparrow}$ are obtained from $H_1^{\uparrow\uparrow}$ by replacing the translation operator $T_{+x}^{A \to B}$ by $T_{-y}^{B \to A}$ for $H_2^{\uparrow\uparrow}$, by $T_{-x}^{A \to B}$ for $H_3^{\uparrow\uparrow}$, and by $T_{+y}^{B \to A}$ for $H_4^{\uparrow\uparrow}$. Hamiltonians $H_\alpha^{\downarrow\downarrow}$ for spin $\downarrow$ states are deduced by TRI that

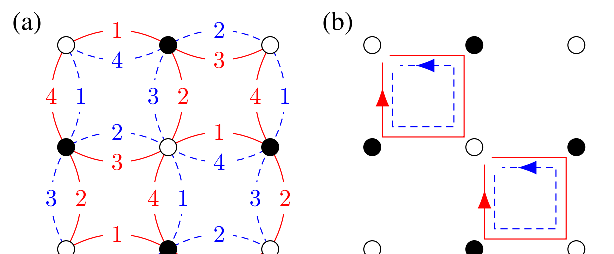

FIG. 2 (color online). Representation of the time evolution. (a) The only nonvanishing hopping amplitudes between sites are represented for each time step $\alpha = 1, \ldots, 4$ as a link labeled with α. This is done for spin-up (red solid lines) and spin-down (blue dashed lines). (b) The time sequence of U_α is summarized around plaquettes, which mimics cyclotron orbits. Full and empty circles represent the two sublattices.

