Solution to the Quantum Zermelo Navigation Problem

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The solution to the problem of finding a time-optimal control Hamiltonian to generate a given unitary gate, in an environment in which there exists an uncontrollable ambient Hamiltonian (e.g., a background field), is obtained. In the classical context, finding the time-optimal way to steer a ship in the presence of a background wind or current is known as the Zermelo navigation problem, whose solution can be obtained by working out geodesic curves on a space equipped with a Randers metric. The solution to the quantum Zermelo problem, which is shown here to take a remarkably simple form, is likewise obtained by finding explicit solutions to the geodesic equations of motion associated with a Randers metric on the space of unitary operators. The result reveals that the optimal control in a sense "goes along with the wind."

The problem of finding the optimal Hamiltonian for processing a given quantum state, or implementing a quantum operation (gate), in the shortest possible time subject to certain constraints, has attracted considerable attention over the past decade [1–8]. Broadly speaking, the task can be classified into two closely related categories: (a) transforming one quantum state into another; and (b) transforming one unitary operator into another, in the shortest possible time. If the constraint is concerned merely with a limit on the energy resource, then the optimal Hamiltonian is time independent, and can be found easily by noting that under a unitary motion, the shortest path coincides with the path along which the speed of evolution is also maximized [9,10]. If there are further constraints, for example, the choice of the Hamiltonian is limited, then often a time-dependent Hamiltonian that minimizes an action has to be determined by variational approaches [11,12]. Finding a solution to such a variational problem is in general difficult; however, an efficient numerical regularization scheme to obtain an approximate solution has been proposed more recently [13].

For many problems related to controlling quantum systems considered in the literature, it is assumed that the experimentalist has full control over the allowable range of Hamiltonians within the constraint, whereas in a laboratory there can often be situations in which the system is immersed in an external field or potential that is beyond control (e.g., gravitational or electromagnetic field), since a complete elimination of external fields in a laboratory can be prohibitively expensive for the given task. Evidently, this is a generic issue that needs to be addressed adequately to be able to accurately implement a rapid quantum processing.

In the present Letter we address this issue by finding the time-optimal control Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$ that generates a unitary motion to transform one unitary

operator $\hat{U}_I$ into another operator $\hat{U}_F$, subject to the constraints (i) that the background Hamiltonian $\hat{H}_0$ cannot be controlled, (ii) that the control Hamiltonian fulfills the energy resource bound of the form $\mathrm{tr}(H_1^2) = 1$ at all time, and (iii) that the background Hamiltonian is not dominating in the sense that $\mathrm{tr}(\hat{H}_0^2) < \mathrm{tr}(\hat{H}_1^2)$. This is the quantum counterpart of a well-known classical navigation problem posed by Zermelo: given the present location in the ocean, with a given wind and/or current distribution characterized by a location-dependent vector field, one wishes to find the optimal control of the vessel so as to reach the destination in the shortest possible time [14,15]. The vector field generated by the reference Hamiltonian $\hat{H}_0$ can be thought of as representing the background wind or current, whereas $\hat{H}_1$ determines the control.

In the classical context, it was observed by Shen [16] that the solution to the Zermelo navigation problem can be obtained by finding the geodesic curves associated with a Randers metric on the configuration space. Motivated by this result, more recently Russell and Stepney [17] introduced the quantum Zermelo navigation problem stated above, and analyzed the shortest time required to realize the transformation $\hat{U}_I \rightarrow \hat{U}_F$. Their observation that quantum Zermelo navigation problems can be solved by finding the geodesics of Randers metrics opens up the possibility of addressing a wide range of realistic quantum control problems where the environmental influence cannot be eliminated. However, analyses involving Randers spaces are generally difficult, and finding solutions to the geodesic equations is not straightforward [18,19]. Indeed, the only examples considered in [17] concern the time-independent cases, while the optimal navigation is realized by a time-independent Hamiltonian only if the background Hamiltonian $\hat{H}_0$ happens to be the one that realizes the

operation $\hat{U}_I \rightarrow \hat{U}_F$ [20]. But since $\hat{U}_I$ and $\hat{U}_F$ are arbitrary given unitary gates one wishes to implement, such a scenario will not prevail in real laboratories.

