Sound Velocity Bound and Neutron Stars

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It has been conjectured that the velocity of sound in any medium is smaller than the velocity of light in vacuum divided by $\sqrt{3}$. Simple arguments support this bound in nonrelativistic and/or weakly coupled theories. The bound has been demonstrated in several classes of strongly coupled theories with gravity duals and is saturated only in conformal theories. We point out that the existence of neutron stars with masses around two solar masses combined with the knowledge of the equation of state of hadronic matter at "low" densities is in strong tension with this bound.

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Introduction.—The nature of matter at high baryon number density is one of the outstanding open problems of nuclear and astrophysics. In principle, the properties of matter at densities comparable to the nuclear saturation density ($n_0 \approx 0.16/\mathrm{fm}^3$) are determined by QCD. In practice, it has been very difficult to extract the QCD predictions for dense matter except at extremely high densities where asymptotic freedom allows for perturbative calculations. The structure of large nuclei provides some information about densities around the nuclear saturation density. Above the nuclear saturation density, all known theoretical methods break down: nuclear effective theories break down due to the high Fermi momentum, and lattice calculations are plagued by sign problems. The only empirical evidence we have about matter at higher baryon densities comes from the study of neutron stars which contain matter up to 5–8 times the saturation density.

General relativity connects the equation of state of dense matter with the relation between the radius and the mass of neutron stars. Rotation, magnetic fields, and finite temperature make only small corrections to the mass-radius relation. Also, we assume, in this Letter, that the ground state of matter at low densities is well described by laboratory nuclei. Thus, the mass-radius relation is essentially unique, and the measurement of radii and masses of several neutron stars determines the equation of state at high energy density. For each equation of state, there is a maximum mass beyond which no stable configuration is possible, regardless of the radius, since a more massive star would collapse into a black hole. The higher the pressure for a given energy density, the larger is the maximum supported (gravitational) mass. In the last few years, two stars were observed with a mass around two solar masses with very small error bars. One is a millisecond pulsar in a binary system whose mass was determined through Shapiro delay [1]; the other has a white dwarf companion whose spectroscopy allowed a precise determination of the neutron star mass [2]. These two observations currently provide the strictest empirical constraints on the equation of state of dense matter.

One way of characterizing dense matter is through the velocity of sound given by $v_s^2 = dp/d\epsilon$ (We use a system of units where $\hbar = c = 1$.), where p is the pressure and ϵ the energy density (including the rest mass of the particles). Causality implies an absolute bound $v_s \leq 1$ and thermodynamic stability guarantees that $v_s^2 > 0$. There are reasons, however, to expect more stringent bounds applicable to all, or, at least, a large class of materials [3]. Nonrelativistic models, at least in the range of densities where they are applicable, predict, obviously, $v_s \ll 1$. On the other extreme, we have gases composed of ultrarelativistic (massless) particles where $v_s^2 = 1/3$. The inclusion of a mass for the particles lowers the speed of sound to $v_s^2 < 1/3$. Interactions among the particles, if perturbative, also lead to $v_s < 1/3$. This is the case of QCD at asymptotically high densities (or temperatures) where a weak coupling expansion is valid. Thus, it is natural to speculate that the speed of sound at intermediate densities will interpolate between these two limits and stay at all densities below the $v_s^2 = 1/3$ value, at least in asymptotically free theories like QCD. The alternative would be the presence of a bump in the speed of sound at intermediate densities before its value approaches $v_s^2 = 1/3$ from below, asymptotically, implying the existence of a maximum and a local minimum of v_s as a function of μ.

There are other reasons, to believe that the $v_s^2 < 1/3$ bound is valid, even in other theories besides QCD. The value $v_s^2 = 1/3$ is common to all systems with conformal symmetry, of which free massless gases are just one example. In fact, the vanishing of the trace of the momentum-energy tensor—the hallmark of conformal theories—implies that the energy density ϵ and the pressure p are related by $\epsilon = 3p$ and, consequently, that $v_s^2 = 1/3$, even in the case of strongly interacting systems.

