Spin Glass in a Field: A New Zero-Temperature Fixed Point in Finite Dimensions

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By using real-space renormalization group (RG) methods, we show that spin glasses in a field display a new kind of transition in high dimensions. The corresponding critical properties and the spin-glass phase are governed by two nonperturbative zero-temperature fixed points of the RG flow. We compute the critical exponents and discuss the RG flow and its relevance for three-dimensional systems. The new spin-glass phase we discovered has unusual properties, which are intermediate between the ones conjectured by droplet and full replica symmetry-breaking theories. These results provide a new perspective on the long-standing debate about the behavior of spin glasses in a field.

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Spin glasses have been the focus of intense and successful research activity in the past 40 years. The techniques and the concepts developed to understand them has had an enormous impact in several fields. Moreover, spin-glass (SG) theory led to various spinoffs, even in other branches of science. Amazingly, despite all of these successes and 40 years of effort, there is still no consensus on their physical behavior: the low-temperature phase as well as the out-of-equilibrium aging dynamics remain matters of strong debates. On one side, there is the line of research starting from mean-field (MF) models, such as the one introduced by Sherrington and Kirkpatrick [1]. These are solved by the full replica symmetry-breaking (FRSB) theory [2], predicting a SG phase characterized by an infinite number of pure states organized in an ultrametric structure. On the other side stands droplet theory (DT), which is a low energy scaling theory based on the existence of only two pure states related by spins flip [3,4]. Although the two approaches provide different predictions, contrasting them has proved to be very difficult in both numerical simulations and experiments due to severe finite-size and finite-time effects [5]. The most clear-cut difference between them concerns the fate of the SG phase in the presence of an external magnetic field: the SG phase remains stable up to the so-called de Almeida–Thouless (AT) line $h_{\mathrm{AT}}(T)$ within MF theory [6], whereas according to the DT it is wiped out by even an infinitesimal magnetic field [3]. In consequence, much of the debate crystallized in proving (or disproving) the existence of the AT line in finite-dimensional SGs.

Field theoretical analysis showed that the Gaussian fixed point (FP) that controls the critical behavior of the AT line for the MF model becomes unstable for $d < 6$, and its basin of attraction shrinks to zero as $d{\downarrow}6$ [7,8] (see also Ref. [9]). These findings have two important consequences. First, *if* there is a transition in a field below six dimensions, then it necessarily corresponds to a nonperturbative (NP) fixed point. Second, this NP FP could be relevant even well above six dimensions: it depends on in which basin of attraction the initial condition of the RG flow, corresponding to finite-dimensional SG, lies. As a matter of fact, the MF behavior could be recovered in very high dimensions only. On the numerical side, the most recent numerical results obtained with the use of the Janus dedicated computer found that no phase transition can be identified with traditional data analysis in three dimensions [10,11]; however, highly nontrivial signals are detected, such as a growing correlation length, peaks in the susceptibility, and a wide probability distribution function of the overlap, as expected from FRSB. Numerical studies performed on one-dimensional long-range (LR) models [12], proxies for three-dimensional short-range SGs, support the absence of the AT line, even though there are some particular observables that are compatible with a transition in nonzero field. Finally, renormalization group (RG) studies performed by the Migdal-Kadanoff (MK) approximation [13] also find no SG phase in a field: the renormalized couplings initially grow for sufficiently small temperatures and fields but eventually vanish when the paramagnetic (PM) FP is reached, as expected from the DT [14]. In conclusion, it is fair to say that the state of the art on SGs in a field, whose study was supposed to clarify the situation, is as intricate as the zero-field case [15].

In this work, by using real-space RG methods, we show that SGs in a field have a new kind of transition for sufficiently large dimensions ($d > 8$). By studying the RG flow we identify two different zero-temperature FPs, one governing the critical properties and the other the low-temperature SG phase. These NP FPs, whose existence was hinted at by the perturbative RG study discussed above, are absent in three dimensions. Nevertheless, they still affect the RG flow and, hence, are relevant for the physical behavior.