Here we solve this problem by deriving the Euler-Poincaré equation of motion for the control Hamiltonian $\hat{H}_1(t)$, and obtain the solution in closed form. Remarkably, we find that the solution to the quantum Zermelo navigation problem takes the simple form

$$\hat{H}_1(t) = e^{-i\hat{H}_0 t}\hat{H}_1(0)e^{i\hat{H}_0 t}, \qquad (1)$$

where $\hat{H}_1(0)$ is the initial condition such that the motion generated by the total Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$ takes $\hat{U}_I$ to $\hat{U}_F$. Thus, the optimal control is obtained by finding the initial direction $H_1(0)$ for the maneuver and drift along the "wind" $\hat{H}_0$. We then provide a scheme for finding the initial condition $\hat{H}_1(0)$. The results are illustrated in terms of a spin-$\frac{1}{2}$ system. We shall also indicate how the analysis presented here can be applied to situations where there are further constraints on the control Hamiltonian.

Since the mathematical machinery required for solving the navigation problem is perhaps not widely accessible to the broader physics community, we begin with a brief discussion of the background concepts before proceeding to derive (1). To address such navigation problems in the calculus of variation, it is often the case that one requires the notion of a distance that depends not only on the location but also on the direction—a concept that goes outside of the realm of Riemannian geometry. Specifically, for a given curve $x^i(t)$ on the configuration space $\mathfrak{M}$, equipped with a Riemannian metric, we consider the integral of the form

$$T = \int_{t_0}^{t_1} F(x^i, \dot{x}^i)dt \qquad (2)$$

for some positive function F, which is assumed to be homogeneous of first degree in $\dot{x}^i$, $F(x^i, \lambda\dot{x}^i) = \lambda F(x^i, \dot{x}^i)$ for any $\lambda > 0$, so that T is independent of the choice of the parameter t along $x^i(t)$. Thus, $F(x^i, \dot{x}^i)$ defines, for each fixed point $x^i \in \mathfrak{M}$, a distance on the tangent space of $\mathfrak{M}$. In particular, the level surface $F(x, \dot{x}) = 1$ on the tangent space of $\mathfrak{M}$ at x defines the indicatrix [15]. Now, for a fixed x and an arbitrary point ξ on the tangent space of $\mathfrak{M}$ at x, the ray $\vec{x\xi}$ clearly intersects the indicatrix at a point ρ_ξ. Thus, conversely, for each point ξ if we define a function F according to $F(\xi) = |\xi|/|\rho_\xi|$, where $|\cdot|$ denotes the Euclidean norm, then we can introduce a metric, known as the Minkowski metric [21], as follows: For ξ, ξ' on the tangent space of $\mathfrak{M}$ at x the distance between these points is defined by $D(\xi, \xi') = F(\xi - \xi')$. In particular, the metric tensor defined on $\mathfrak{M}$ induced by the distance function D associated with the fundamental function F can be expressed in the form

$$g_{ij}(x, \dot{x}) = \frac{1}{2}\frac{\partial^2}{\partial\dot{x}^i\partial\dot{x}^j}F^2(x^i, \dot{x}^i), \qquad (3)$$

and we have $F^2 = g_{ij}\dot{x}^i\dot{x}^j$.

In classical mechanics, often the fundamental function takes the form of the kinetic energy: $F^2 = \gamma_{ij}(x)\dot{x}^i\dot{x}^j$. Thus, the resulting metric $g_{ij}(x, \dot{x}) = \gamma_{ij}(x)$ is independent of the direction $\dot{x}$; i.e., it defines a Riemannian metric on $\mathfrak{M}$, since the indicatrix is just a sphere. In many problems with engineering applications, such as a navigation problem, however, the relevant function takes a different form, and one is required to go beyond the techniques of Riemannian geometry. Realizing this, Carathéodory suggested to his then Ph.D. student Finsler to investigate the geometry of spaces equipped with direction-dependent metrics [22] such as the one discussed above. Subsequently, spaces endowed with locally Minkowski metrics were referred to as Finsler spaces [23].