In order to find a violation of the speed of sound bound, we should, then, look at strongly interacting relativistic systems away from conformality. Strongly coupled theories are difficult to analyze, but several calculations of the speed of sound in several different models were performed in the strong coupling limit using the AdS/CFT correspondence. The speed of sound was computed at high temperatures in the single scalar model [3,4], the Sakai-Sugimoto model [5] (a close analogue to QCD), the $D3/D7$ system [6], and the $\mathcal{N} = 2^*$ gauge theory [7], and, in all cases, the bound $v_s^2 < 1/3$ is respected. The bound was also verified in the $D3/D7$ system at finite baryon and isospin chemical potential. Each of these holographic models corresponds to a whole family of four dimensional field theories. It is unclear, however, how broad the set of theories covered by these examples actually is.

Some additional insight into the physical origins of the apparent $v_s^2 < 1/3$ bound can be obtained writing the (baryon number) density n as $n = N(\mu)\mu^3/(6\pi^2)$. For a free ultrarelativistic fermionic gas, $N(\mu)$ is independent of μ and equal to the number of "degrees of freedom" of the system (different species, polarizations, etc.). In general, $N(\mu)$ depends on μ, but we will still refer to $N(\mu)$ as the number of effective degrees of freedom relevant at chemical potential μ. Simple thermodynamics arguments lead to the relation $v_s^2 = \frac{1}{3}[1 + \frac{\mu}{3}N'(\mu)/N(\mu)]^{-1}$, so as long as the number of effective degrees of freedom increases with μ (and the density), the velocity bound is valid. A similar argument can be made for the finite temperature case by substituting μ and $n(\mu)$ by the temperature T and the entropy density $s(T)$. In finite temperature QCD, the degrees of freedom at small temperatures are the pions and, at high temperatures, the much more numerous gluons and quarks. Lattice QCD calculations show that $N(T)$ is, indeed, a monotonically increasing function of T [8], and the bound $v_s^2 < 1/3$ is valid. It is much less clear whether a similar thing happens at finite chemical potential. Some arguments [9] suggest that the related quantity $\tilde{N}(T) = -f(T)/T^4$, where $f(T)$ is the free energy density, is an increasing function of T in asymptotically free theories, a result similar in spirit to the "a theorem" [10] valid for all local, unitary field theories.

There are counterexamples to the bound $v_s^2 < 1/3$. Nonrelativistic models lead to $v_s^2 > 1/3$, and even $v_s^2 > 1$, at high densities where they are not applicable. The well-known counterexample of Zeldovich [11] relies on semiclassical arguments, mean field approximations, and the neglect of retardation effects. Perhaps a better counterexample is the case of QCD with an isospin chemical potential μ_I larger than the pion mass but smaller than QCD scales. The isospin chemical potential drives the formation of a pion condensate (one also has to assume that electromagnetism is "turned off" to allow for charged pion condensation), and the energy density oscillations on top of the condensate violates the velocity bound, as a simple chiral perturbation theory calculation shows [12]. Notice that, in this case, the medium is comprised of a condensate of bosons, and there is no net baryon number, a situation physically very different to the finite density of the baryon number we are interested in.

The purpose of this Letter is to demonstrate that there is an acute tension between the $v_s^2 < 1/3$ conjecture and the existence of neutron stars with masses $M \approx 2M_\odot$ for all reasonable low density equations of state. This tension, for two equations of state, was already observed in [13] (see, also, related earlier work in Refs. [14–16]). Assuming the validity of the sound speed bound, the properties of strongly interacting matter at low density are known well enough to put a bound on the largest star mass achievable. Because the equation of state is very constrained up to baryon number densities about $2n_0$, the increase of the pressure with the density is limited by the assumption $v_s^2 = dp/d\epsilon < 1/3$. In this case, the equation of state with the largest maximum mass is that with the largest pressure above $2n_0$ [17,18]. As a consequence, there is a bound on the largest neutron star mass consistent with fairly well established facts about the low density behavior of mass and the bound $v_s^2 < 1/3$. The remainder of this Letter will demonstrate that the numerical value of this bound is near $2M_\odot$ and will quantify the uncertainties.

