Integrability and Maximally Helicity Violating Diagrams in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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We apply maximally helicity violating (MHV) diagrams to the derivation of the one-loop dilatation operator of $\mathcal{N}=4$ supersymmetric Yang-Mills theory in the SO(6) sector. We find that in this approach the calculation reduces to the evaluation of a single MHV diagram in dimensional regularization. This provides the first application of MHV diagrams to an off-shell quantity. We also discuss other applications of the method and future directions.

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Introduction.—The study of $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory has led to the discovery of integrability in the planar limit, providing the tools to compute the anomalous dimensions of local operators for any value of the coupling. In an initially independent line of research into this theory, the study of its on-shell scattering amplitudes has uncovered a rich structure and simplified calculations dramatically. It is widely expected that the integrability of the planar anomalous dimension problem and the hidden structures and symmetries of scattering amplitudes are related in some interesting way. In this Letter we take a first step towards unraveling this potential connection.

Specifically, our goal here is to apply a method originally devised for computing amplitudes known as MHV diagrams [1] to the derivation of the one-loop dilatation operator Γ in the SO(6) sector of $\mathcal{N}=4$ SYM theory, originally computed by Minahan and Zarembo (MZ) in [2]. It is known that MHV diagrams are obtained from a particular axial gauge choice, followed by a field redefinition [3,4]; hence, the validity of the method not only applies to on-shell amplitudes, but also to off-shell quantities such as correlation functions. This paper provides the first application of the MHV diagram method to the computation of correlation functions.

There are several reasons to pursue an approach based on MHV diagrams. First, it is interesting to consider the application of this method to the computation of fully off-shell quantities such as correlation functions. Second, in the MHV diagram method there is a natural way to regulate the divergences arising from loop integrations, namely, dimensional regularization, used in conjunction with the four-dimensional expressions for the vertices. In this respect, we recall that one-loop amplitudes were calculated with MHV diagrams in [5], where the infinite sequence of MHV amplitudes in $\mathcal{N}=4$ SYM theory was rederived. One-loop amplitudes in $\mathcal{N}=1$ SYM theory were subsequently computed in [6,7], while in [8] the cut-constructible part of the infinite sequence of MHV amplitudes in pure Yang-Mills theory at one loop was presented. The $\mathcal{N}=1$ and $\mathcal{N}=0$ amplitudes have ultraviolet (UV) divergences (in addition to infrared ones), which are also regulated in dimensional regularization. The two-point correlation function we compute in this Letter also exhibits UV divergences, which we regulate in exactly the same way as in the case of amplitudes. The reader may consult [9,10] for further applications of the MHV diagram method to the calculation of loop amplitudes.

An additional motivation for our work is provided by the interesting recent papers [11,12]. In particular, [11] successfully computed Γ using $\mathcal{N}=4$ supersymmetric twistor actions [13–15]. It is known that such actions, in conjunction with a particular axial gauge choice, generate the MHV rules in twistor space [14], and the question naturally arises as to whether one could derive the dilatation operator directly using MHV diagrams in momentum space, without passing through twistor space. The answer to this question is positive, and furthermore we find that the calculation is very simple—it amounts to the evaluation of a single MHV diagram in dimensional regularization, leading to a single UV-divergent integral, identical to that appearing in [2].

The rest of the Letter is organized as follows. In the next section we briefly review the result of [2] for the integrable Hamiltonian describing the one-loop dilatation operator Γ in the SO(6) sector. In Sec. 3 we address the calculation of Γ using MHV diagrams. We present our conclusions and suggestions for future research in Sec. 4.

The one-loop dilatation operator.—The computation of the dilatation operator in the SO(6) sector of the $\mathcal{N}=4$ SYM theory is equivalent to extracting the UV-divergent part of the two-point function $\langle \mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)\rangle$, where $\mathcal{O}$ is a single-trace scalar operator of the form

$$\mathcal{O}_{A_1B_1,A_2B_2,\ldots,A_LB_L}(x) := \mathrm{Tr}(\phi_{A_1B_1}(x)\cdots\phi_{A_LB_L}(x)). \quad (2.1)$$

At one loop and in the planar limit, only nearest neighbor scalar fields can be connected by vertices. This simplifies the calculation to that of $\langle(\phi_{AB}\phi_{CD})(x_1)(\phi_{A'B'}\phi_{C'D'})(x_2)\rangle$. The expected flavor structure of this correlation function is

$$\langle(\phi_{AB}\phi_{CD})(x_1)(\phi_{A'B'}\phi_{C'D'})(x_2)\rangle = \mathcal{A}\epsilon_{ABCD}\epsilon_{A'B'C'D'} + \mathcal{B}\epsilon_{ABA'B'}\epsilon_{CDC'D'} + \mathcal{C}\epsilon_{ABC'D'}\epsilon_{A'B'CD}. \tag{2.2}$$