Let us now turn to the classical Zermelo navigation problem of reaching a target on a manifold $\mathfrak{M}$ equipped with a Riemannian metric h_{ij} in the shortest possible time, in the presence of background wind w^i. The analysis of the problem simplifies if we observe that it suffices to find the locally optimal solution on the tangent space [16]. Specifically, for any vector $\vec{\xi}$ on the tangent space to $\mathfrak{M}$ at x we can regard $|\vec{\xi}|_h = \sqrt{h_{ij}\xi^i\xi^j}$ as representing the time it takes to reach the end point of $\vec{\xi}$ in the absence of wind. Now suppose that the time it takes to reach the destination $\vec{u}$ at full throttle is 1 in a suitable unit (e.g., seconds), i.e., $|\vec{u}|_h = 1$. In the presence of wind, with $|\vec{w}|_h < 1$, however, after a journey of one second at full throttle the vessel will reach the point $\vec{v} = \vec{u} + \vec{w}$, instead of the destination $\vec{u}$. In other words, the unit sphere $|\vec{u}|_h = 1$ has been displaced by the wind, but since $|\vec{w}|_h < 1$ by assumption, the center point x remains in the interior of the sphere. Therefore, for any vector $\vec{\xi}$ on the tangent space the ray $\vec{x\xi}$ intersects the indicatrix at a point ρ_ξ; working out the Euclidean norms of $\vec{\xi}$ and $\vec{\rho}_\xi$ and taking the ratio, a short calculation shows that the fundamental function takes the form (see also [24,25])

$$F(x, \xi) = \frac{\sqrt{\langle\vec{w}, \vec{\xi}\rangle_h^2 + |\vec{\xi}|_h^2(1 - |\vec{w}|_h^2)} - \langle\vec{w}, \vec{\xi}\rangle_h}{1 - |\vec{w}|_h^2}, \quad (4)$$

where $|\vec{\xi}|_h^2 = h_{ij}\xi^i\xi^j$ and $\langle\vec{w}, \vec{\xi}\rangle_h = h_{ij}w^i\xi^j$. Making use of (3), an explicit form of the metric on $\mathfrak{M}$ can be obtained. The calculation simplifies if one writes

$$\alpha_{ij} = \frac{h_{ij}}{1 - |\vec{w}|_h^2} + \frac{w_i w_j}{(1 - |\vec{w}|_h^2)^2}, \qquad \beta_i = -\frac{w_i}{1 - |\vec{w}|_h^2}, \qquad (5)$$

where $w_i = h_{ij}w^j$, so that we have $F = \sqrt{\alpha_{ij}\xi^i\xi^j} + \beta_i\xi^i$. The solution curves to the Zermelo navigation problem are then found by working out the geodesics of the metric.

We remark that the metric of the type $\sqrt{\alpha_{ij}\xi^i\xi^j} + \beta_i\xi^i$ was introduced by Randers in the context of a unified theory of gravitation and electromagnetism [26]. However, Randers was unaware of the Finslerian nature of the metric, and attempted to interpret it in the Riemannian sense in the context of a five-dimensional Kaluza-Klein theory. Randers metrics are perhaps the most commonly investigated Finsler metrics in physical applications such as the electron microscope [27] and in propagation of sound and light rays in a moving medium [25,28,29].

