Ground-State Degeneracy of Topological Phases on Open Surfaces

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We relate the ground state degeneracy of a non-Abelian topological phase on a surface with boundaries to the anyon condensates that break the topological phase into a trivial phase. Specifically, we propose that gapped boundary conditions of the surface are in one-to-one correspondence with the sets of condensates, each being able to completely break the phase, and we substantiate this by examples. The ground state degeneracy resulting from a particular boundary condition coincides with the number of confined topological sectors due to the corresponding condensation. These lead to a generalization of the Laughlin-Tao-Wu charge-pumping argument for Abelian fractional quantum Hall states to encompass non-Abelian topological phases, in the sense that an anyon loop of a confined anyon winding a nontrivial cycle can pump a condensed anyon from one boundary to another. Such generalized pumping may find applications in quantum control of anyons, eventually realizing topological quantum computation.

Introduction.—A key feature of intrinsic topological orders is the existence of protected ground state degeneracy (GSD). On closed spatial 2-surfaces, its genus number and the fusion rules between anyon excitations determine the GSD [1–8]. Protected GSD is vital to topological quantum computation, and yet realizing GSD on high genus closed surfaces is unfeasible in experiments. Obviously it is much more natural to build finite open systems. Yet, it is necessary that any boundary massless modes that often appear can be gapped to have a well-defined GSD. The gapping conditions of Abelian phases have recently been understood very well in terms of the concept of Lagrangian subsets [9–14], and subsequently the GSDs of these Abelian phases on open surfaces with multiple boundaries were computed [13,15], based on the idea of anyon transport across boundaries. Experiments detecting and applying the topological degeneracy with gapped boundaries were proposed in Refs. [16,17]. Nevertheless, non-Abelian phases bear much richer sets of degenerate ground states, and the braiding of non-Abelian anyons serves as the best known candidate that may realize universal topological quantum computing. It is also hopeful to realize or simulate non-Abelian anyons [18]. Despite the importance of understanding non-Abelian phases on open surfaces, it remains a big open problem, a problem we shall solve here via the method of anyon condensation. In summary, our main results are as follows:

(a) We find the condition for gapped boundaries of nonchiral non-Abelian phases by identifying each such boundary to a set of anyon condensates; the condition allows one to classify all gapped boundaries for the given phase. Our results also encompass situations in which a defect or phase boundary separates arbitrary phases because any such system can always be mapped back to one where a phase ends on the trivial vacuum by the folding trick.

(b) For any given boundary conditions on some arbitrary open system, we describe the computation of the GSD, dictated by a reduced set of conserved topological sectors—the set of *confined anyons*—anyons mutually nonlocal with the boundary condensate. Typically, on a cylinder,

$$\text{GSD} = \text{Number of confined anyons.} \qquad (1)$$

We show an explicit nontrivial example of a GSD counting on a cylinder whose two ends have distinct boundary conditions. Note that such reduction in the number of conserved anyons suggests a novel way to engineer desired conserved bulk anyons suitable for specific quantum computations. Our method is corroborated using the correspondence between a generic non-Abelian phase $\mathcal{A}$ and its double $\mathcal{A} \times \bar{\mathcal{A}}$ via the folding trick.

(c) We find connections between our counting and prior works on Abelian phases that make use of charge transport. This generalizes the Laughlin-Tao-Wu (LTW) charge-pump argument in Abelian fractional quantum Hall states (FQHSs) [1,19] to generic anyon-pump in non-Abelian phases, hinting at novel ways of braiding and controlling non-Abelian anyons. Such anyon transport may also help experimentally discern different topological phases [16,17].

We shall make heavy use of the technologies studying anyon condensation [20–22]. The basic premise of anyon condensation is that certain types of anyons cease to have a conserved particle number across a phase transition; they thus effectively condense, exactly as how Cooper pairs condense, in the process breaking some symmetry. As in

usual Bose-Einstein condensation, the condensable anyons should have bosonic self-statistics. There are various constraints that determine the properties of the condensed phase, such as the types of anyons that remain conserved and their fusion rules. We shall refer the reader to the original references for details. Some prior attempts on non-Abelian phases are found in Refs. [23,24].