In the following, we present first the analysis performed by the MK RG method and then complement it by using the Dyson hierarchical RG method [16]. In a nutshell, the MK procedure applied to a hypercubic lattice in d dimensions consists in replacing it with a hierarchical diamond one, for which the MK RG is exact [13,17]. Hierarchical diamond lattices (HL) are generated iteratively. The procedure starts at the step $G = 0$ with two spins connected by a single link. At each step G, for each link of step $G - 1$, p parallel branches, made of 2 bonds in series each, are added, creating p new spins. The relationship between the dimension of the hypercubic lattice and the number of branches is $d = 1 + \ln(p)/\ln(2)$. The SG Hamiltonian on HL is the usual one:

$$H = -\frac{1}{p}\left(\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j + \sum_i \sigma_i h_i\right),$$

where J_{ij} and h_i are independent random variables extracted from a Gaussian distribution with variance v_J^2 and v_h^2, respectively ($v_J = 1$ in the following). The sum over i and j runs over nearest neighbors on the lattice. Without loss of generality, we focus on a random external magnetic field. The RG procedure is exactly the opposite of the iterative procedure to construct the HL. For instance, in step 1, the p spins generated at the last level are integrated out, generating new effective couplings between the remaining spins. By integrating out the spins connecting, say σ_1 and σ_2, one gets

$$\tilde{E}^i_{1,2} = \tilde{J}^i_{12}\sigma_1\sigma_2 + \vec{h}_1^{\,i}\sigma_1 + \overleftarrow{h}_2^{\,i}\sigma_2 + c_{12}.$$

New fields ($\vec{h}_1^{\,i}$ and $\overleftarrow{h}_2^{\,i}$) associated with each link and an effective coupling between σ_1 and σ_2 are generated in addition to a constant c_{12}. As anticipated, in the presence of external fields there is a difference between HL and bond-moving MK. In the MK approximation, the spins in the lattice are divided in blocks of size ℓ. Then all the couplings internal to the blocks are moved to the spins at the edges of the blocks. At this point a decimation of the spins at the edges, except those on the corners, is performed. As for the fields, we follow Ref. [14] and move them coherently with the bonds on the spins placed on the edges of the blocks that are traced out in the RG step. In this way, the RG iteration is exactly the same one of a HL except that the fields associated with the links are moved from the external spins to the internal ones for all p branches but one. The unmoved fields represent the ones on the original link. None of the original site fields is moved. This change in the renormalization procedure is important to have a correct interpretation in terms of bond moving and to avoid pathological behaviors. The exact equations for the flow of the probability distribution of fields and couplings are reported in the Supplemental Material (SM) [18]. We analyzed them by using the population dynamics method [20] (see SM [18] for more details).

In the following we present our results on the RG flow. Let us recall first the zero-field results [21]. For $d \geq 2.58$ ($p \geq 3$), the model has a phase transition from a PM to a SG phase at T_c^0. The critical temperature is p dependent and is equal to $(1/\sqrt{p})$ in the large p limit [22]. The critical FP related to the transition corresponds to a finite value of T/v_J. The corresponding SG phase is associated to a nontrivial zero-temperature FP at which the typical value of the couplings after n iterations scales as $v_J^{(n)} \propto \ell^{\theta_0}$, where ℓ is the renormalization length after n RG steps: $\ell = 2^n$ (see Fig. 2). The behavior of θ_0 as a function of the dimension approximatively follows $(d - 2.5/2)$, which is consistent with the lower critical dimension $d_L = 2.5$ found in usual short-range SG without field [23]. Applying a small field for $T < T_c^0$, the system first approaches the zero-temperature FP in zero field but eventually flows away from it since the external field corresponds to a relevant perturbation. Correspondingly, the variances of coupling v_J and bond field $v_{\vec{h}}$ grow as $v_J^{(n)} \propto \ell^{\theta_0}$, $v_{\vec{h}}^{(n)} \propto \ell^{d/2}$ with $\theta_0 < (d/2)$, as predicted by the DT. The exponent $d/2$ is expected on general grounds because the field couples in a random way to the SG phase. No matter how small the initial value of v_h, the renormalized field eventually becomes larger than the coupling. On this basis the DT concluded that any infinitesimal field destroys the SG phase. This would take place when $v_{\vec{h}}^{(n)} \propto v_J^{(n)}$ and is indeed what we obtain for $d < 8.066$. In agreement with previous results that focused on the three-dimensional case [14], one finds that the ratio $v_{\vec{h}}^{(n)}/v_J^{(n)}$ increases, but when it exceeds a certain value r, $v_J^{(n)}$ starts to decrease and $v_{\vec{h}}^{(n)}$ tends to a constant value. However, for $d \geq d_L = 8.066$, when the ratio $v_{\vec{h}}^{(n)}/v_J^{(n)}$ exceeds r, the growth of $v_{\vec{h}}^{(n)}$ and ${v_J}^{(n)}$ changes: $v_{\vec{h}}^{(n)} \propto \ell^{\theta}$, $v_J^{(n)} \propto \ell^{\theta}$, with $\theta < \theta_0$. In Fig. 1 we show the RG flow in the plane (T/v_J) versus $(v_{\vec{h}}/v_J)$.