The relevance of Finsler geometry to problems in quantum control has been observed in [30,31]. In the presence of background fields, more recently Russell and Stepney [17] proposed the technique of Shen [16] to be applied to the manifold $\mathfrak{M}$ of special unitary matrices endowed with the bi-invariant Hilbert-Schmidt norm. Specifically, working with the elements of the Lie algebra $\hat{\xi}, \hat{\xi}' \in \mathfrak{su}(N)$ we have

$$\langle \hat{\xi}, \hat{\xi}'\rangle_h = \mathrm{tr}(\hat{\xi}^\dagger \hat{\xi}'). \qquad (6)$$

With this setup we wish to minimize the journey time (2) in the presence of wind given in $\mathfrak{su}(N)$ by $-i\hat{H}_0$, when $\hat{\xi} = -i\hat{H}(t) = -i(\hat{H}_0 + \hat{H}_1(t))$. The fundamental function (4) in this quantum context thus reads

$$F(\hat{\xi}) = i\frac{\sqrt{[\mathrm{tr}(\hat{\xi}\hat{H}_0)]^2 + \mathrm{tr}(\hat{\xi}^2)(1 - \mathrm{tr}(\hat{H}_0^2))} - \mathrm{tr}(\hat{\xi}\hat{H}_0)}{1 - \mathrm{tr}(\hat{H}_0^2)}, \qquad (7)$$

which is just the Finslerian norm $\|\hat{\xi}\|$.

To proceed we find it convenient to minimize the kinetic energy $\frac{1}{2}\int F^2 dt$ along the path, instead of $\int F dt$. It should be evident that the optimal path $\hat{\xi}(t)$ that minimizes the latter also minimizes the former. Writing $F^2(\hat{\xi}) = \|\hat{\xi}\|^2$ we have

$$\delta\|\hat{\xi}\|^2 = \left\langle \frac{\delta\|\hat{\xi}\|^2}{\delta\hat{\xi}}, \delta\hat{\xi} \right\rangle = 2\|\hat{\xi}\|\left\langle \frac{\delta\|\hat{\xi}\|}{\delta\hat{\xi}}, \delta\hat{\xi} \right\rangle, \qquad (8)$$

where we have written, for any $\hat{\nu} \in \mathfrak{su}(N)$ and any $f(\hat{\xi})$,

$$\left\langle \frac{\delta f(\hat{\xi})}{\delta\hat{\xi}}, \hat{\nu} \right\rangle = \frac{d}{d\epsilon} f(\hat{\xi} + \epsilon\hat{\nu})\Big|_{\epsilon=0}, \qquad (9)$$

and on account of (7) we have

$$\left\langle \frac{\delta\|\hat{\xi}\|}{\delta\hat{\xi}}, \hat{\nu} \right\rangle = -i\frac{\mathrm{tr}(\hat{\nu}\hat{H}_0)}{1 - \mathrm{tr}(\hat{H}_0^2)} + i\frac{\mathrm{tr}(\hat{\xi}\hat{H}_0)\mathrm{tr}(\hat{\nu}\hat{H}_0) + (1 - \mathrm{tr}(\hat{H}_0^2))\mathrm{tr}(\hat{\xi}\hat{\nu})}{(1 - \mathrm{tr}(\hat{H}_0^2))\sqrt{[\mathrm{tr}(\hat{\xi}\hat{H}_0)]^2 + \mathrm{tr}(\hat{\xi}^2)(1 - \mathrm{tr}(\hat{H}_0^2))}}. \qquad (10)$$

Our aim is to solve

$$0 = \delta\left(\frac{1}{2}\int_0^1 \|\hat{\xi}\|^2 dt\right) = \int_0^1 \|\hat{\xi}\|\left\langle \frac{\delta\|\hat{\xi}\|}{\delta\hat{\xi}}, \delta\hat{\xi} \right\rangle dt \qquad (11)$$

with fixed end points of the curve on $SU(N)$. The constraints on the end points restrict admissible variations $\delta\hat{\xi}$. In particular, a standard result of Euler-Poincaré reduction [32] asserts that

$$\delta\hat{\xi} = \dot{\hat{\eta}} - [\hat{\xi}, \hat{\eta}], \qquad (12)$$

where $\hat{\eta}$ is an arbitrary curve in $\mathfrak{su}(N)$ with $\hat{\eta}(0) = \hat{\eta}(1) = 0$. Substituting (12) and (10) in (11) and rearranging terms, we are thus led to the relation