GSD of the $\mathbb{Z}_2$ toric code on open surfaces revisited.—To explain the generalization of Lagrangian subsets and the subsequent GSD counting in Abelian phases, we revisit these concepts from the perspective of anyon condensation by taking the $\mathbb{Z}_2$ toric code as an example. Recall that the $\mathbb{Z}_2$ toric code has four topological sectors $\{1, e, m, f\}$, where e, m are self-bosons, and f a fermion. Any two distinct nontrivial anyons are mutual fermions. The nontrivial fusion rules are $e^2 = m^2 = f^2 = 1$ and $e \times m = f$. Each boundary of the system admits two distinct gapping conditions, respectively characterized by the two *Lagrangian subsets*:

$$L_e = \{1, e\}, L_m = \{1, m\}. \quad (2)$$

A Lagrangian subset L is a maximal collection of anyons that are self-bosons with trivial mutual statistics and that all remaining anyons excluded from the set are nonlocal with respect to at least one member of L [9–12]. This is clearly satisfied by both sets in Eq. (2). One crucial observation [9,10,13,20–22,25,26] is that for the boundary condition characterized by a set L_i, an anyon in the set ceases to be conserved and can be either created or annihilated at the boundary. Therefore, a gapped boundary condition L_i is equivalent to the condensation of the anyons in L_i right at the boundary. Any anyon not in a condensed L_i would be *confined* at the boundary [20–22,25–27]. For example, m and f are confined in the L_e condensate and thus are mobile in the bulk but fail to cross the boundary into the vacuum. Equally important, in the vicinity of the L_e condensate, m and f are indistinguishable, like 1 and e become identified, by fusion with any number of e's freely supplied by the boundary condensate [20–22,25–27]. This leads to an easy GSD counting. Consider a cylinder with both boundaries characterized by L_e. A convenient basis for ground states consists of uncontractible anyon loops winding the cylinder. For the L_e boundaries, only two distinct anyon loops exist: 1 and m. One then infers that

$$\text{GSD}^{L_e \text{boundaries on cylinder}}_{\mathbb{Z}_2 \text{toric code}} = 2, \quad (3)$$

in accord with the result of Ref. [13]. This is illustrated in Fig. 1. The LTW charge-pump argument for FQHS [1,19] applies here: If one threads a magnetic flux loop around the cylinder and adiabatically increases it from zero to a unit m flux, a charge e can be pumped from one boundary to the other of the cylinder, as depicted in Fig. 1.
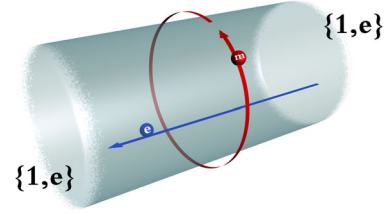

FIG. 1 (color online). Ground state basis specified by a conserved anyon line m winding the cylinder where the boundaries are characterized by the L_e condensate. A unit of e is transferred across the boundaries via the LTW charge pumping mechanism if the unit m line can be changed to two units adiabatically.

To deal with more boundaries potentially characterized by different L_i's, we need only to work out the remaining conserved (i.e., confined) distinct uncontractible anyon loops, and the anyon loops around different cycles must admit at least a fusion channel. We now generalize the procedure described above to non-Abelian phases.

GSD of Non-Abelian phases on open surfaces.—We now lay down the general procedure to obtain the GSD of a generic non-Abelian topological order with boundaries. We will illustrate each step with the example of the doubled Fibonacci model. To avoid clutter, the topological data of the doubled Fibonacci model and notations are reviewed in the Supplemental Material [28].

Defining boundary conditions: First we have to decide upon the boundary condition on each boundary. Each boundary whose edge modes could be completely gapped is characterized by a generalized Lagrangian subset L, which is a collection of anyons that could condense simultaneously at the boundary, and that the resultant phase after the condensation $\mathcal{T}_L$ contains only confined anyons as well as the trivial sector. As reviewed in the Supplemental Material [28], in the case of $F \times \bar{F}$, it has $L_{\tau\bar{\tau}} = \{1, \tau\bar{\tau}\}$, leading to $\mathcal{T}_{L_{\tau\bar{\tau}}} = \{1, \chi\}$, where χ behaves like a Fibonacci anyon τ except for its lack of a well-defined topological spin.

Counting GSD via confined charges: If all the boundaries are characterized by the same L, the GSD is obtained as follows. We first find out the fusion rules of all the confined anyons in $\mathcal{T}_L$. Then we count all possible basis states constructed from loops of confined anyons winding nontrivial cycles. This must satisfy the consistency condition on anyons wrapping cycles that merge have to fuse to the anyon wrapping the resultant merged cycle. This is to ensure no net charge exists in the bulk.