The equation of state for $n < 2n_0$.—For densities below $2n_0$, a nonrelativistic model of nucleons interacting through a (possibly momentum-dependent) potential is adequate. The interactions of the nucleons in the relevant energy regime are well known experimentally and are well fit by several potential models. Modern Monte Carlo methods are capable of using those to determine the spectrum of light nuclei and bulk matter with negligible numerical error. The hierarchy observed between two- and three-body forces, as well as different components of the three-body force, follow the expectation of effective theory power counting arguments (for a review, see, for instance, Ref. [19]).

The two-body force obtained from the chiral low momentum expansion fits the scattering data very well. Many-body calculations using the two-and three-body forces up to next-to-next-to-leading order in the low momentum expansion were argued to be perturbative in [20] and the neutron matter equation of state computed in [21,22]. Similarly, the equation of state of pure neutron matter with the AV8′ two-body force (which fits all s-and p-wave phase shifts up to energies in excess of the ones found in back-to-back scattering of neutrons on the Fermi surface at $n = 2n_0$) and a variety of three-nucleon forces fit to reproduce the binding energy of nuclear matter was computed in Refs. [23–25] with a numerical error smaller than 2%. The different three-body forces lead to different equations of state at high densities, but, up to densities $n < 2n_0$, their effect is modest. We can see, in Ref. [23], that the difference in the energy per neutron at $n = 2n_0$ between two extreme models (no three-body force and the strongly repulsive Urbana IX three-body force) is about 2 MeV (when the three-body forces are tuned so the binding energy of nuclear matter at saturation is fixed) to 12 MeV (when the three-body forces change to cover the range of empirically allowed values of nuclear binding). This is to be compared to the total energy per neutron which is dominated by the rest mass $M_N = 939$ MeV. This approach gives, for densities $n < 2n_0$, very similar results, and with similar uncertainties, to the one in Refs. [21,22].

In a real star, the weak interactions allow for the β decay of neutrons into protons and a small proton fraction, $x = n_P/n < 6\%$, is expected. In order to incorporate this information into the small extrapolation from neutron matter (with $x = 0$) to β-equilibrated matter, we use the Skyrme-like parametrization [22,26]

$$\begin{aligned}\frac{\epsilon(n,x)}{n_0} &= (1-x)M_N + xM_P \\ &+ \frac{3T_0}{5}[x^{5/3} + (1-x)^{5/3}]\left(\frac{2n}{n_0}\right)^{2/3} \\ &- T_0[(2\alpha - 4\alpha_L)x(1-x) + \alpha_L]\frac{n}{n_0} \\ &+ T_0[(2\eta - 4\eta_L)x(1-x) + \eta_L]\left(\frac{n}{n_0}\right)^{\gamma}, \quad (1)\end{aligned}$$

with $T_0 = (3\pi^2 n_0/2)^{2/3}/2M_N$. When reduced to pure neutron matter ($x = 0$), Eq. (1) fits the results of Refs. [21–25] very well, and it is a convenient manner in which to parametrize them. Choosing the parametrization of Ref. [23] would give similar results to those we report.