$$0 = -\partial_t(\|\hat{\xi}\|)\hat{H}_0 - \|\hat{\xi}\|[\hat{H}_0, \hat{\xi}]$$
$$+ \partial_t\left(\frac{\|\hat{\xi}\|}{\sqrt{\cdots}}\mathrm{tr}(\hat{\xi}\hat{H}_0)\right)\hat{H}_0 + \frac{\|\hat{\xi}\|}{\sqrt{\cdots}}\mathrm{tr}(\hat{\xi}\hat{H}_0)[\hat{H}_0, \hat{\xi}]$$
$$+ (1 - \mathrm{tr}(\hat{H}_0^2))\partial_t\left(\frac{\|\hat{\xi}\|\hat{\xi}}{\sqrt{\cdots}}\right), \qquad (13)$$

where we have written $\sqrt{\cdots}$ for the square-root term appearing in the numerator of (7). This result appears unduly complicated; however, if we take note of the fact that we are interested in the quantum navigation at full throttle, i.e., $\|\hat{\xi}\| = 1$, then by taking the relevant time

derivatives in (13) we deduce the Euler-Poincaré equation of the form $\dot{\hat{\xi}} + i[\hat{H}_0, \hat{\xi}] - (i\hat{H}_0 + \hat{\xi})\mathrm{tr}(\dot{\hat{\xi}}\hat{H}_0)/\sqrt{\cdots} = 0$. Substituting $\hat{\xi} = -i(\hat{H}_0 + \hat{H}_1(t))$ in here we thus obtain the relevant equation of motion for the control Hamiltonian: $-i\dot{\hat{H}}_1 + [\hat{H}_0, \hat{H}_1] + \hat{H}_1\mathrm{tr}(\hat{H}_0\dot{\hat{H}}_1)/\sqrt{\cdots} = 0$. If we eliminate the square-root term using (7) along with $F(\hat{\xi}) = \|\hat{\xi}\| = 1$, which gives us $i\sqrt{\cdots} = 1 + \mathrm{tr}(\hat{H}_0\hat{H}_1)$, then we deduce that

$$\dot{\hat{H}}_1 + i[\hat{H}_0, \hat{H}_1] - \frac{\hat{H}_1}{1 + \mathrm{tr}(\hat{H}_1\hat{H}_0)}\mathrm{tr}(\hat{H}_0\dot{\hat{H}}_1) = 0, \qquad (14)$$

where we have made use of the constraint that $\mathrm{tr}(\hat{H}_1^2) = 1$. Multiplying (14) with $\hat{H}_0$ and taking the trace, we thus deduce that $\mathrm{tr}(\hat{H}_0\hat{H}_1) = 0$. We therefore conclude from (14) that the quantum Zermelo-Euler-Poincaré equation takes the simple form

$$\dot{\hat{H}}_1 + i[\hat{H}_0, \hat{H}_1] = 0. \qquad (15)$$

This, however, is just the equation for a co-adjoint motion, so it can be solved, with the solution (1).

It is interesting to observe that, after some lengthy but straightforward algebra, we are led to a simple and intuitive solution to the quantum navigation problem, namely, that we must pick the initial direction $\hat{H}_1(0)$ and let it be

advected by the prevailing field $\hat{H}_0$. To hit the right target $\hat{U}_F$ starting from the initial point $\hat{U}_I$, however, the initial direction $\hat{H}_1(0)$ has to be chosen appropriately. In what follows we shall derive an ordinary differential equation satisfied by the initial direction.