These three terms are usually referred to as trace, permutation, and identity. We are only interested in computing the leading UV-divergent contributions to the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, which, according to [2], are expected to be [in the definitions of $\mathcal{A}_{\mathrm{UV}}$, $\mathcal{B}_{\mathrm{UV}}$, and $\mathcal{C}_{\mathrm{UV}}$ we omit a factor of $\lambda/(8\pi^2)\times[1/(4\pi^2x_{12}^2)]^2\times(1/\epsilon)$].

$$\mathcal{A}_{\mathrm{UV}} = \frac{1}{2}, \qquad \mathcal{B}_{\mathrm{UV}} = -1, \qquad \mathcal{C}_{\mathrm{UV}} = 1. \tag{2.3}$$

This leads to the famous result of [2] for the one-loop dilatation operator Γ in the SO(6) sector,

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{n=1}^{L}\left(\mathbb{1} - \mathbb{P}_{n,n+1} + \frac{1}{2}\mathrm{Tr}_{n,n+1}\right), \tag{2.4}$$

where $\mathbb{P}$ and Tr are the permutation and trace operators, respectively. L is the number of scalar fields in the operator, and λ the ’t Hooft coupling.

In the MZ calculation, one particular integral plays a central role, depicted in Fig. 1. It is given by

$$I(x_{12}) = \int d^Dz\,\Delta^2(x_1 - z)\Delta^2(x_2 - z), \tag{2.5}$$

where $x_{12} := x_1 - x_2$ and

$$\Delta(x) := -\frac{\pi^{2-(D/2)}}{4\pi^2}\Gamma\left(\frac{D}{2} - 1\right)\frac{1}{(-x^2 + i\varepsilon)^{(D/2)-1}}, \tag{2.6}$$

is the scalar propagator in D dimensions. Note that $I(x_{12})$ has UV divergences arising from the regions $z \to x_1$ and $z \to x_2$.

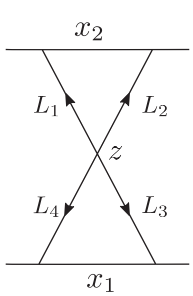

FIG. 1. The particular one-loop integral in configuration space contributing to the dilatation operator.

Because the MHV diagram method is formulated in momentum space, it is useful to recast $I(x_{12})$ as an integral in momentum space. Doing so, one finds that

$$\begin{aligned} I(x_{12}) &= \int \prod_{i=1}^{4}\frac{d^DL_i}{(2\pi)^D}\frac{e^{i(L_1+L_2)x_{12}}}{L_1^2L_2^2L_3^2L_4^2}(2\pi)^D\delta^{(D)}\left(\sum_{i=1}^{4}L_i\right) \\ &= \int\frac{d^DL}{(2\pi)^D}e^{iLx_{12}} \\ &\quad\times\int\frac{d^DL_1}{(2\pi)^D}\frac{d^DL_3}{(2\pi)^D}\frac{1}{L_1^2(L-L_1)^2L_3^2(L+L_3)^2}, \end{aligned} \tag{2.7}$$

where $L := L_1 + L_2$. The integral over L_1 and L_3 is the product of two bubble integrals with momenta as in Fig. 2, which are separately UV divergent. These divergences arise from the region $L_1, L_3 \to \infty$. The leading UV divergence of (2.7) is equal to

$$I(x_{12})|_{\mathrm{UV}} = \frac{1}{\epsilon}\frac{1}{8\pi^2}\frac{1}{(4\pi^2x_{12}^2)^2}. \tag{2.8}$$

The one-loop dilatation operator from MHV rules.— In this section we compute the UV-divergent part of the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ defined in (2.2), representing the trace, permutation, and identity flavor structures, respectively. In order to compute these three coefficients, it is sufficient to consider one representative configuration for each one. We will choose the following helicity [or SU(4)] assignments:

	$ABCD$	$A'B'C'D'$
Tr	1234	2413
$\mathbb{P}$	1213	3424
$\mathbb{1}$	1213	2434

(3.1)

There is a single MHV diagram to compute, represented in Fig. 3. It consists of one supersymmetric four-point MHV vertex,

$$V_{\mathrm{MHV}}(1,2,3,4) = \frac{\delta^{(4)}(\sum_{i=1}^4 L_i)\delta^{(8)}(\sum_{i=1}^4 \ell_i\eta_i)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \tag{3.2}$$

and four scalar propagators $1/(L_1^2\cdots L_4^2)$ connecting it to the four scalars in the operators. Here L_i are the (off-shell) momenta of the four particles in the vertex. The off-shell

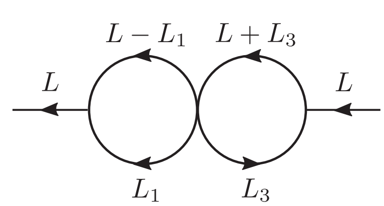

FIG. 2. The double-bubble integral relevant for the computation of $I(x_{12})$.