Consider for example $F \times \bar{F}$ on a cylinder (see Fig. 2), where both boundaries must be characterized by $L_{\tau\bar{\tau}}$. As a result, the only conserved nontrivial topological sector must be the confined $\chi \in \mathcal{T}_{L_{\tau\bar{\tau}}}$, as it cannot leak through the boundaries into the vacuum. We conclude that the two distinct sectors in $\mathcal{T}_L$ imply that
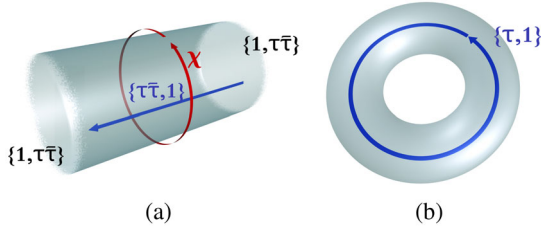
FIG. 2 (color online). (a) $F \times \bar{F}$ on a cylinder with both boundaries characterized by the Lagrangian subset $\{1, \tau\bar{\tau}\}$. (b) Single Fibonacci phase on a torus. The two systems are equivalent.

$$\mathrm{GSD}_{F \times \bar{F}}^{\mathrm{cylinder}} = |\mathcal{T}_{L_{\tau\bar{\tau}}}| = 2. \qquad (4)$$

To check our claims, note that it is expected that

$$\mathrm{GSD}_{F \times \bar{F}}^{\mathrm{cylinder}} = \mathrm{GSD}_{F}^{\mathrm{torus}}. \qquad (5)$$

This is because $F \times \bar{F}$ ending at a boundary can be thought of as folding up a Fibonacci phase characterized by a single copy of the anyons $\{1, \tau\}$. Thus the $F \times \bar{F}$ on a cylinder is in fact equivalent to the Fibonacci phase itself residing on two different cylinders, yet joined at both boundaries because of the gapped boundary condition we imposed on $F \times \bar{F}$. That is, we have in fact F on a torus. Similarly, we can consider placing the $F \times \bar{F}$ on a surface with three holes, or the "pants diagram" [a special case of Fig. 3(a)]. The three boundaries must again be characterized by $L_{\tau\bar{\tau}}$. On this surface, when two confined anyon loops a and b, respectively, winding two of the three cycles (holes) merge to an anyon loop c winding the third cycle, the three loops of anyons must admit a fusion channel. The GSD counting in this scenario then boils down to the formula

$$\mathrm{GSD}_{F \times \bar{F}}^{\mathrm{pants}} = \sum_{a,b,c \in \mathcal{T}_{L_{\tau\bar{\tau}}}} N_{ab}^{c} = 5, \qquad (6)$$

where for a given c, the fusion matrix element N_{ab}^{c} is the multiplicity of c in the fusion product of a and b. This again agrees with the expected result following from

$$\mathrm{GSD}_{F \times \bar{F}}^{\mathrm{pants}} = \mathrm{GSD}_{F}^{\mathrm{genus\text{-}2\ torus}}. \qquad (7)$$

We note that such a correspondence between a "doubled phase" on a surface with gapped boundaries characterized
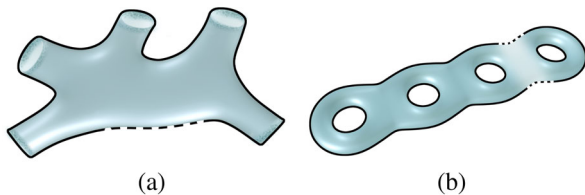

FIG. 3 (color online). (a) An open surface with $M > 3$ boundaries. (b) A genus-$(M - 1)$ torus.

by condensates of all the *diagonal* pair—the analogues of $\tau\bar{\tau}$—and the undoubled phase on a closed surface is in fact generic. This correspondence offers a nontrivial check of our methods in large classes of non-Abelian phases expressible as a doubled phase using well-known results of GSD of phases on closed surfaces, and by which we find perfect agreement.