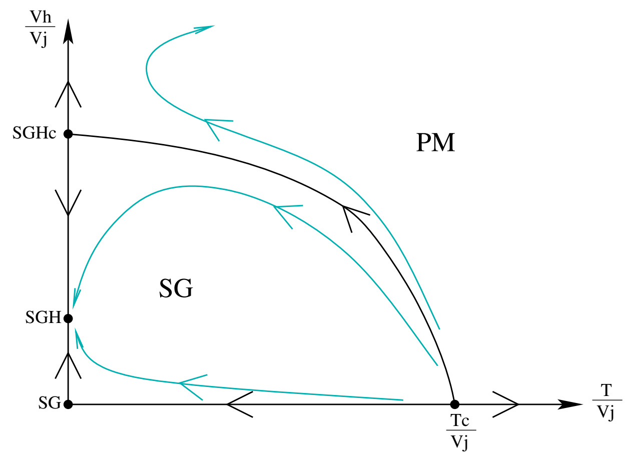

FIG. 1 (color online). Renormalization flow in the plane T/v_J-$v_{\vec{h}}/v_J$ for $d \geq d_L$.

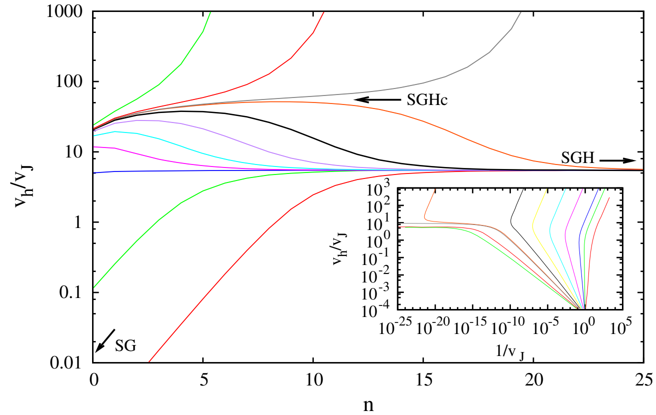

FIG. 2 (color online). Evolution of the observable $v_{\bar{h}}/v_J$ as a function of the renormalization step at $T = 0$ for $d = 10$ starting from different v_h. Inset: Renormalization flow in the plane $1/v_J$-$v_{\bar{h}}/v_J$ at $T = 0$ starting from $v_h = 0.0001$ for dimensions $d = 2, 2.58, 3, 4, \ldots, 8, 8.066, 9.23, 9.97$ from right to left.

The system flows towards a new zero-temperature stable FP $(T/v_J, v_{\bar{h}}/v_J) = (0, (v_{\bar{h}}/v_J)^*)$, called SGH in Fig. 1, which rules the behavior of the SG phase in a field.

Since at high temperature or for strong fields the system has to flow to the PM FP $(T/v_J, v_{\bar{h}}/v_J) = (\infty, \infty)$, there is necessarily an unstable FP, which we denote SGH_c, separating the disordered (PM) and the ordered (SGH) ones. As shown in Fig. 2, this is also at zero temperature and governs the transition of SGs in a field: when approaching it the couplings and the fields grow as $v_J^{(n)} \propto \ell^{\theta_U}$, $v_{\bar{h}}^{(n)} \propto \ell^{\theta_U}$. The three zero-temperature FPs are visible in Fig. 2, where the evolution of $v_{\bar{h}}^{(n)}/v_J^{(n)}$ is shown as a function of n at $T = 0$, starting from different initial v_h. We checked that no other FP exists.