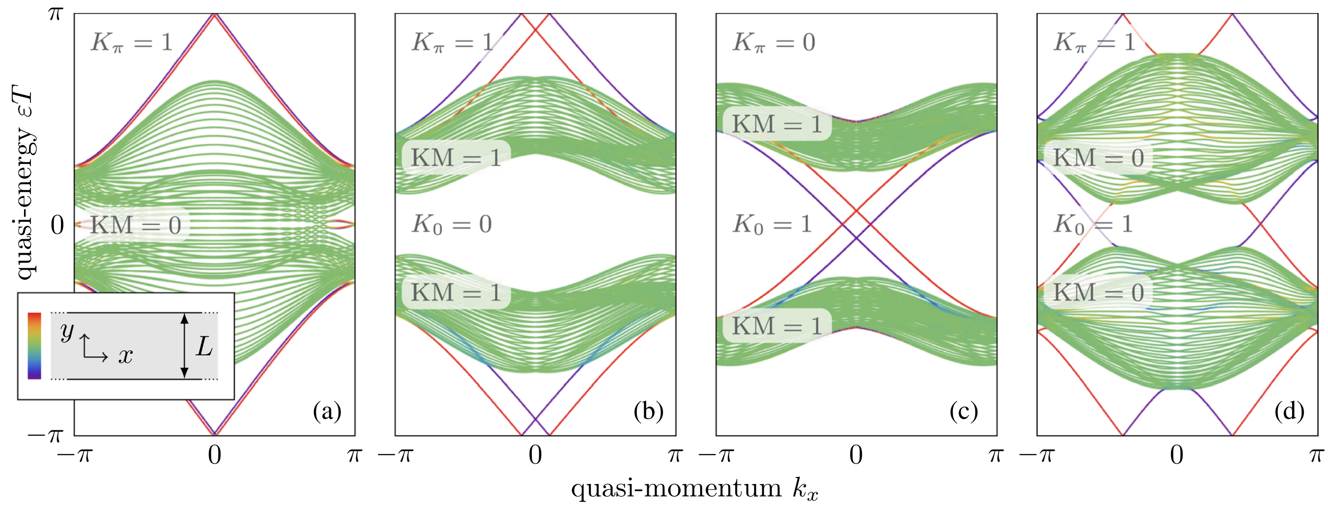

FIG. 3 (color online). Quasienergy spectra of the TRI periodic evolution for a strip geometry (inset). The spectra reveal helical edge states in the gap $\epsilon = 0$ or π. The parity of the number of pairs of edge states in a bulk gap localized on each boundary is given by the corresponding value K_ϵ. Colors correspond to the density of states along y: red and purple states are localized at opposite edges (see inset). The parameters are (a) $J = 3\pi$, $J' = \pi$, $\Delta = 0$, (b) $J = 3\pi/2$, $J' = 1/2$, $\Delta = \pi/2$, (c) $J = -5\pi$, $J' = 1/2$, $\Delta = 9\pi/2$, (d) $J = 15\pi/2$, $J' = \pi$, $\Delta = 2\pi$, For clarity, in the case (a), a small boundary mass term was added to distinguish edge states.

imposes $H^{\downarrow\downarrow}(t,k) = \bar{H}^{\uparrow\uparrow}(-t,-k)$. They correspond to quantum evolution along orbits cycling in opposite direction. In this initial model, the index K_ϵ associated to a spectral gap of $U(T)$ identifies modulo 2 with the index W_ϵ of the evolution of the $\uparrow$ states; see (8). This is no longer the case when spin-flipping terms are added to the dynamics. They are incorporated into the model by adding to the Hamiltonian off-diagonal terms $H_\alpha^{\downarrow\uparrow}$ obtained from $H_\alpha^{\uparrow\uparrow}$ by replacing J by J', together with $H^{\uparrow\downarrow}(t,k) = -\bar{H}^{\downarrow\uparrow}(-t,-k)$, as imposed by the TRI constraint.

The evolution operator over one period $U(T)$ is diagonalized for a strip geometry periodic in the x direction. The quasienergies of the strip, together with the localization of the eigenstates from each edge, are shown in Fig. 3. Figure 3(a) clearly shows in the only gap at $T\epsilon = \pi$ the existence of one pair of helical boundary states at each edge of the strip, although the KM index associated with the unique band is necessarily zero. The quasienergy gap does not close when J' decreases to zero (see Supplemental Material [36]). Consequently, the value K_π in the π gap remains unchanged and can be calculated from (8). We find $K_\pi = 1$, in agreement with the number of edge states.

Next, we consider a case with two bands, i.e., with two gaps. To do so, we add a staggered on-site potential Δ and a fifth step $\alpha = 5$ with $U_5 = I$ [18]. Two typical situations are found and illustrate our formula (1), relating the Kane-Mele invariant associated to each quasienergy band and the index K associated to the 0 and π gaps. In Fig. 3(b) [respectively, (c)] one pair of helical edge states appears in the π gap (respectively, 0 gap). We find in perfect agreement $K_0 = 0$ and $K_\pi = 1$ [Fig. 3(b)] and $K_0 = 1$, $K_\pi = 0$ [Fig. 3(c)] and the Kane-Mele invariants are nonzero for each band. In contrast, Fig. 3(d) illustrates a situation where one pair of edge states lies in each gap, in agreement with $K_0 = 1$ and $K_\pi = 1$, so the Kane-Mele band invariants vanish. This provides a TRI analog of the periodically driven TR breaking phases characterized with zero Chern band invariants but nonzero W gap invariants [18].

Conclusions.—We defined a new index characterizing topological properties of periodically driven 2D crystals constrained by time-reversal invariance, relating it to the Kane-Mele invariant of quasienergy bands for which we found a new expression in terms of the Wess-Zumino amplitude. This paves the way for a study of physical properties associated to these topological states, such as the out-of-equilibrium dc transport [16,40,41] by the edge states that our analysis reveals. In particular, the search for experimental signatures in electronic systems via transport properties in a multiterminal geometry is a particularly interesting direction for future investigations, but other proposals such as a local probe by means of tunneling current [42] are also conceivable. Various physical setups, like anisotropic metamaterials [43], dielectric ring resonators [44], or shaken optical lattices [45], seem to be good candidates to achieve experimentally such topological states as long as dissipation is not too strong. In particular, a Floquet analog of the Kane-Mele model seems to be achievable experimentally in a near future with cold atoms by modulating simultaneously in time the trapping lattice and a magnetic field gradient [29].

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