More generically, the boundaries could also be characterized by different L_i's, which would contain anyons not mutually local. One would have to work out the condensed phase $\mathcal{T}_{L_i}$ at each boundary, and then further reduce the number of conserved anyons, which correspond roughly to finding an intersection of the $\mathcal{T}_{L_i}$'s. The fusion between the remaining conserved anyons again determine the GSD. There is not, to date, a fully systematic procedure dealing with multiple sets of nonmutually local condensates, but we will exemplify how this is to work by a nontrivial example in the next section. In summary (Fig. 3), assuming that there are $M > 3$ boundaries, respectively characterized by (potentially identical) Lagrangian subsets L_i's,

$$\mathrm{GSD}_{\{L_i\}} = \sum_{\{a_i, b_i\}} N_{a_1 a_2}^{b_1} \prod_{i=1}^{M-4} N_{a_{i+2} b_i}^{b_{i+1}} N_{a_{M-1} b_{M-3}}^{\bar{a}_M}, \qquad (8)$$

where $\{\mathcal{T}_{L_i}\} := \cap_{i=1}^{M} \mathcal{T}_{L_i}$, b_i's are the intermediate fusion channels, and $\{a_i, b_i\}$ refers to summing over $\{\mathcal{T}_{L_i}\}$. Note that the product term exists only for $M \geq 5$.

Counting GSD via charge transport: As alluded to at the beginning and also in the discussion of Abelian phases, we can alternatively count the GSD by considering charge transport across boundaries. This amounts to counting the fusion channels producing the trivial sector, between anyons across Lagrangian subsets L_i's characterizing the boundaries. Recall that $F \times \bar{F}$ on a cylinder has a GSD given in Eq. (4). The same result can be obtained by counting the fusion channels between condensed anyons on the two boundaries to the trivial sector. Specifically, since there is only one trivial fusion channel in $\tau\bar{\tau} \times \tau\bar{\tau} = 1 + 1\bar{\tau} + \tau\bar{1} + \tau\bar{\tau}$, together with the obvious trivial fusion channel $1 \times 1 = 1$, there are exactly 2 trivial fusion channels. Such an agreement between the two different ways of counting the GSD motivates a generalization of the aforementioned LTW argument for FQHS to the case of non-Abelian topological phases. That is, in a non-Abelian (gauge) theory, there should exist some *adiabatically changing* Wilson (anyon) loop, e.g., χ around the cylinder, which pumps a unit of the condensed $\tau\bar{\tau}$ from one boundary to the other [Fig. 2(a)].

As will be seen in the next section, such a flux-charge correspondence would work only if each L_i also includes the *multiplicity* data of a condensed anyon—the number of condensed sectors contained in each anyon in L_i that splits under condensation. This extra twist in the story makes GSD counting by confined anyons more natural.

$\mathbb{Z}_2^3$ *twisted quantum double.*—We now present here a
fascinating example—the $\mathbb{Z}_2^3$ twisted quantum double
(TQD) [29–31]—that bears more than one set of nontrivial
gapped boundary conditions. As a twisted version of the
$G = \mathbb{Z}_2^3$ Kitaev model, this model contains 22 distinct
anyons. As in other gauge theories, the electric charges are
representations of the gauge group G, and as such they
come in 8 distinct types. We can denote them by $E_{e_1e_2e_3}$,
where $e_i \in \{0,1\}$, corresponding to the trivial and non-
trivial one dimensional representations of each of the
three $\mathbb{Z}_2$ groups in G. The rest of the anyons are 14 non-
Abelian dyons all with quantum dimension 2, denoted
by $D_{m_1m_2m_3}^{\pm}$, where $m_i \in \{0,1\}$, and $m_1m_2m_3 \neq 000$.
Their properties, such as fusion rules, are detailed
in the Supplemental Material [28]. We focus on two
distinct admissible Lagrangian subsets, L_E and L_D. Set
$L_E = \{E_{e_1e_2e_3}\}$ contains all the electric charges. As
explained in the Supplemental Material [28], the con-
densed phase $\mathcal{T}_E$ contains the new trivial sector and 7
nontrivial confined anyons that descend from the dyons,
where $D_{m_1m_2m_3}^{\pm} \rightarrow 2d_{m_1m_2m_3}$. This gives, on a cylinder
with both boundaries characterized by L_E,

$$\text{GSD}_{\mathbb{Z}_2^3\text{TQD}}^{L_E\text{boundaries on cylinder}} = 8. \tag{9}$$