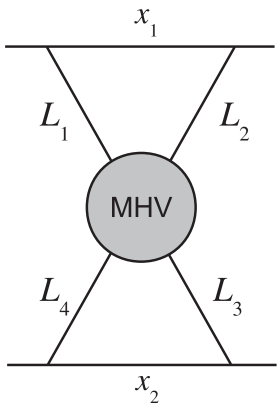

FIG. 3. The single MHV diagram contributing to the dilatation operator at one loop.

continuations of the spinors associated to the internal legs are defined using the prescription of [1], namely,

$$\ell_{i\alpha} := L_{i\alpha\dot{\alpha}} \xi^{\dot{\alpha}}. \qquad (3.3)$$

Here, $\xi^{\dot{\alpha}}$ is a constant reference spinor. As we mentioned earlier, MHV diagrams were derived in [3,4] from a change of variables in the Yang-Mills action quantized in the light cone gauge. The spinor $\xi^{\dot{\alpha}}$ is precisely related to this gauge choice. Next we extract the relevant component vertices for the three flavour assignments in (3.1). These turn out to be as follows:

$$\begin{aligned} \mathrm{Tr}&: \frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle},\\ \mathbb{P}&: -1,\\ \mathbb{1}&: \frac{\langle 13\rangle\langle 24\rangle}{\langle 23\rangle\langle 14\rangle}. \end{aligned} \qquad (3.4)$$

Hence, in the case of $\mathbb{P}$ the resulting loop integral is precisely the double-bubble integral $I(x_{12})$ of (2.7) (up to a sign), while in the other two cases the double-bubble integrand is dressed with the vertex factors in (3.4). In the following we discuss the additional contributions from the vertex for the three configurations Tr, $\mathbb{P}$, and $\mathbb{1}$.

The Tr integrand.—We begin our analysis with the vertex factor for the trace configuration, first line of (3.4). Using the off-shell prescription for MHV diagrams we can rewrite it as

$$T := \frac{[\xi|L_1L_3|\xi][\xi|L_2L_4|\xi]}{[\xi|L_1L_2|\xi][\xi|L_3L_4|\xi]}. \qquad (3.5)$$

Using momentum conservation to eliminate L_2 and L_4, this can be recast as a sum of three terms,

$$T = -\frac{[\xi|L_1L_3|\xi]}{[\xi|L_3L|\xi]} - \frac{[\xi|L_1L_3|\xi]}{[\xi|L_1L|\xi]} - \frac{[\xi|L_1L_3|\xi]^2}{[\xi|L_1L|\xi][\xi|L_3L|\xi]}, \qquad (3.6)$$

where $L := L_1 + L_2$. The first two terms correspond to linear bubble integrals in L_1 and L_3, respectively. We will study separately the contribution arising from the last term. The linear bubble integral is

$$\int \frac{d^DK}{(2\pi)^D}\frac{K^\mu}{K^2(K\pm L)^2} = \mp \frac{L^\mu}{2}\mathrm{Bub}(L^2), \qquad (3.7)$$

where

$$\mathrm{Bub}(L^2) := \int \frac{d^DK}{(2\pi)^D}\frac{1}{K^2(K+L)^2}. \qquad (3.8)$$

This is one of the two scalar bubbles comprising the MZ integral of Fig. 2. In the following we will then only quote the coefficient dressing the MZ integral. Doing so, the first term in (3.6) becomes, after the reduction,

$$-\frac{[\xi|LL_3|\xi]}{[\xi|L_3L|\xi]}\frac{1}{2} = \frac{1}{2}. \qquad (3.9)$$

Similarly, the second term in (3.6) gives a result of $+1/2$. Next, we move to the third term. To simplify its expression, we first notice that the bubble integral in L_1 is symmetric under the transformation $L_1 \to L - L_1$. The idea is then to simplify the integrand by using this symmetry. Thus, we rewrite the quantity $[\xi|L_1L_3|\xi]$ in the numerator as $[\xi|L_1L_3|\xi] = [\xi|(L_1 - \frac{1}{2}L)L_3|\xi] + \frac{1}{2}[\xi|LL_3|\xi]$. Doing so, we get

$$\begin{aligned} -\frac{[\xi|L_1L_3|\xi]^2}{[\xi|L_1L|\xi][\xi|L_3L|\xi]} = &-\frac{[\xi|(L_1 - \frac{L}{2})L_3|\xi]^2}{[\xi|L_1L|\xi][\xi|L_3L|\xi]} + \frac{1}{4}\frac{[\xi|LL_3|\xi]}{[\xi|L_1L|\xi]}\\ &+ \frac{[\xi|(L_1 - \frac{1}{2}L)L_3|\xi]}{[\xi|L_1L|\xi]}. \end{aligned} \qquad (3.10)$$