The five parameters $\alpha, \alpha_L, \eta, \eta_L$, and γ can be determined by the empirical knowledge of five quantities

$$\begin{aligned}-B &= \frac{\epsilon(n_0, 1/2)}{n_0} - \frac{M_N + M_P}{2}, \\ p &= n^2 \frac{\partial(\epsilon/n)}{\partial n}\bigg|_{n=n_0, x=1/2} = 0, \\ K &= 9n_0^2 \frac{\partial^2(\epsilon/n)}{\partial n^2}\bigg|_{n=n_0, x=1/2}, \\ S &= \frac{1}{8n_0}\frac{\partial^2 \epsilon}{\partial x^2}\bigg|_{n=n_0, x=1/2}, \\ L &= \frac{3n_0}{8}\frac{\partial^3(\epsilon/n)}{\partial n \partial x^2}\bigg|_{n=n_0, x=1/2}. \quad (2)\end{aligned}$$

The analysis of nuclear masses predicts $B = 16 \pm 0.1$ MeV and $n_0 = 0.16 \pm 0.01\ \text{fm}^{-3}$ [27], and the study of giant resonances implies $K = 235 \pm 25$ MeV for the nuclear incompressibility. Finally, a wide range of experimental data from nuclear masses, dipole polarizabilities, and giant resonances implies $S = 32 \pm 2$ MeV for the symmetry energy and $L = 50 \pm 15$ MeV (see [28,29] and references therein). Given values of B, n_0, and K, one can determine α, η, and γ, and then S and L can be used to obtain α_L and η_L. After a set of parameters is chosen, the β-equilibrated state is found by minimizing $\epsilon(n,x)$ in relation to x for any given value of n. At the highest density considered, and for all parameters used, $x < 6\%$, confirming that only a slight extrapolation for the pure neutron case is necessary.

Bound on neutron star masses.—We will now determine the highest neutron mass achievable assuming the validity of the bound $v_s^2 < 1/3$ and the knowledge on the low density equation of state discussed in the previous section. Within the set of equations of state satisfying the low density and the $v_s^2 < 1/3$ constraints, the equation of state with the largest pressure is given by

$$\epsilon(p) = \begin{cases} \min_x \epsilon(n(p), x), & n < 2n_0 \\ \min_x \epsilon(2n_0, x) + 3p, & n > 2n_0 \end{cases}. \quad (3)$$

To reflect our uncertainty of the low density equation of state, we choose the parameters $\alpha, \alpha_L, \eta, \eta_L$, and γ in Eq. (1) by selecting values for K, S, and L at random with a Gaussian distribution centered around their empirical central values and standard deviation given by uncertainty of their empirical determination. Note that increasing the transition density ($2n_0$) would require massive stars to have a larger sound speed, and lowering it significantly would conflict microscopic calculations of the equation of state. The small uncertainties in B and n_0 do not affect our results. Notice that each of these equations of state are not meant to be realistic at high densities; they are continuous, but the speed of sound has a sudden jump at $n = 2n_0$. Rather, they provide an upper bound on the pressure for each value of the pressure and, by the result in Ref. [17], an upper bound on the maximum mass of the star. For each of these equations of state (namely, for each value of $\alpha, \alpha_L, \eta, \eta_L$, and γ) the Tollman-Oppenheimer-Volkov equations, describing the structure of a spherically symmetric star, is solved and the maximum mass allowed is determined. The result is shown in the histogram in Fig. 1.

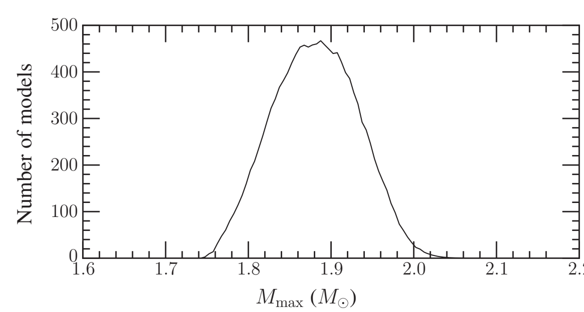

FIG. 1. Histogram of the number of models as a function of the maximum mass supported.