Now we fully characterize the critical properties. For zero-temperature FPs, there are three independent critical exponents, one more than for standard phase transitions [24]. The additional one is θ that we have already introduced. The other two exponents we focus on are x and ν, following the notation of Ref. [24]. The exponent x describes the rescaling of an infinitesimal symmetry-breaking field under renormalization; hence it is related to the anomalous dimension of the order parameter. The exponent ν is the one associated to the divergence of the correlation length. In the case of SGs, the order parameter introduced by Edwards and Anderson [25] corresponds to the overlap between two different replicas subjected to the same quenched disorder. Correspondingly, the symmetry-breaking field ϵ is an effective attraction (or repulsion) between two different replicas $\{\sigma^1\}$ and $\{\sigma^2\}$. We proceed as for the random field Ising model [26]: we introduce a field ϵ at the extremities of each bond and analyze how its average is renormalized in one RG step, $x = [\ln(d\bar{\epsilon}^R/d\epsilon)]/\ln(2)$. The calculation of x at the zero-field FP can be performed analytically, leading to $x_0 = d$. In order to compute ν, we measured how two renormalized flows of the observable $v_{\bar{h}}/v_J$ corresponding to different original v_h distance themselves. The values of θ, x, and ν as a function of d are reported in Table I; all other exponents can be obtained by scaling relations [24], e.g., $\beta = (d - x)\nu$, $\alpha = 2 - (d - \theta)\nu$ (we use the standard notation of critical phenomena). We find that ν increases and possibly diverges at $d = 8$, as expected since the FPs disappear below eight dimensions. The fact that $x_{\mathrm{SGH}} < d$ implies that the SG phase in a field has a very different nature from its zero-field counterpart: the system is ordered but only on a fractal system-size set (accordingly, the transition induced by changing ϵ from 0^+ to 0^- is second order instead of being first order). Let us finally discuss the behavior of correlation functions. As it is known for zero-temperature FPs, two different correlation functions are critical [24]. One is associated to thermal fluctuations,

$$G_c(r) = \overline{\langle\sigma_0\sigma_r\rangle^2 - \langle\sigma_0\rangle^2\langle\sigma_r\rangle^2} = \frac{T}{r^{d-2+\eta}} g(r/\xi). \quad (1)$$

while the other is associated to disorder fluctuations,

$$G_d(r) = \overline{\langle\sigma_0\rangle^2\langle\sigma_r\rangle^2} - \overline{\langle\sigma_0\rangle^2}\,\overline{\langle\sigma_r\rangle^2} = \frac{1}{r^{d-4+\tilde{\eta}}} g_{\mathrm{dis}}(r/\xi). \quad (2)$$

The exponents η and $\tilde{\eta}$ are linked by the relation $\tilde{\eta} - \eta = 2 - \theta$ and $\tilde{\eta} = d + 4 - 2x$. Since $\theta > 0$, the two correlation functions decay with different power laws (the disordered one more slowly than the thermal one). Note that the system is not only critical at the transition, but also in the whole SG phase in a field.

We have also studied analytically the large d limit of the RG equations, as done for the zero-field case in Ref. [22]. We found that $(v_{\bar{h}}/v_J)^*|_{\mathrm{SGH}} \simeq 5.045$ and $\theta_{\mathrm{SGH}}(d) \simeq (d-1)/2 - 2.425$, which are actually good approximations for all dimensions larger than eight. For $d \to \infty$, the transition loses its zero-temperature character since $\theta_{\mathrm{SGH}_c} \to 0$ [27].

TABLE I. Critical values for systems with different d.

d	h_c	θ_{SGH}	θ_{SGH_c}	ν	x_{SGH}	x_{SGH_c}
8.066	23.4(1)	0.6222(1)	0.4833(6)	9.1(6)	3.7611(2)	3.4795(1)
9.229	85.15(10)	1.5824(2)	0.1203(3)	1.72(3)	5.5509(4)	2.8640(2)
9.966	151.05(10)	2.0044(9)	0.060(1)	1.60(16)	6.36295(9)	2.8139(2)

We now turn to general considerations about our results. First, let us discuss their relevance for systems in dimensions less than d_L. In the inset of Fig. 2 we show the flow diagram at $T = 0$ for different dimensions, starting from a very small field. For $d < d_L$, the flow still feels the vestige of the SGH FP and is initially attracted towards it, closer and closer as d approaches d_L. However, when the ratio $(v_{\bar{h}}/v_J)$ becomes larger than the value at the stable FP, the transition is avoided and the system finally escapes from the SGH FP and flows away towards the PM fixed point.