This immediately agrees with the result obtained by
considering allowed charge transport across the bounda-
ries, i.e., the number of fusion channels between the
condensed anyons in the top and bottom boundary that
fuse to the trivial sector. More interesting is the boundary
characterized by L_D, where

$$L_D = \{2D_{100}^+, E_{0e_2e_3}\}. \tag{10}$$

The resultant condensed phase $\mathcal{T}_D$ contains again 8 distinct
sectors, 7 of which confined and descended from the other
dyons and electric charges of the form $E_{0e_2e_3}$. A very special
thing arises here: The condensed anyon $D_{100}^+ \rightarrow 1 + 1$ splits
into two copies of the vacuum in the condensed phase $\mathcal{T}_D$.
This is unlike the examples encountered above, where each
sector appearing in the condensate only splits into the trivial
sector once. To make that information explicit, we have
included the dyon D_{100}^+ twice in defining the set L_D above.
Now the number of confined sectors in $\mathcal{T}_D$ would indicate
that on a cylinder,

$$\text{GSD}_{\mathbb{Z}_2^3\text{TQD}}^{L_D\text{boundaries on cylinder}} = 8 \tag{11}$$

again. However, a naive count of allowed charge transport
across the boundaries gives only 5 channels, if we count
D_{100}^+ only once. The only way to recover a match between
these two ways of counting is to take D_{100}^+ literally as
appearing twice, so that they alone contribute $2^2 = 4$ fusion
channels between the top and bottom boundaries to the
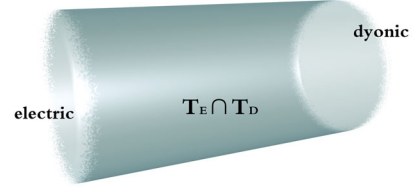trivial sector, instead of only one as in the naive count. Then

FIG. 4 (color online). $\mathbb{Z}_2^3$ twisted quantum double on a cylinder
with the left boundary characterized by electric condensation L_E
and the right one by dyonic condensation L_D. GSD $= |T_E \cap T_D|$.

we recover a $D = 4 + 4 = 8$. We therefore *postulate* that the
generalization of the Lagrangian subset in non-Abelian
phases must include specifying the *multiplicity* of a con-
densed anyon—the number of condensates that is actually
contained in the splitting of the anyon after anyon con-
densation. We have tested this postulate in this model in
surfaces with more boundaries and found that the counting
via charge-transport across boundaries continue to match the
analysis via confined sectors in the condensed phase. We
have also checked our postulate in the quantum double
model with group $G = D_3$. The GSD due to a condensate
involving multiplicities greater than 1 again supports our
postulate.

To end the section, we return to the $\mathbb{Z}_2^3$ TQD model on a
cylinder, with, now, the left boundary characterized by L_E
and the right by L_D. The two sets of anyons have exactly 4
fusion channels that can fuse to one, namely, the fusion of
the four shared electric charges. So the charge transport
reasoning leads to the interesting result

$$\text{GSD}_{\mathbb{Z}_2^3\text{TQD}}^{L_E\text{left},L_D\text{right}} = 4. \tag{12}$$

The same result can be obtained via counting the confined
sectors, as depicted in Fig. 4. As aforementioned and
detailed in the Supplemental Material [28], $\mathcal{T}_D$ has 8
sectors, conveniently denoted by

$$\{1, d_{m_2m_3}^{1,2}, d | m_2m_3 \neq 00\}, \tag{13}$$

satisfying the following important fusion rule:

$$d_{m_2m_3}^{1(2)} \otimes d = d_{m_2m_3}^{2(1)}. \tag{14}$$

Since d descends from $E_{1e_1e_2}$, it is is no longer conserved in
the other boundary where all electric charges are in L_E and
condense. Hence, $d_{m_2m_3}^{\pm}$ becomes indistinguishable, and the
GSD is determined by the following four states each with
an anyon line winding the nontrivial cycle

$$\{|1\rangle, |d_{01}\rangle, |d_{10}\rangle, |d_{11}\rangle\}, \tag{15}$$

leading again to a precise match. This is the first example to
date of a non-Abelian phase whose GSD on an open surface
with multiple boundary conditions is computed.

Conclusion.—We have achieved the long sought goal to count the GSD of a generic nonchiral non-Abelian topological order with boundaries, making use of insights and techniques developed in the past [9,13,20–22,25,26,32]. We note that very much analogous to prior analysis of gapped boundary conditions via the introduction of explicit boundary gapping terms, one could imagine that such an analysis is also possible for non-Abelian phases. Some preliminary work has been done for example in Ref. [33], in which explicit terms can be written down, whenever the bulk non-Abelian phase adopts a simple construction based on orbifolding Abelian ones. It would be useful to generalize these studies to other non-Abelian phases. Given the importance of robust GSD as a resource in topological quantum computation, our new understanding will be crucial towards finding experimental realizations and applications of topological orders.

Note added.—Recently we became aware of a related work by Lan, Wang, and Wen [34].

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