We proceed by first solving the navigation problem in the absence of the wind: $\hat{H}_0 = 0$. In this case, the optimal control $\hat{H}_1$ is time independent, and the initial condition $\hat{H}_1(0)$ can thus be obtained by taking the matrix logarithm of $\hat{U}_F \hat{U}_I^{-1}$. The idea behind our scheme is to gradually increase $\hat{H}_0$ from zero to the level specified by the problem, while calculating, for each increment of $\hat{H}_0$, the optimal control Hamiltonian that solves the Zermelo problem with that wind. Clearly, as $\hat{H}_0$ is increased, $\hat{H}_1(0)$ has to be adjusted as well, or else the target gate will be missed. Moreover, the trajectory might take slightly more or slightly less time. Hence, the duration of the trajectory needs also be adapted.

With this in mind, let us calculate how the final gate varies when the wind, the initial control, and the terminal time are adjusted infinitesimally. Let $\hat{U}(t)$ be a curve in $SU(N)$ starting at $\hat{U}_I$ satisfying $\partial_t \hat{U} = \hat{\xi} \hat{U}$ for some curve $\hat{\xi}$ in $\mathfrak{su}(N)$, and fix a time s. If $\hat{\xi}(t) + \epsilon \delta\hat{\xi}(t)$ is a variation of $\hat{\xi}$, then $\hat{U}(s)$ varies as

$$\delta\hat{U}(s) = \hat{U}(s) \int_0^s \hat{U}(t)^{-1} \delta\hat{\xi}(t) \hat{U}(t) dt. \qquad (16)$$

This follows from adapting Lemma 2.4 of [33] to the present context. To proceed, let us write $\hat{H}_1(0, \lambda)$ for the optimal initial control and T_λ for the duration of the trajectory when the wind is given by $\lambda\hat{H}_0$, $\lambda \in [0, 1]$. Let us further denote by $\hat{U}_\lambda(t)$ the corresponding geodesic curve in $SU(N)$. In what follows we shall write derivatives with respect to λ as T', $\hat{U}'_\lambda$, and so on. Notice that $\hat{U}_\lambda(T_\lambda)$ equals the target gate $\hat{U}_F$ for all λ. Hence, $\hat{U}'_\lambda(T_\lambda) = 0$. Using (16), we thus obtain

$$0 = \hat{U}_\lambda^{-1}(T_\lambda) \hat{U}'_\lambda(T_\lambda)$$
$$= \int_0^{T_\lambda} \hat{U}_\lambda(t)^{-1} \hat{\xi}'_\lambda(t) \hat{U}_\lambda(t) dt + T'_\lambda \hat{U}_\lambda^{-1}(T_\lambda) \hat{\xi}_\lambda(T_\lambda) \hat{U}_\lambda(T_\lambda). \qquad (17)$$

Recall that $\hat{\xi}_\lambda(t) = -i(\lambda\hat{H}_0 + \hat{H}_1(t, \lambda))$, where $\hat{H}_1(t, \lambda)$ is given by (1). Therefore, upon differentiation, $\hat{\xi}'_\lambda(t) = -ie^{-i\hat{H}_0\lambda t}(\hat{H}_0 + it[\hat{H}_1(0, \lambda), \hat{H}_0] + \hat{H}_1'(0, \lambda))e^{i\hat{H}_0\lambda t}$, from which it follows that (17) is a linear equation in T'_λ and $\hat{H}_1'(0, \lambda)$, admitting a unique solution for each λ once the linear constraint $\mathrm{tr}(\hat{H}_1(0, \lambda)\hat{H}_1'(0, \lambda)) = 0$ is taken into account. Finally, T'_λ and $\hat{H}_1'(0, \lambda)$ can be integrated up to $\lambda = 1$ starting from the wind-free solution $\lambda = 0$. The optimal initial control is then given by $\hat{H}_1(0, 1)$, and the trajectory is traversed in time T_1.
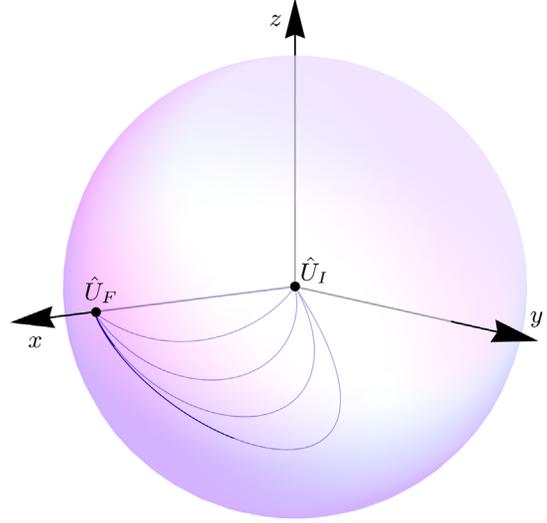
FIG. 1 (color online). Optimal generation of target unitary gate. The time-optimal trajectories $\hat{U}(t)$ are shown for various wind strengths $\omega = 0, 0.25, 0.5, 0.75, 1$, as curves in the rotation group using the standard covering map. The center of the sphere corresponds to the initial gate $\hat{U}_I = \mathbb{1}$, while the terminal point that lies on the surface of the sphere is the target gate $\hat{U}_F = -i\hat{\sigma}_x$. The direction of the vector joining the center $\mathbb{1}$ to a point $\hat{U}(t)$ on a given curve represents the axis of rotation, whereas the length of the vector represents the angle of rotation. The sphere, upon which the target gate lies, thus has radius π.