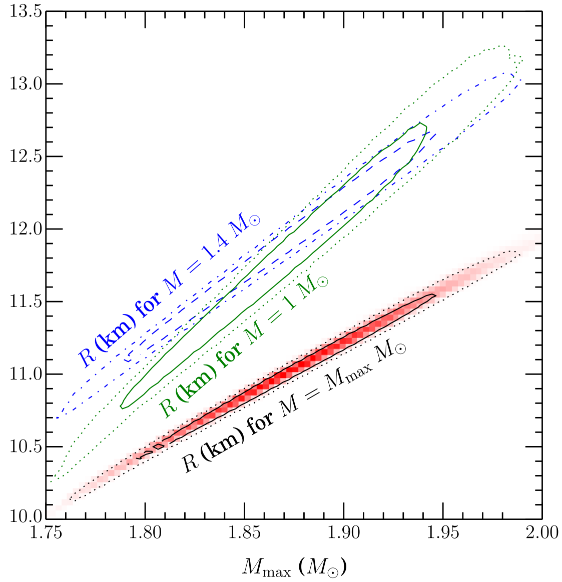

FIG. 2 (color online). Radii versus maximum mass for configurations with the largest possible pressure subject to the velocity bound. The 68% and 95% confidence regions are outlines for the radii for three different masses.

The most important feature of Fig. 1, and the main point of this Letter, is the abrupt disappearance of viable models at masses larger than about $2M_\odot$. We will refrain from identifying the number of models capable of sustaining masses above $2M_\odot$ to a probability as the error bars in the input parameters of the low density equation of state are dominated by systematic errors. Still, Fig. 1 makes clear that the $v_s^2 < 1/3$ bound is in strong tension with known empirical facts. This conclusion is even more believable if one notices that we have intentionally left out phenomena —like the appearance of hyperons and other degrees of freedom—that would further decrease the pressure but that are less certain and harder to quantify. Also, since our purpose was to establish an upper bound on the maximum mass, we used equations of state where the speed of sound changes suddenly from its value at $n = 2n_0$ to $v_s^2 = 1/3$. A smoother, more realistic transition would further reduce the maximum mass.

The observation of neutron stars with small radii tends to strengthen the argument that the velocity bound must be violated. The correlation between the radius of a 1.0 $M_\odot$ neutron star and the maximum mass is displayed in Fig. 2. The observation of a 1.0 $M_\odot$ neutron star with a radius smaller than 13 km, or the observation of any neutron star with a radius less than 11.8 km, means that the velocity bound must be violated. In particular, the neutron star in the globular cluster NGC 6397 already suggests that the velocity bound must be violated, but there are several systematic uncertainties which make this connection less clear [30,31].

If the bound on the speed of sound is actually violated—as is strongly suggested by our results—the speed of sound, as a function of the energy density, has a peculiar shape. It raises from small values, reaches a maximum with $v_s^2 > 1/3$, lowers to a local minimum with $v_s^2 < 1/3$, and then raises again, approaching $v_s^2 = 1/3$ from below at high densities. We find it remarkable that such a conclusion can be derived from well established facts.

There is, however, another way of looking at our result. If a proof of the speed of sound bound is obtained, either by adapting the arguments in Refs. [9,10] or by other means, our results imply that the equation of state of QCD at finite density would essentially be known up to several times nuclear saturation densities, as only models that, at low density, are the hardest allowed by empirical evidence and rapidly transition to one with $v_s^2 = 1/3$ can support stars as heavy as two solar masses. Of course, the determination of the equation of state within such a narrow range has been a "holy grail" of nuclear and astrophysics since the discovery of pulsars. In addition, such a result would imply that other degrees of freedom, like Λ hyperons, cannot appear in neutron stars in any significant numbers, which requires strong repulsion between Λ and neutrons [32]. We would also know that neutron stars have radii on the upper range of the current estimates with important consequences for the detection of gravitational waves generated in neutron star collisions [33]. The importance of all these questions seems to warrant further field theoretical studies on the status of the speed of sound bound. Hopefully the present Letter, by pointing out the phenomenological consequences that such a proof would have, will spark an interest in this question.

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