We then notice that the first and the second term are antisymmetric under the transformation $L_1 \to L - L_1$ and hence vanish upon integration. The third term is a sum of two linear bubbles in L_3, and the corresponding contributions are quickly seen to be equal to $-1/2$ and zero, respectively.

Summarizing, the trace integral gives a contribution of $1/2$ times the dimensionally regularized MZ integral. Thus, $\mathcal{A}_{\mathrm{UV}} = 1/2$.

The $\mathbb{P}$ integrand.— In this case the vertex is simply -1 and the corresponding result is -1 times the MZ integral, or $\mathcal{B}_{\mathrm{UV}} = -1$.

The $\mathbb{1}$ integrand.—The relevant vertex factor is written in the third line of (3.4). In this case we observe that

$$\frac{\langle 13\rangle\langle 24\rangle}{\langle 23\rangle\langle 14\rangle} = 1 + \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle}. \qquad (3.11)$$

The first term gives a contribution equal to the MZ integral, and we will now argue that the second term is UV finite, and hence does not contribute to the dilatation operator. Indeed, we can write

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle} = \frac{[\xi|L_1 L|\xi][\xi|L_3 L|\xi]}{[\xi|(L-L_1)L_3|\xi][\xi|L_1(L+L_3)|\xi]}. \quad (3.12)$$

The UV divergences we are after arise when L_1 and L_3 are large. The integrand (3.12) provides one extra power of momentum per integration, which makes each of the two bubbles in the MZ integral finite. (One may also notice that for large L_1 and L_3 the integrand becomes an odd function of these two variables, and thus the integral should be suppressed even further than expected from power counting.) Thus. $\mathcal{C}_{\rm UV} = 1$.

We end this section with a couple of comments. 1. Since MHV diagrams are obtained from a particular axial gauge choice, combined with a field redefinition [3,4], it is guaranteed that ξ dependence drops out at the end of the calculation. In the present case one can see this directly as follows. Lorentz invariance ensures that the result of the L_1 and L_3 integrations can only depend on L^2, as the other Lorentz-invariant quantity $[\xi|L^2|\xi]$ vanishes (note that $L\xi$ cannot appear as our integrands only depend on the antiholomorphic spinor $\xi_{\dot\alpha}$). 2. We point out that in the MHV diagram formalism there are no self-energy corrections to the propagator, as already observed in Sec. 6 of [5]. Presumably this is also the case for the self-energy evaluated with the twistor action of [14] employed in [11] for the calculation of the one-loop dilatation operator. It is interesting to note that the superfield renormalization in the light cone gauge formalism of [16] is finite in the $\mathcal{N} = 4$ theory.

Conclusions.—We conclude with some suggestions for future investigations. First, it would be interesting to apply MHV diagrams to the calculation of the dilatation operator in other sectors of $\mathcal{N} = 4$ SYM theory, also containing fermions and derivatives. Applications to different Yang-Mills theories with less supersymmetry can also be considered, given the validity of the MHV diagram method beyond $\mathcal{N} = 4$ SYM theory.

An obvious goal is the extension of our calculation to higher loops. This has proved difficult for amplitudes, but addressing the calculation of just the UV-divergent part of the two-point correlation function may simplify this task enormously. At one loop the complete dilatation operator is known [17], while direct perturbative calculations at higher loops—without the assumption of integrability—have been performed only up to two [18–20], three [21–23], and four loops [24] in particular sectors. A simplified route to such a calculation would be greatly desirable, and would provide further verification of this crucial assumption. The expected structure remains that of (2.7), with the double-bubble integral replaced by more complicated (but still single-scale) loop integrals.

It would also be very interesting if one could apply other on-shell methods such as generalized unitarity [25,26] to the direct calculation of two-point functions, and hence to the dilatation operator of $\mathcal{N} = 4$ SYM theory.

Finally, our result hints at a link between the Yangian symmetry of amplitudes in $\mathcal{N} = 4$ SYM theory[27] and integrability of the dilatation operator of the theory [2,17,28–31]. It would be interesting to explore this point further. We hope to be able to report on some of these ideas in the very near future.

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