The MK renormalization has pros and cons: for example, it correctly captures the zero-temperature FP of the random field Ising model [26], which is highly nontrivial. On the other hand, it becomes less quantitatively accurate in high dimensions and sometimes even fails qualitatively [28]. Thus, in order to test the robustness of our results, it is crucial to complement the previous analysis with another one that uses a completely different real-space RG scheme. We focus on the one based on the Dyson hierarchical lattice, which is able to emulate a short-range model in different dimensions just by changing a parameter. We solved the RG equations via an approximation, the real-space ensemble renormalization group method, that was introduced and tested for SGs without field in Ref. [16]. Within this framework the relation between original and renormalized parameters (the variance of couplings and fields) is obtained by imposing the equivalence between the average of some particular observables over an ensemble of 2^n-spin lattices and an ensemble of 2^{n-1}-spin lattices [16] (see SM [18] for more details). The ensemble renormalization group method was shown to be able to capture the high-dimensional behavior correctly. For instance, it identifies the upper critical dimension of SGs in zero field [16]. Moreover, it does not suffer of the ambiguity in the treatment of the external magnetic fields. The results we found using essentially the same method and observable as in Ref. [16] (in particular, taking $n = 4$) agree with the MK results. The corresponding renormalization flow is shown in Fig. 3 for effective dimension $d = 20$. A stable zero-temperature FP is clearly visible and MF critical behavior is not recovered. In the inset of Fig. 3, the behavior for $d = 6 < d_L$ is shown. As for MK renormalization, the flow feels the vestige of the SGH FP and is initially attracted towards it; however, the transition is avoided and the system finally flows away towards the PM fixed point. For more details see Ref. [29].

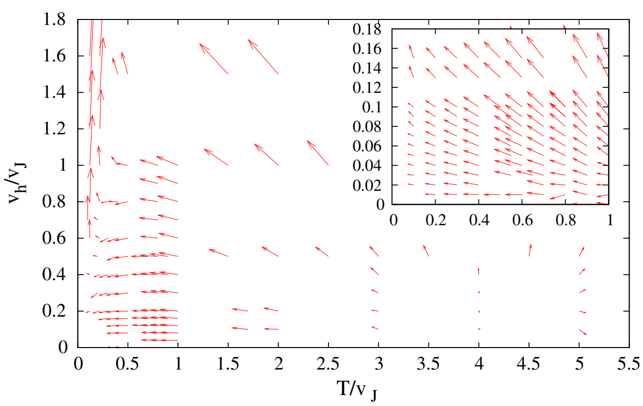

FIG. 3 (color online). Renormalization flow for the Dyson model in dimension $d = 20$ ($d = 6$ in the inset).

In summary, these two very different complementary real-space RG methods lead to the same conclusion: the initial condition corresponding to microscopic SG models in a field does not lie in the basin of attraction of the Gaussian (MF) fixed point except possibly for very high dimensions. Contrary to what was argued by DT, there is no general argument against the existence of a transition for SG in a field. The flaw in the DT argument is that even though the SG FP is unstable in the presence of an external field, the system can nevertheless flow toward a new fixed point SGH_c. Indeed, we have unveiled here that the very same method used as a basis for DT, the MK RG approach, shows precisely that in high enough dimensions.

The peculiarity of the SG transition in a field is the absence of $\mathcal{Z}_2$ symmetry. In consequence, contrary to the zero field case where two possibilities were conjectured, that are the existence of just two pure states (related by spin-flip) or of an infinite number, in the presence of an external field the only possibility is the latter one [30]. Whether this is related to FRSB physics is nevertheless unclear. Indeed, we do not find any sign of states characterized by extensive free-energy differences of the order of 1, a hallmark of FRSB. Moreover, the MF transition is not governed by a zero-temperature FP. However, it might be that our RG methods are too crude to address this issue. For this reason and in order to get a better understanding of the new SG phase discovered in this work and obtain more precise quantitative results (e.g., the value of d_L), it would be very interesting to develop and apply a more refined nonperturbative RG method such as the Wetterich one [31,32]. Numerical simulations of high-dimensional systems would also be instrumental. In particular, it is worth performing new simulations for LR models, proxies of short-range models in large dimensions. Previous works already found a SG transition in nonzero field for these systems [33,34]. By using those data, we have analyzed the transition of LR models corresponding to $d = 10$ and compared the quality of mean-field finite size scaling (FSS) to the non-mean-field one (in the former case, the RG flow is governed by the Gaussian FP, whereas in the latter case, it is governed by a nontrivial one). We have found that the non-mean-field FSS is at least comparable, if not even better [35]. In future analysis it would be interesting to check whether this transition is associated to a $T = 0$ FP, in particular, whether disorder fluctuations are much stronger than thermal ones. The same thing could be checked in short-range models in four dimensions, where a transition can be identified performing a particular

FSS analysis [36]. As for three-dimensional SGs in a field, we notice that the results of numerical simulation can indeed be interpreted in terms of an avoided transition where time and length scales are exponentially related [10,12], exactly as would expected from the RG flow we obtained. Finally, it would also be interesting to identify the consequences of the phase transition we found for the problem of the glass transition, for which an analogy to the Ising SG in a field was already proposed [37].

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