In summary, we have derived the Euler-Poincaré equation (15) associated with the quantum Zermelo navigation problem introduced in [17]. The equation of motion is surprisingly simple, and admits an elementary solution (1). We have provided a scheme which allows for the determination of the initial control Hamiltonian $\hat{H}_1(0)$ required to hit the correct target gate $\hat{U}_F$. On account of linearity, our scheme can easily be implemented in practice. With the solution (1) at hand, optimal quantum control with finite energy resources becomes feasible under the presence of external field or potential that might be difficult to eliminate in laboratories. As an illustrative example let us consider the control of a spin-$\frac{1}{2}$ system, where the objective is to transform $\hat{U}_I = \mathbb{1}$ into $\hat{U}_F = e^{-i\pi\hat{\sigma}_x/2} = -i\hat{\sigma}_x$, under the influence of an external field $\hat{H}_0 = -\omega\hat{\sigma}_z$, where $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are the Pauli matrices. In this example a closed-form expression for the optimal initial Hamiltonian $\hat{H}_1(0)$ can be obtained on account of the relation (cf. [34,35]) $\hat{U}_F = -i\hat{\sigma}_x = e^{i\omega\hat{\sigma}_z T} e^{-i\hat{H}_1(0)T}$, which follows from (1). Specifically, a calculation shows that $\hat{H}_1(0) = \frac{1}{\sqrt{2}}\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}$ and $T = \pi/\sqrt{2}$, where the unit vector $\boldsymbol{n}$ is given by $\boldsymbol{n} = [\cos(\omega T), \sin(\omega T), 0]$. The resulting unitary orbit $\hat{U}(t)$ is sketched in Fig. 1 for a range of values of ω.

We conclude by remarking that in the presence of additional constraints on the control Hamiltonian that limit the implementability of (1), it suffices to include them in

the maximization of F^2 by use of Lagrange multipliers. It then follows that the solution (1) remains valid, except that the initial control $H_1(0)$ is replaced by a time-dependent one (cf. [34]). More precisely, what the solution (1) shows is that it is possible to switch to a frame that moves in the counter direction to the wind so that the analysis of constrained optimization performed, for example, in [11] with time-dependent constraints, becomes applicable. In this manner the solution to the Zermelo navigation problem presented here can be extended straightforwardly to accommodate further constraints that one might encounter for instance in systems involving a large number of coupled spins where controllable degrees of freedom are typically limited.

We thank Gary Gibbons for drawing our attention to [17,25,29].

Note added.—Russell and Stepney have independently obtained the solution (1) to the quantum Zermelo navigation problem [35], using a theorem of Ref. [36] on geodesics of Randers spaces (rather than deriving and solving the Euler-Poincaré equation as we have done here).

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